Working with algebraic fractions

An algebraic fraction involves division and is normally indicated by fraction notation such as
\[
\frac{2x}{x + 1}
\]
As with a numerical fraction, the top and bottom of the algebraic fraction are called the numerator and the denominator, respectively.

When you want to write an algebraic fraction as part of a line of text, you can replace the horizontal line by a slash symbol, but remember that you may need to enclose the numerator and/or denominator in brackets to make it clear what is divided by what.

Simplifying algebraic fractions

The process of dividing the numerator and denominator of a fraction by a common factor is known as cancelling the factor, or cancelling down the fraction. You should simplify a fraction as much as possible. To simplify a fraction cancel the highest common factor of the numerator and denominator. It is often easiest to do this by cancelling in stages: first you cancel one common factor, then another, and so on, until eventually the overall effect is that you have cancelled the highest common factor.

Example 1

Simplify the following algebraic fractions.

(a) \( \frac{12a^2b^4}{9a^3b} \)

(b) \( \frac{x^3(x + 1)}{x(x + 1)^2} \)

Solution

(a) Divide top and bottom by 3 (the highest common factor of the coefficients), then by \( a^2 \) (the highest common factor of the powers of \( a \)), and finally by \( b \) (the highest common factor of the powers of \( b \)).

\[
\frac{12a^2b^4}{9a^3b} = \frac{4}{3} \frac{2a^2b^4}{a^3b} = \frac{4}{3} \frac{1}{a} \frac{b^4}{b} = \frac{4b^3}{3a} \quad (b \neq 0)
\]

(b) Divide top and bottom by \( x \) (the highest common factor of the powers of \( x \)), then by \( x + 1 \) (the highest common factor of the powers of \( x + 1 \)).

\[
\frac{x^3(x + 1)}{x(x + 1)^2} = \frac{x^2}{1} \frac{(x + 1)}{x} = \frac{x^2}{x} \frac{1}{1} \frac{(x + 1)}{x + 1} = \frac{x^2}{x + 1} \quad (x \neq 0)
\]
Example 2  \textit{Simplifying more algebraic fractions}

Simplify the following algebraic fractions.

(a) \[ \frac{x^2 + 2x}{5x^3 - 3x} \quad (b) \frac{u^2 - 2u}{3u - 6} \]

Solution

\begin{itemize}
\item Factorise the numerator and denominator to check for common factors. Cancel any common factors. \hfill \lozenge \hfill \lozenge \\
\end{itemize}

\begin{align*}
(a) & \quad \frac{x^2 + 2x}{5x^3 - 3x} = \frac{x(x+2)}{x(5x^2-3)} = \frac{x+2}{5x^2-3} \quad (x \neq 0) \\
(b) & \quad \frac{u^2 - 2u}{3u - 6} = \frac{u(u-2)}{3(u-2)} = \frac{u}{3} \quad (u \neq 2)
\end{align*}

\section*{Adding and subtracting algebraic fractions}

If the denominators are the same just add the numerators.

For example,
\[ \frac{2x}{bc} + \frac{19y}{bc} - \frac{2h}{bc} = \frac{2x + 19y - 2h}{bc} \].

If the denominators are different you need do some work to write each fraction with the same denominator. The correct denominator has to be a common multiple of the denominators of all the fractions to be added or subtracted. The easiest way to find a suitable denominator is to multiply all the denominators together.

For example,
\[ \frac{3}{x} + \frac{7}{y} \].

The new denominator will be $xy$ when we multiply both denominators together. Remember that we also have to multiply each numerator by the other fraction’s denominator
\[ \frac{3y}{xy} + \frac{7x}{xy} \].

Look carefully at the new expression above and make sure that you can see that it is exactly the same as the original expression if $x$ and $y$ are cancelled.

Now the denominators are the same we can add the fractions to get
\[ \frac{3y + 7x}{xy} \].
**Example 3  Adding and subtracting algebraic fractions**

Write each of the following expressions as a single algebraic fraction.

(a) \( \frac{x + 2}{x^2} - \frac{2}{x^2} \)
(b) \( \frac{1}{ab} + \frac{2}{a} - \frac{3}{b} \)
(c) \( \frac{x}{1 - x} - \frac{1}{x} \)
(d) \( \frac{5}{d} + c \)

**Solution**

(a) The denominators are the same, so subtract the second numerator from the first. Simplify the answer.

\[
\frac{x + 2}{x^2} - \frac{2}{x^2} = \frac{x + 2 - 2}{x^2} = \frac{x}{x^2} = \frac{1}{x}
\]

(b) First write the fractions with the same denominator, by multiplying the top and bottom of each fraction by an appropriate expression. Then add and subtract the numerators.

\[
\frac{1}{ab} + \frac{2}{a} - \frac{3}{b} = \frac{1}{ab} + \frac{2b}{ab} - \frac{3a}{ab} = \frac{1 + 2b - 3a}{ab}
\]

(c) Proceed in a similar way to part (b). Simplify the answer.

\[
\frac{x}{1 - x} - \frac{1}{x} = \frac{x^2}{x(1 - x)} - \frac{1 - x}{x(1 - x)} = \frac{x^2 - (1 - x)}{x(1 - x)} = \frac{x^2 + x - 1}{x(1 - x)}
\]

(d) First write \( c \) as a fraction, then proceed as before.

\[
\frac{5}{d} + c = \frac{5}{d} + \frac{c}{1} = \frac{5 + cd}{d}
\]

**Multiplying algebraic fractions**

To multiply two or more algebraic fractions, multiply the numerators together and multiply the denominators together.

For example,

\[
\frac{2x}{3wk} \times \frac{wy}{k} = \frac{2xwy}{3wk^2}
\]

Notice that \( w \) is common to numerator and denominator so it cancels to give

\[
\frac{2xy}{3k^2}
\]
Dividing algebraic fractions

To divide by an algebraic fraction, multiply by its reciprocal.

For example,
\[
\frac{2r}{gt} ÷ \frac{ry}{a} = \frac{2r}{gt} \times \frac{a}{ry} = \frac{2ra}{gt\cdot ry} = \frac{2a}{gt\cdot y}
\]

Remember to always simplify your answer if possible.

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**Example 4**  **Multiplying and dividing algebraic fractions**

Simplify the following expressions.

(a) \( \frac{2x - 5}{(x - 1)^2} \times \frac{x - 1}{4} \)

(b) \( \frac{9P^2}{Q^8} ÷ \frac{3P^3}{Q^9} \)

**Solution**

(a) To multiply the fractions, multiply the numerators and multiply the denominators. Cross-cancel any common factors first.

\[
\frac{2x - 5}{(x - 1)^2} \times \frac{x - 1}{4} = \frac{2x - 5}{(x - 1)^2} \times \frac{x - 1}{4} = \frac{2x - 5}{4(x - 1)}
\]

(b) To divide by a fraction, multiply by the reciprocal. Cross-cancel any common factors before doing the multiplication.

\[
\frac{9P^2}{Q^8} ÷ \frac{3P^3}{Q^9} = \frac{9P^2}{Q^8} \times \frac{Q^9}{3P^3} = \frac{3P^2}{Q^8} \times \frac{Q^9}{P^3} = \frac{3Q^9}{P^3} = \frac{Q^9}{P^3}
\]