**Finding the magnitude and direction of a vector**

**Finding the magnitude of a vector from its components**

Consider any two-dimensional vector \( \mathbf{v} \), not parallel to an axis. If you draw a right-angled triangle whose hypotenuse is \( \mathbf{v} \), and whose two shorter sides are parallel to the axes, then the lengths of these two shorter sides are the **magnitudes** of the components of \( \mathbf{v} \). This is illustrated in Figure 1, for a vector \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} \) whose components \( a \) and \( b \) are positive.

![Figure 1](image)

**Figure 1** A two-dimensional vector with positive components

By Pythagoras’ theorem, the magnitude of the vector \( \mathbf{v} \) in Figure 1 is given by

\[
|\mathbf{v}| = \sqrt{a^2 + b^2}.
\]

You can see that this formula will hold for any vector \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} \), no matter whether the components \( a \) and \( b \) are positive, negative or zero, for reasons similar to those that you saw for the distance formula. So we have the following fact.

**The magnitude of a two-dimensional vector in terms of its components**

The magnitude of the vector \( a \mathbf{i} + b \mathbf{j} \) is \( \sqrt{a^2 + b^2} \).

There is a similar formula for three-dimensional vectors.

**The magnitude of a three-dimensional vector in terms of its components**

The magnitude of the vector \( a \mathbf{i} + b \mathbf{j} + c \mathbf{k} \) is \( \sqrt{a^2 + b^2 + c^2} \).

**Activity 1** Finding the magnitudes of vectors from their components

Find the magnitudes of the following vectors. Give exact answers.

(a) \( -3 \mathbf{i} + \mathbf{j} \)     (b) \( 2 \mathbf{i} + 4 \mathbf{j} - 3 \mathbf{k} \)     (c) \( \begin{pmatrix} -2 \\ 0 \end{pmatrix} \)     (d) \( \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \)
Solution

(a) $|−3i+j| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$

(b) $|2i+4j−3k| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$

(c) $\left|\begin{array}{c} -2 \\ 0 \end{array}\right| = \sqrt{(-2)^2 + 0^2} = 2$

(d) $\left|\begin{array}{c} 3 \\ -1 \\ 3 \end{array}\right| = \sqrt{3^2 + (-1)^2 + 3^2} = \sqrt{19}$

Finding the direction of a two dimensional vector from its components

As you saw earlier, you can express the direction of a two-dimensional vector by stating the angle (measured clockwise or anticlockwise) from some chosen reference direction to the direction of the vector.

A method of expressing the direction of a two-dimensional vector, which is often used when components are involved, is to state the angle measured from the positive $x$-direction to the direction of the vector. The angle is always measured anticlockwise, and this is usually assumed rather than being stated explicitly. For example, Figure 2 shows two vectors making angles of $35^\circ$ and $200^\circ$, respectively, with the positive $x$-direction.

With this method, you can state the angles in any convenient way – they don’t need to be between $0^\circ$ and $360^\circ$, and they could be measured in radians. Sometimes it is helpful to use negative angles – for example, you can say that the vector $b$ in Figure 2 makes an angle of $-160^\circ$ with the positive direction of the $x$-axis.

![Figure 2](image)

Suppose that you have a two-dimensional vector in component form, and you want to find the angle that it makes with some reference direction such as the north or the positive $x$-direction. The first thing to do is to sketch the vector. If it is parallel to an axis, then it should be straightforward to find the angle that you want. Otherwise, you can sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the axes. The lengths of the shorter sides are the magnitudes of the components of the vector.

You can use basic trigonometry to find an acute angle in the triangle, and you can then use this acute angle to find the angle that you want. This method is demonstrated in the following example.
Example 1  Finding the magnitude and direction of a vector from its components

Find the magnitude of the vector $4\mathbf{i} - 3\mathbf{j}$, and the angle that it makes with the positive $x$-direction. Give the angle to the nearest degree.

Solution

Use the standard formula to find the magnitude. The magnitude of the vector is

$$\sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$ 

To find the required angle, first draw a diagram.

Find the acute angle $\phi$, and hence find the required angle. From the diagram,

$$\tan \phi = \frac{3}{4},$$

so

$$\phi = \tan^{-1} \left( \frac{3}{4} \right) = 36.86\ldots^\circ.$$ 

Hence the angle, labelled $\theta$, that the vector makes with the positive $x$-direction is

$$360^\circ - 36.86\ldots^\circ = 323.13\ldots^\circ$$

$$= 323^\circ$$ (to the nearest degree).

Activity 2  Finding magnitudes and directions of vectors from their components

Find the magnitudes and directions of the following vectors. Give the magnitudes as exact values, and give the directions as the angles that the vectors make with the positive $x$-direction, to the nearest degree.

(a)  $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$  (b)  $\mathbf{b} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}$
Solution

(a) The magnitude of the vector $a = -2i + 3j$ is

$$|a| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

The vector $a$ is shown below.

The angle $\phi$ is given by

$$\phi = \tan^{-1}\left(\frac{2}{3}\right) = 34^\circ \text{ (to the nearest degree)}.$$

So the angle $\theta$ that $a$ makes with the positive $x$-direction is $34^\circ + 90^\circ = 124^\circ \text{ (to the nearest degree)}.$

(b) The magnitude of the vector $b = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}$ is

$$|b| = \sqrt{0^2 + (-3.5)^2} = 3.5.$$

The vector $b$ has $i$-component 0, so it is parallel to the $y$-axis. Its $j$-component is negative, so $b$ points in the negative $y$-direction. Hence the angle that it makes with the positive $x$-direction is $270^\circ$.

To find the direction of a vector as a bearing, you need to find the angle measured clockwise from north to the direction of the vector. You can try this in the next activity.

Activity 3  Finding magnitudes, and directions as bearings, from components

Find the magnitude and bearing of each of the following velocity vectors. Assume that $i$ and $j$ are taken to point east and north, respectively, and that the units are m s$^{-1}$. Give the magnitudes of the vectors in m s$^{-1}$ to one decimal place, and their bearings to the nearest degree.

(a) $u = 4i - 2j$  (b) $v = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$
Solution

(a) The magnitude of the vector \( \mathbf{u} = 4 \hat{i} - 2 \hat{j} \) is
\[
|\mathbf{u}| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 4.5 \text{ m s}^{-1} \text{ (to 1 d.p.)}.
\]

The angle \( \phi \) is given by
\[
\phi = \tan^{-1} \left( \frac{4}{2} \right) = \tan^{-1}(2) = 63.43 \ldots ^\circ.
\]
So the bearing of \( \mathbf{u} \) is \( \theta = 180^\circ - 63.43 \ldots ^\circ = 117^\circ \) (to the nearest degree).

(b) The magnitude of the vector \( \mathbf{v} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \) is
\[
|\mathbf{v}| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17} = 4.1 \text{ m s}^{-1} \text{ (to 1 d.p.)}.
\]

The angle \( \phi \) is given by
\[
\phi = \tan^{-1} \left( \frac{1}{4} \right) = 14.03 \ldots ^\circ.
\]
So the bearing of \( \mathbf{v} \) is \( \theta = 180^\circ + 14.03 \ldots ^\circ = 194^\circ \) (to the nearest degree).