**Refresh Quadratic and polynomial functions**

**Quadratic functions**

Any function whose rule is of form $f(x) = ax^2 + bx + c$ where $a, b, c$ are constants is called a **quadratic function**. The graph is a parabola that is u-shaped if $a > 0$, for example see Figure 1(a) and 1(b) to see the shape of typical u-shaped parabolas. The graph is n-shaped if $a < 0$, see Figure 1(c) for the form of a typical n-shaped parabola. Notice that $c$ indicates the intercept on the $y$ axis.

![Graphs of quadratic functions](image)

**Figure 1** The graphs of the functions (a) $f(x) = x^2$ (b) $f(x) = \frac{1}{5}x^2 + \frac{2}{5}x - 3$ (c) $f(x) = -4x^2 + 8x - 1$

**Polynomial functions**

Any expression that is a sum of finitely many terms, each of which is of the form $ax^n$ where $a$ is a number and $n$ is a non-negative integer is called a polynomial expression in $x$. If the right-hand side of the rule of a function is a polynomial expression in $x$, then the function is called a polynomial function. The linear and quadratic functions are special cases of polynomial functions. The highest power of the variable $x$ in a function is called the degree of the function. So a linear function has degree 1 and a quadratic function has degree 2. Polynomial functions of degrees 3, 4 and 5 are called **cubic**, **quartic** and **quintic** functions, respectively.
**Activity 1**

What is the degree of the function \( f(x) = x^5 + 5x^3 + 7x + 10 \)? Is it a polynomial?

**Solution**

\( f(x) = x^5 + 5x^3 + 7x + 10 \) is a polynomial function. The degree is 5 because the largest power of \( x \) that occurs is \( x^5 \).

**Activity 2**

Is \( f(x) = x^2 + \sin x \) a polynomial function? What about \( f(x) = x^2 + e^{-x} \)?

**Solution**

Neither of these functions are polynomial functions because neither \( \sin x \) nor \( e^{-x} \) is a polynomial expression.
Dominant terms

You often need to know what happens to a function as $x \to \pm\infty$. You can get this from the dominant term. This is the one that outweighs all other terms in the expression. For example, if $f(x) = x^5 + 5x^3 + 7x + 10$ then as $x$ gets very large (either positive or negative) the size of the $x^5$ term dominates all other terms. We can say that the dominant behaviour of $f(x) = x^5 + 5x^3 + 7x + 10$ as $x \to \pm\infty$ is approximately given by $f(x) \simeq x^5$ where $\simeq$ means approximately equal to.

Example 1

What is the dominant behaviour of $f(x) = x^2 + \sin x$ as $x \to \infty$?

Solution

You know that $-1 \leq \sin x \leq +1$ for all values of $x$. You also know that $x^2$ gets larger and larger as $x$ increases. As $x$ is squared this is true regardless of whether $x$ is positive or negative. This means that $f(x) = x^2 + \sin x \simeq x^2 \to \infty$ as $x \to \infty$.

Example 2

What is the dominant behaviour of $f(x) = x^2 + e^{-x}$ as $x \to \infty$?

Solution

If you need help with the exponential function please see the exponentials and logarithms help. For $f(x) = x^2 + e^{-x}$ you know that as $x$ increases $e^{-x}$ gets smaller and smaller. Because $x^2$ gets larger and larger as $x$ increases this means that $f(x) = x^2 + e^{-x} \simeq x^2 \to \infty$ as $x \to \infty$.

Example 3

What is the dominant behaviour of $f(x) = x^2 + e^{-x}$ as $x \to -\infty$?

Solution

You know that as $x$ increases in the negative direction we end up with an exponential of an increasingly large number, so $e^{-x}$ gets larger and larger. We also know that $x^2$ gets larger and larger as $x$ increases but an exponential grows faster than a power. This means that $f(x) = x^2 + e^{-x} \simeq e^{-x} \to \infty$ as $x \to -\infty$. 