**Refresh Position vectors**

If \( P \) is any point, either in the coordinate plane or in three-dimensional space, then the position vector of \( P \) is the displacement vector \( \overrightarrow{OP} \), where \( O \) is the origin. This is illustrated in Figure 1, in the case of two dimensions.

![Figure 1](image1.png)

**Figure 1** The position vector \( \overrightarrow{OP} \) in two dimensions

The components of the position vector of a point are the same as the coordinates of the point. That is, the position vector of the point \( P(x, y) \) in two dimensions is

\[
\overrightarrow{OP} = xi + yj = \begin{pmatrix} x \\ y \end{pmatrix},
\]

and similarly the position vector of the point \( Q(x, y, z) \) in three dimensions is

\[
\overrightarrow{OQ} = xi + yj + zk = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

If a point is denoted by a capital letter, as is usual, then it’s often convenient to denote its position vector by the corresponding lower-case, bold (or underlined) letter. For example, we often denote the position vector of the point \( P \) by \( \mathbf{p} \), the position vector of the point \( A \) by \( \mathbf{a} \), and so on.

There is a simple equation that expresses any displacement vector \( \overrightarrow{AB} \) in terms of the position vectors of the points \( A \) and \( B \). Consider Figure 2, which shows two points \( A \) and \( B \), the displacement vector \( \overrightarrow{AB} \), and the position vectors of \( A \) and \( B \).

![Figure 2](image2.png)

**Figure 2** The position vectors of two points \( A \) and \( B \), and the vector \( \overrightarrow{AB} \)

Since

\[
\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB},
\]

it follows that

\[
\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB};
\]

that is,

\[
\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.
\]


This equation, which applies in either two or three dimensions, can be stated as below.

If the points \( A \) and \( B \) have position vectors \( \mathbf{a} \) and \( \mathbf{b} \), respectively, then

\[
\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.
\]

For example, Figure 3 shows two points \( A \) and \( B \), with position vectors \( 2\mathbf{i} + \mathbf{j} \) and \( 3\mathbf{i} + 5\mathbf{j} \), respectively. The vector \( \overrightarrow{AB} \) is

\[
(3\mathbf{i} + 5\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = \mathbf{i} + 4\mathbf{j}.
\]

Activity 1  Using position vectors

Consider the points \( A(5, 3) \) and \( B(-2, 4) \). Find the vector \( \overrightarrow{AB} \) in component form.

Solution

In component form, \( \overrightarrow{OA} = 5\mathbf{i} + 3\mathbf{j} \) and \( \overrightarrow{OB} = -2\mathbf{i} + 4\mathbf{j} \), so

\[
\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\mathbf{i} + 4\mathbf{j} - (5\mathbf{i} + 3\mathbf{j}) = -7\mathbf{i} + \mathbf{j}.
\]

Vectors, and position vectors in particular, often provide a convenient means of solving problems and proving facts in coordinate geometry. The next example gives an alternative proof of the formula that you met earlier for the coordinates of the midpoint of a line segment in terms of the coordinates of the endpoints.
Example 1  Working with position vectors

Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $M$ be the midpoint of the line segment $AB$, and let $a$, $b$ and $m$ be the position vectors of $A$, $B$ and $M$, respectively.

(a) Express $m$ in terms of $a$ and $b$.

(b) Hence find the coordinates of $M$, in terms of the coordinates of $A$ and $B$.

Solution

(a) First draw a diagram.

We have

$$ \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}; $$

that is,

$$ m = a + \overrightarrow{AM}. $$

Also, $\overrightarrow{AM}$ has the same direction as $\overrightarrow{AB}$ but half its magnitude. So

$$ \overrightarrow{AM} = \frac{1}{2} \overrightarrow{AB} = \frac{1}{2}(b - a). $$

Hence

$$ m = a + \frac{1}{2}(b - a) $$

$$ = a + \frac{1}{2}b - \frac{1}{2}a $$

$$ = \frac{1}{2}a + \frac{1}{2}b $$

$$ = \frac{1}{2}(a + b). $$

(b) Use the components of $a$ and $b$ to find the components of $m$ and hence the coordinates of $M$. Since $a = x_1i + y_1j$ and $b = x_2i + y_2j$, we have

$$ m = \frac{1}{2}(x_1i + y_1j + x_2i + y_2j) $$

$$ = \left(\frac{x_1 + x_2}{2}\right)i + \left(\frac{y_1 + y_2}{2}\right)j. $$

Hence the coordinates of $M$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. 
Example 1 gives the following useful fact.

**Midpoint formula in terms of position vectors**

If the points \( A \) and \( B \) have position vectors \( \mathbf{a} \) and \( \mathbf{b} \), respectively, then the midpoint of the line segment \( AB \) has position vector \( \frac{1}{2}(\mathbf{a} + \mathbf{b}) \).

Here is a similar example for you to try.

**Activity 2  Working with position vectors**

Consider the regular hexagon \( ABCDEF \), with centre the origin, shown below. Here \( \overrightarrow{OA} = \mathbf{a} \) and \( \overrightarrow{OB} = \mathbf{b} \).

![Hexagon diagram](image)

Express the following vectors in terms of \( \mathbf{a} \) and \( \mathbf{b} \).

(a) \( \overrightarrow{AB} \)  (b) \( \overrightarrow{OC} \)  (c) \( \overrightarrow{BC} \)  (d) \( \overrightarrow{AD} \)  (e) \( \overrightarrow{AE} \)

(Remember that each of the six triangles \( OAB, OBC, OCD, ODE, OEF \) and \( OFA \) forming the hexagon is equilateral.)

**Solution**

(a) \( \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} \)

(b) Since both \( OAB \) and \( OBC \) are equilateral triangles, \( \angle ABO = \angle BOC = 60^\circ \). So \( \overrightarrow{OC} \) is parallel to \( \overrightarrow{AB} \), and \( \overrightarrow{OC} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} \).

(c) \( \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a} \). Alternatively, notice that \( BC \) is parallel to \( OA \), so \( \overrightarrow{BC} = -\overrightarrow{OA} = -\mathbf{a} \).

(d) \( OD \) is parallel to \( OA \) and of equal length, so \( \overrightarrow{AD} = -2\overrightarrow{OA} = -2\mathbf{a} \)

(e) \( \overrightarrow{AE} = \overrightarrow{AO} + \overrightarrow{OE} = -\mathbf{a} - \mathbf{b} \).
In the next activity you are asked to use position vectors to prove the following basic property of every triangle: the three lines that join a vertex to the midpoint of the opposite side meet at a common point.

**Activity 3  Proving a geometric property of triangles**

In the triangle $OAB$ shown below, the points $C$, $D$ and $E$ are midpoints of sides of the triangle. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

\[\begin{array}{c}
O & C & D & E & B \\
\end{array}\]

(a) Show that $\overrightarrow{AC} = \frac{1}{2}\mathbf{b} - \mathbf{a}$ and $\overrightarrow{BD} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.

(b) Find the position vectors of each of the following points, expressed in terms of the position vectors $\mathbf{a}$ and $\mathbf{b}$.

(i) The point two thirds of the way along $OE$ from $O$
(ii) The point two thirds of the way along $AC$ from $A$
(iii) The point two thirds of the way along $BD$ from $B$

(c) Deduce that the lines $OE$, $AC$ and $BD$ are concurrent; that is, they meet at a common point.

**Solution**

(a) The point $C$ is the midpoint of $OB$, so its position vector is $\frac{1}{2}\mathbf{b}$. Hence $\overrightarrow{AC} = \frac{1}{2}\mathbf{b} - \mathbf{a}$.

The point $D$ is the midpoint of $OA$, so its position vector is $\frac{1}{2}\mathbf{a}$. Hence $\overrightarrow{BD} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.

(b) The point $E$ is the midpoint of $AB$, so its position vector is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Hence the position vector of the point $\frac{2}{3}$ of the way along $OE$ from $O$ is $\frac{2}{3} \times \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{3}(\mathbf{a} + \mathbf{b})$.

The position vector of the point $\frac{2}{3}$ of the way along $AC$ from $A$ is

$$\overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC} = \mathbf{a} + \frac{2}{3} \times (\frac{1}{2}\mathbf{b} - \mathbf{a})$$

$$= \mathbf{a} + \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}$$

$$= \frac{1}{3}(\mathbf{a} + \mathbf{b})$$.

The position vector of the point $\frac{2}{3}$ of the way along $BD$ from $B$ is $\overrightarrow{OB} + \frac{2}{3}\overrightarrow{BD} = \mathbf{b} + \frac{2}{3} \times (\frac{1}{2}\mathbf{a} - \mathbf{b})$

$$= \mathbf{b} + \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$$

$$= \frac{1}{3}(\mathbf{a} + \mathbf{b})$$.

(c) The three points in part (b) all have the same position vector, so they are all the same point. Hence the lines $OE$, $AC$ and $BD$ are concurrent.