Refresh Particular antiderivatives

Sometimes when you’re working with a function, you need to find a particular antiderivative that it has, rather than any antiderivative, or its indefinite integral. For example, this is often what you need to do when the function is a mathematical model of a real-life situation. In this section, you’ll learn how to find such particular antiderivatives, and see some examples of how you can use them.

You can work out which of the infinitely many antiderivatives of a function is the right one if you have some appropriate extra information. Usually the extra information that you have is the value taken by the antiderivative at some value of the input variable. Here’s an example.

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**Example 1  Finding a particular antiderivative**

Find the antiderivative $F$ of the function $f(x) = x^2 + 5$ such that $F(3) = 20$.

**Solution**

Find the indefinite integral.

The indefinite integral is

$$F(x) = \frac{1}{3}x^3 + 5x + c.$$  

Use the extra information to find the required value of the constant $c$.

Using the fact that $F(3) = 20$ gives

$$\frac{1}{3} \times 3^3 + 5 \times 3 + c = 20$$

$$9 + 15 + c = 20$$

$$c = -4.$$  

Hence the required antiderivative is

$$F(x) = \frac{1}{3}x^3 + 5x - 4.$$  

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**Activity 1  Finding particular antiderivatives**

(a) Find the antiderivative $F$ of the function $f(x) = x + 2$ such that $F(1) = \frac{3}{2}$.

(b) Find the antiderivative $F$ of the function

$$f(x) = \frac{1}{x^3} \quad (x \in (0, \infty))$$

such that $F(3) = -\frac{1}{9}$.
Solution
(a) The function is \( f(x) = x + 2 \). Its indefinite integral is
\[
F(x) = \frac{1}{2}x^2 + 2x + c.
\]
Using the fact that \( F(1) = \frac{9}{2} \) gives
\[
\frac{1}{2} \times 1^2 + 2 \times 1 + c = \frac{9}{2}
\]
\[
\frac{5}{2} + c = \frac{9}{2}
\]
\[
c = \frac{4}{2} = 2.
\]
Hence the required antiderivative is
\[
F(x) = \frac{1}{2}x^2 + 2x + 2.
\]
(b) The function is \( f(x) = \frac{1}{x^3} = x^{-3} \) with domain \((0, \infty)\). Its indefinite integral is
\[
F(x) = -\frac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c.
\]
Using the fact that \( F(3) = -\frac{1}{9} \) gives
\[
-\frac{1}{2 \times 3^2} + c = -\frac{1}{9}
\]
\[
-\frac{1}{18} + c = -\frac{1}{9}
\]
\[
c = -\frac{1}{9} + \frac{1}{18} = \frac{1}{18}.
\]
Hence the required antiderivative is
\[
F(x) = -\frac{1}{2x^2} - \frac{1}{18}
\]
\[
= -\frac{9 + x^2}{18x^2} \quad (x \in (0, \infty)).
\]

As mentioned at the start of this section, it’s often useful to find a particular antiderivative when you’re working with a function that models a real-life situation. This is illustrated in the example below.

Example 2  Using integration to deduce displacement from velocity
Suppose that a man walks along a straight path, and his velocity \( v \) (in kilometres per hour) at time \( t \) (in hours) after he begins his walk is given by the equation
\[
v = 6 - \frac{5}{4}t.
\]
Let \( s \) be his displacement in kilometres from his starting point.

(a) Find an equation that expresses \( s \) in terms of \( t \).
(b) Hence find the man’s displacement two hours after he began his walk.
Solution

(a) Write down the given equation for the man’s velocity.

The man’s velocity \( v \) at time \( t \) is given by

\[ v = 6 - \frac{5}{4}t. \]

Use integration to find an equation for his displacement. It will contain an arbitrary constant, because at this stage we don’t know which antiderivative is the right one.

Hence his displacement \( s \) at time \( t \) is given by

\[ s = 6t - \frac{5}{4} \times \frac{1}{2} t^2 + c, \]

that is,

\[ s = 6t - \frac{5}{8}t^2 + c, \]

where \( c \) is a constant.

Use extra information to find the value of the constant \( c \).

When the man begins his walk, his displacement from his starting point is 0 km. That is, when \( t = 0 \), \( s = 0 \). Substituting these values into the equation above gives

\[ 0 = 6 \times 0 - \frac{5}{8} \times 0^2 + c, \]

that is,

\[ c = 0. \]

Hence the required equation for \( s \) in terms of \( t \) is

\[ s = 6t - \frac{5}{8}t^2. \]

(b) When \( t = 2 \),

\[ s = 6 \times 2 - \frac{5}{8} \times 2^2 = 12 - \frac{5}{2} = \frac{19}{2} = 9.5. \]

So the man’s displacement two hours after he began his walk is 9.5 km.

Activity 2 Using integration to deduce displacement from velocity

Suppose that a marble is rolled in a straight line down a long slope, and its velocity \( v \) (in metres per second) at time \( t \) (in seconds) after it begins rolling is given by the equation

\[ v = \frac{1}{10}t. \]

The marble starts rolling from a position that is 0.3 metres down the slope. Let \( s \) be its displacement in metres from the top of the slope.

(a) Find an equation that expresses \( s \) in terms of \( t \).

(Hint: First find an equation for \( s \) in terms of \( t \) that contains an arbitrary constant, then use the value of \( s \) when \( t = 0 \).)

(b) Hence find the displacement of the marble from the top of the slope 4 seconds after it begins rolling.
Solution

(a) The given equation for \( v \) in terms of \( t \) is

\[
v = \frac{1}{10} t.
\]

Hence the equation for \( s \) in terms of \( t \) is

\[
s = \frac{1}{10} \times \frac{1}{2} t^2 + c,
\]

that is,

\[
s = \frac{1}{20} t^2 + c,
\]

where \( c \) is a constant.

The marble starts rolling from a position 0.3 metres down the slope, so when \( t = 0 \), \( s = 0.3 \).

Substituting these values of \( s \) and \( t \) into the equation above gives

\[
0.3 = \frac{1}{20} \times 0^2 + c
\]

\[
c = 0.3.
\]

So the equation for \( s \) in terms of \( t \) is

\[
s = \frac{1}{20} t^2 + 0.3.
\]

(b) Substituting \( t = 4 \) into the equation from part (a) gives

\[
s = \frac{1}{20} \times 4^2 + 0.3 = 0.8 + 0.3 = 1.1.
\]

That is, the displacement of the marble from the top of the slope after it has been rolling for 4 seconds is 1.1 metres.

If an object falls from rest under the influence of gravity, and the effects of air resistance are negligible, then its displacement \( s \) (in metres) at time \( t \) (in seconds) after it began falling is modelled by the equation

\[
s = -4.9t^2.
\]

Here the displacement of the object is measured from the point from which it starts to fall, and the positive direction along the line of motion is upwards.

You saw that if you differentiate this equation, to obtain an equation for the object’s velocity in terms of time, and then differentiate again to obtain an equation for its acceleration in terms of time, then you obtain the fact that the object is moving with a constant acceleration of \(-9.8 \text{ m s}^{-2}\). The magnitude of this acceleration, \(9.8 \text{ m s}^{-2}\), is known as the acceleration due to gravity.

**Example 3 Using integration to deduce displacement from acceleration**

Assume that an object falling from rest has a constant acceleration of \(-9.8 \text{ m s}^{-2}\), where the positive direction is upwards.

Let \( s \) be the displacement in metres of the object from its initial position and let \( t \) be the time in seconds since it began falling.

Find an equation that expresses \( s \) in terms of \( t \).
Solution

Define the variables that you intend to use. Let \( v \) be the velocity of the object in m s\(^{-1} \) and let \( a \) be its acceleration in m s\(^{-2} \).

Write down the equation for the acceleration of the object. The equation for \( a \) in terms of \( t \) is

\[ a = -9.8. \]

The variable \( t \) doesn’t appear on the right-hand side of this equation because the acceleration is constant.

Use integration to find an equation for \( v \) in terms of \( t \), containing an arbitrary constant. Hence the equation for \( v \) in terms of \( t \) is

\[ v = -9.8t + c, \]

where \( c \) is a constant.

Use extra information to find the value of the constant \( c \). The object falls from rest, so when it begins falling its velocity is 0 m s\(^{-1} \). That is, when \( t = 0 \), \( v = 0 \). Substituting these values into the equation above gives

\[ 0 = -9.8 \times 0 + c, \quad \text{that is,} \quad c = 0. \]

So the equation for \( v \) in terms of \( t \) is

\[ v = -9.8t. \]

Use integration again to find an equation for \( s \) in terms of \( t \), containing another arbitrary constant. Don’t use the letter \( c \) for the arbitrary constant, as it’s been used already. Choose a different letter, such as \( b \).

It follows that the equation for \( s \) in terms of \( t \) is

\[ s = -9.8 \times \frac{1}{2}t^2 + b, \]

that is,

\[ s = -4.9t^2 + b, \]

where \( b \) is a constant.

Use extra information to find the value of the constant \( b \). When the object begins falling, its displacement is 0 m. That is, when \( t = 0 \), \( s = 0 \). Substituting these values into the equation gives

\[ 0 = -4.9 \times 0^2 + b, \quad \text{that is,} \quad b = 0. \]

So the equation for \( s \) in terms of \( t \) is

\[ s = -4.9t^2. \]

This is the required equation for the displacement of the object in terms of time.
The next activity is about the motion of a ball thrown vertically into the air. The ball is an example of a \textit{vertical projectile}. A \textbf{projectile} is an object that’s launched into the air by a force that ceases after launch, and a \textbf{vertical projectile} is a projectile that’s launched vertically upwards.

Provided that the effects of air resistance are negligible, a vertical projectile has a constant downwards acceleration due to gravity of about 9.8 m s\(^{-2}\) at all times after its launch, just as an object falling from rest has. The only thing that makes its motion different from that of an object falling from rest is that at the start of its motion its velocity is not zero. In the next activity you’re asked to use a process similar to that in Example 3 to find a formula for the displacement of a ball thrown vertically into the air, in terms of time.

**Activity 3  Using integration to deduce displacement from acceleration**

Suppose that a ball is thrown vertically upwards with initial speed 12 m s\(^{-1}\). Assume that its subsequent motion is modelled as having a constant acceleration of \(-9.8\) m s\(^{-2}\), where the positive direction along the line of motion is upwards.

Let the acceleration, velocity and displacement of the ball at time \(t\) (in seconds) after it was thrown be \(a\) (in m s\(^{-2}\)), \(v\) (in m s\(^{-1}\)) and \(s\) (in m), respectively, where displacement is measured from the point from which the ball was thrown.

(a) Find an equation that expresses \(v\) in terms of \(t\).

(b) Hence find an equation that expresses \(s\) in terms of \(t\).

(c) Use the formulas that you found in parts (a) and (b) to find the velocity and the displacement of the ball 1 second after it was thrown.

(d) Use the formula that you found in part (b) to determine how long it takes for the ball to fall back to the point from which it was thrown.

\textit{Hint: At the time when the ball has fallen back to this point, what is the value of \(s\)?}

\textbf{Solution}

(a) The equation for \(a\) in terms of \(t\) is

\[ a = -9.8. \]

Integrating this equation gives the following equation for \(v\) in terms of \(t\):

\[ v = -9.8t + c, \]

where \(c\) is a constant.

At the start of the motion, the velocity of the ball is 12 m s\(^{-1}\). That is, when \(t = 0\), \(v = 12\).

Substituting these values into the equation for \(v\) above gives

\[ 12 = -9.8 \times 0 + c, \quad \text{that is,} \quad c = 12. \]

So the equation for \(v\) in terms of \(t\) is

\[ v = 12 - 9.8t. \]
(b) Integrating the equation found in part (a) gives the following equation for $s$ in terms of $t$:

$$s = 12t - 9.8 \times \frac{1}{2}t^2 + b,$$

that is,

$$s = 12t - 4.9t^2 + b,$$

where again $b$ is a constant.

At the start of the motion, the displacement of the ball is 0 m. That is, when $t = 0$, $s = 0$.

Substituting these values into the equation for $s$ above gives

$$0 = 12 \times 0 - 4.9 \times 0^2 + b,$$

that is, $b = 0$.

So the equation for $s$ in terms of $t$ is

$$s = 12t - 4.9t^2.$$

(c) When $t = 1$,

$$v = 12 - 9.8 \times 1 = 12 - 9.8 = 2.2,$$

and

$$s = 12 \times 1 - 4.9 \times 1^2 = 12 - 4.9 = 7.1.$$

So the velocity and the displacement of the ball 1 second after it was thrown are $2.2 \text{ m s}^{-1}$ and $7.1 \text{ m}$, respectively.

(d) When the ball has fallen back to the point from which it was thrown, $s = 0$. Substituting this value of $s$ into the equation found in part (b) gives

$$0 = 12t - 4.9t^2$$

$$12t - 4.9t^2 = 0$$

$$t(12 - 4.9t) = 0$$

$$t = 0 \text{ or } t = 12/4.9 = 2.4 \text{ (to 1 d.p.)}.$$  

So the ball takes about 2.4 seconds to fall back to the point from which it was thrown.  

(The displacement–time graph for the ball is shown below. You can see that the ball first rises and then falls, as expected. You can also see that it reaches a height of about 7 m after 1 second, and that it returns to the point from which it was thrown after about 2.4 seconds, as you would expect from the answers found in parts (c) and (d).)
Activity 4  Using integration to work with marginal cost and total cost

A company that produces agricultural fertiliser has developed a model for its production costs. According to the model, if the amount of fertiliser that the company is currently producing each week is \( q \) (in tonnes), then the marginal weekly cost \( m \) (in £ per tonne) of producing extra fertiliser is given by the equation

\[
m = 250 - 0.5q.
\]

The model applies for values of \( q \) between 50 and 200.

Currently the company produces 80 tonnes of fertiliser each week, at a total weekly cost of £30 400.

Let the total weekly cost of producing \( q \) tonnes of fertiliser be \( t \) (in £).

(a) By using the fact that marginal cost is the derivative of total cost, find a formula for \( t \) in terms of \( q \).

(b) Use the formula that you found in part (a) to find the total weekly cost of producing 160 tonnes of fertiliser.

(c) What is the cost per tonne (in other words, the unit cost, not the marginal cost) of producing the fertiliser if the company produces 80 tonnes of fertiliser each week? What is it if it produces 160 tonnes each week?

Solution

(a) The given equation for \( m \) in terms of \( q \) is

\[
m = 250 - 0.5q.
\]

Since marginal cost is the derivative of total cost, the equation for \( t \) in terms of \( q \) is

\[
t = 250q - 0.5 \times \frac{1}{2}q^2 + c,
\]

that is,

\[
t = 250q - 0.25q^2 + c,
\]

where \( c \) is a constant.

According to the information given in the question, when \( q = 80 \),

\[
t = 30 400.
\]

Substituting these values into the equation found for \( t \) above gives

\[
30 400 = 250 \times 80 - 0.25 \times 80^2 + c
\]

\[
30 400 = 18 400 + c
\]

\[
c = 12 000.
\]

Hence the equation for \( t \) in terms of \( q \) is

\[
t = 250q - 0.25q^2 + 12 000.
\]

(b) The total weekly cost of producing 160 tonnes of fertiliser each week is

\[
£(250 \times 160 - 0.25 \times 160^2 + 12 000)
\]

\[
= £45 600.
\]
(c) The question states that the total weekly cost of producing 80 tonnes of fertiliser each week is £30 400. Hence the cost per tonne of producing 80 tonnes of fertiliser each week is

\[
\frac{\£30 400}{80} = \£380.
\]

It was calculated in part (b) that the total weekly cost of producing 160 tonnes of fertiliser each week is £45 600. Hence the cost per tonne of producing 160 tonnes of fertiliser each week is

\[
\frac{\£45 600}{160} = \£285.
\]

(The graph of \( t \) against \( q \) is shown below. You can see that the total weekly cost of producing 160 tonnes of fertiliser each week is about £45 000, as expected.)