Matrix notation and terminology

A matrix (pronounced ‘may-tricks’) is a rectangular array of numbers, usually enclosed in brackets. Here are some examples:

\[
\begin{pmatrix}
1 & 2 & -1 \\
5 & 3 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
3 & 1
\end{pmatrix},
\begin{pmatrix}
\frac{1}{3} & \frac{1}{2} & 4 \\
-6 & 0
\end{pmatrix}.
\]

Some texts use square brackets for matrices, like this:

\[
\begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix}.
\]

In this module we’ll always use round brackets.

We often use capital letters to denote matrices. For example, we might write

\[
A = \begin{pmatrix} 1 & 2 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{3} & 4 \\ \frac{1}{2} & -6 \end{pmatrix}.
\]

In the text in this module, the capital letters that denote matrices are in bold font, but you don’t need to do anything special when you handwrite them. In particular, you don’t need to underline them, unlike letters that denote vectors.

The numbers in a matrix are called its **elements** (or its **entries**, in some texts). For example, matrices \( A \) and \( C \) above each have six elements, and matrix \( B \) has four elements. In this module, the elements of matrices are always real numbers.

A horizontal line of numbers in a matrix is called a **row**, and a vertical line of numbers is called a **column**. For instance, in the matrix \( A \) above the first row is \((1 \ 2 \ -1)\) and the second row is \((5 \ 3 \ 0)\). Likewise, the first column is \(\begin{pmatrix} 1 \\ 5 \end{pmatrix}\), the second column is \(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\) and the third column is \(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\).

The matrix \( A \) has two rows and three columns: we say that it is a \(2 \times 3\) matrix. In general, a matrix with \(m\) rows and \(n\) columns is described as an \(m \times n\) matrix, or a matrix of **size** \(m \times n\). The first number in this notation is always the number of rows, and the second number is always the number of columns. The size \(m \times n\) of a matrix is read as ‘\(m\) by \(n\)’.

A matrix with the same number of rows as columns is called a **square matrix**. Thus, for example, matrix \( B \) above is square.

The column notation for vectors that you met earlier is a particular instance of matrix notation: two-dimensional column vectors are \(2 \times 1\) matrices, and three-dimensional column vectors are \(3 \times 1\) matrices. In general, the word **vector** is used to mean any matrix with a single column, even if it has more than 3 elements and hence has no geometric interpretation in the plane or three-dimensional space. For instance, the following matrices are vectors:

\[
\begin{pmatrix}
1 \\
3 \\
5 \\
7
\end{pmatrix}, \quad \begin{pmatrix}
\sqrt{2} \\
-3 \\
\frac{1}{2} \\
0 \\
7
\end{pmatrix}\quad \text{and} \quad \begin{pmatrix}
w \\
x \\
y \\
z
\end{pmatrix}, \quad \text{where} \ w, x, y, z \ \text{are variables}.
\]
The elements of a vector are often called its components. A vector with \( n \) components is called an \( n \)-dimensional vector.

There is a useful notation for the elements of a matrix in terms of their row and column positions. The element in row \( i \) and column \( j \) of a matrix \( A \) is denoted by \( a_{ij} \). Similarly, the element in row \( i \) and column \( j \) of a matrix \( B \) is denoted by \( b_{ij} \), and so on.

For instance, if

\[
A = \begin{pmatrix}
\frac{1}{3} & 4 \\
7 & -6
\end{pmatrix},
\]

then \( a_{11} = \frac{1}{3}, \ a_{12} = 4, \ a_{21} = 7 \) and \( a_{22} = -6 \).

This notation is used for matrices of any size. Thus, for instance, a general \( 3 \times 4 \) matrix can be denoted by

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{pmatrix}.
\]

**Activity 1 Using matrix notation**

Write down the elements \( a_{33}, \ a_{34}, \ a_{43}, \ a_{44} \) and \( a_{42} \) of the matrix

\[
A = \begin{pmatrix}
2 & 8 & 0 & -1 \\
4 & 7 & 5 & -2 \\
9 & -1 & 3 & -4 \\
6 & -5 & -7 & -9
\end{pmatrix}.
\]

What is the size of \( A \)?

**Solution**

We have \( a_{33} = 3, \ a_{34} = -4, \ a_{43} = -7, \ a_{44} = -9, \ a_{42} = -5. \)

The matrix \( A \) has four rows and four columns, so its size is \( 4 \times 4 \).

Two matrices are equal if they have the same numbers of rows and columns, and all corresponding elements are equal. For example,

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix},
\]

but

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} \neq \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} \neq \begin{pmatrix}
1 & 3 \\
2 & 4
\end{pmatrix}.
\]