Matrix multiplication

Previously you learned how to multiply a number and a matrix. Under certain conditions, it is also possible to multiply two matrices. The way that this is done is less obvious than matrix addition or scalar multiplication.

Here is an example where a matrix is used to store the numbers of copies of three bestselling books that four bookshops ordered from a publisher in a certain week:

\[
\begin{pmatrix}
10 & 25 & 12 \\
5 & 10 & 5 \\
7 & 5 & 0 \\
10 & 8 & 10
\end{pmatrix}
\]

Now suppose that the publisher charges the bookshops £4 per copy for Bestseller 1, £7 per copy for Bestseller 2 and £9 per copy for Bestseller 3. According to the data in the matrix, Bookshop 1 ordered 10 copies of Bestseller 1, 25 copies of Bestseller 2 and 12 copies of Bestseller 3. Therefore the total amount, in £, that Bookshop 1 owes the publisher for that week’s order is

\[10 \times 4 + 25 \times 7 + 12 \times 9 = 323.\]

The expression on the left of this equation is obtained by multiplying each element in the first row of the matrix by the price of the corresponding book, and then adding the results.

If we write the prices (in £) for the three bestsellers as a three-dimensional vector, that is, as a $3 \times 1$ matrix, then we can picture this procedure as shown in Figure 1.

![Figure 1](image)

The procedure for calculating the amount owed by Bookshop 1

If we want to calculate the amount owed by Bookshop 2, then we can carry out the same procedure with the second row of the matrix, and likewise for Bookshops 3 and 4 with the third and fourth rows, respectively. We can write the four amounts that we obtain as a four-dimensional vector. The procedure described above combines the two matrices

\[
\begin{pmatrix}
10 & 25 & 12 \\
5 & 10 & 5 \\
7 & 5 & 0 \\
10 & 8 & 10
\end{pmatrix}
\hspace{1cm}
\begin{pmatrix}
4 \\
7 \\
9
\end{pmatrix}
\]
to give
\[
\begin{pmatrix}
10 \times 4 + 25 \times 7 + 12 \times 9 \\
5 \times 4 + 10 \times 7 + 5 \times 9 \\
7 \times 4 + 5 \times 7 + 0 \times 9 \\
10 \times 4 + 8 \times 7 + 10 \times 9
\end{pmatrix}
= \begin{pmatrix} 323 \\ 135 \\ 63 \\ 186 \end{pmatrix}.
\]
This calculation shows that the amounts owed by the four bookshops are £323, £135, £63 and £186, respectively.

This procedure for combining matrices comes up so often in applications that it is used to define matrix multiplication. We say that

the product of
\[
\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}
\]
and
\[
\begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix}
\]
is
\[
\begin{pmatrix} 323 \\ 135 \\ 63 \\ 186 \end{pmatrix}.
\]

Now suppose that as well as calculating the amount owed by each of the four bookshops, the publisher also wants to calculate the profit made from sales to each individual bookshop. Suppose that the profits made on each copy of Bestsellers 1, 2 and 3 are £1, £2 and £2, respectively. You can use exactly the same procedure as above to calculate the profit made from the sales to each bookshop. You write the profits (in £) made on the three books as a three-dimensional vector, and form

the product of
\[
\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}
\]
and
\[
\begin{pmatrix} 1 \\ 2 \\
1 \\ 2 \\
1 \\ 2 \\
1 \\ 2 \end{pmatrix}
\]
which is
\[
\begin{pmatrix}
10 \times 1 + 25 \times 2 + 12 \times 2 \\
5 \times 1 + 10 \times 2 + 5 \times 2 \\
7 \times 1 + 5 \times 2 + 0 \times 2 \\
10 \times 1 + 8 \times 2 + 10 \times 2
\end{pmatrix}
= \begin{pmatrix} 84 \\ 35 \\ 17 \\ 46 \end{pmatrix}.
\]
So the profits made from sales to the four bookshops are £84, £35, £17 and £46, respectively.

In fact you can view the two matrix multiplications above – the one for the amounts owed, and the one for the profits – as a single matrix multiplication. To do this, you put the two vectors containing the prices and the profits together as a single matrix with two columns, and you put the two vectors that contain the results of the calculations together as a single matrix with two columns, and say that

the product of
\[
\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}
\]
and
\[
\begin{pmatrix} 4 & 1 \\ 7 & 2 \\ 9 & 2 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \end{pmatrix}
\]
is
\[
\begin{pmatrix} 323 & 84 \\ 135 & 35 \\ 63 & 17 \\ 186 & 46 \end{pmatrix}.
\]

Let’s start by considering which pairs of matrices can be multiplied together. Remember that the element in the first row and first column of the product matrix in calculation (1) above was obtained by multiplying each element in the first row of the first matrix by the corresponding element in the first column of the second matrix, and adding the results:

\[
10 \times 4 + 25 \times 7 + 12 \times 9 = 323.
\]

It was possible to do this because the number of columns in the first matrix is the same as the number of rows in the second matrix, so each element in the first row of the first matrix has a corresponding element in the first column of the second matrix. You can multiply two matrices together only if this condition holds.
Thus, you can multiply together a $4 \times 3$ matrix and a $3 \times 2$ matrix. On the other hand, it is not possible to multiply together a $3 \times 4$ matrix and a $2 \times 3$ matrix. When it is not possible to multiply two matrices, we sometimes say that their product is **undefined**.

A convenient way to determine whether two matrices, of sizes $m \times n$ and $p \times q$, say, can be multiplied together is to write their sizes next to each other as follows:

$$m \times n \quad p \times q.$$ 

The matrices can be multiplied only if the two numbers in the middle are equal; that is, if $n = p$. If these numbers are equal, then the size of the product matrix is given by the remaining numbers. So the product of an $m \times n$ matrix and an $n \times q$ matrix is a matrix of size $m \times q$.

For example, a $4 \times 3$ matrix and a $3 \times 2$ matrix can be multiplied together, and the product has size $4 \times 2$. Here's a useful way to picture this fact:

$$4 \times 3 \quad 3 \times 2 \quad \text{gives} \quad 4 \times 2.$$ 

Now let's look at the procedure for multiplying two matrices step by step. It's demonstrated in the next example.

---

**Example 1** *Multiplying matrices*

Let $A = \begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix}$.

Check that the product matrix $AB$ can be formed, determine the size of $AB$, and calculate it.

**Solution**

1. Write down the sizes of $A$ and $B$.

   $A$ has size $3 \times 3$ and $B$ has size $3 \times 2$.

   $$3 \times 3 \quad 3 \times 2$$

   The numbers in the middle are equal, so the product $AB$ can be formed. It has size $3 \times 2$. 

2. Calculate $AB$.
To obtain the element in the first row and first column of $AB$, multiply each element in the first row of $A$ by the corresponding element in the first column of $B$, and add the results.

\[
\begin{pmatrix}
-2 & 1 & 3 \\
2 & 4 & -2 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{2} & 2 \\
-1 & 4
\end{pmatrix}
= \begin{pmatrix}
(-2) \times 1 + 1 \times \frac{1}{2} + 3 \times (-1) \\
\end{pmatrix}
= \begin{pmatrix}
-\frac{9}{2}
\end{pmatrix}
\]

To obtain the element in the first row and second column of $AB$, apply the same procedure to the first row of $A$ and the second column of $B$.

\[
\begin{pmatrix}
-2 & 1 & 3 \\
2 & 4 & -2 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{2} & 2 \\
-1 & 4
\end{pmatrix}
= \begin{pmatrix}
-\frac{9}{2} & (-2) \times 0 + 1 \times 2 + 3 \times 4
\end{pmatrix}
= \begin{pmatrix}
-\frac{9}{2} & 14
\end{pmatrix}
\]

To obtain the element in the second row and first column of $AB$, apply the same procedure to the second row of $A$ and the first column of $B$.

\[
\begin{pmatrix}
-2 & 1 & 3 \\
2 & 4 & -2 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{2} & 2 \\
-1 & 4
\end{pmatrix}
= \begin{pmatrix}
2 \times 1 + 4 \times \frac{1}{2} + (-2) \times (-1)
\end{pmatrix}
= \begin{pmatrix}
-\frac{9}{2} & 14
\end{pmatrix}
\]

In general, to obtain the element in the $i$th row and $j$th column of $AB$, multiply each element in the $i$th row of $A$ by the corresponding element in the $j$th row of $B$, and add the results.

\[
\begin{pmatrix}
-2 & 1 & 3 \\
2 & 4 & -2 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{2} & 2 \\
-1 & 4
\end{pmatrix}
= \begin{pmatrix}
\frac{-9}{2} & 14 \\
6 & 0 \\
\frac{-1}{2} & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \times 1 + (-1) \times \frac{1}{2} + 0 \times (-1) \\
0 \times 0 + (-1) \times 2 + 0 \times 4
\end{pmatrix}
= \begin{pmatrix}
\frac{-9}{2} & 14 \\
6 & 0 \\
\frac{-1}{2} & 2
\end{pmatrix}
\]
Here’s a summary of the procedure for matrix multiplication.

**Matrix multiplication**

Let \( A \) and \( B \) be matrices. Then the product matrix \( AB \) can be formed only if the number of columns of \( A \) is equal to the number of rows of \( B \).

If \( A \) has size \( m \times n \) and \( B \) has size \( n \times p \), then the product \( AB \) has size \( m \times p \).

The element in row \( i \) and column \( j \) of the product matrix \( AB \) is obtained by multiplying each element in the \( i \)th row of \( A \) by the corresponding element in the \( j \)th column of \( B \) and adding the results.

In element notation, if \( c_{ij} \) denotes the element in the \( i \)th row and \( j \)th column of \( AB \), then

\[
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.
\]

Figure 2 illustrates how row \( i \) of a matrix \( A \) and column \( j \) of a matrix \( B \) are combined to give the element in row \( i \) and column \( j \) of the product matrix \( AB \).

Matrix multiplication may seem quite complicated at first, but it is a very useful technique, and with practice you should become proficient at it.

**Activity 1  Multiplying matrices**

In each of parts (a)–(e) below, calculate the matrix product if it exists.

(a) \[
\begin{pmatrix}
2 & 9 \\
7 & 1 \\
-6 & 8
\end{pmatrix}
\begin{pmatrix}
-2 & 3 & 1 \\
4 & 0 & 5
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
9 \\
8 \\
7
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
2 & 9 \\
7 & 1 \\
-6 & 8
\end{pmatrix}
\begin{pmatrix}
0 & 10 \\
4 & 1 \\
-2 & 7
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
9 \\
8 \\
7
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3
\end{pmatrix}
\]

(e) \[
\begin{pmatrix}
1 & 2 & 3 \\
7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 \\
1 & 0 & -1 \\
2 & 1 & 3
\end{pmatrix}
\]
Solution

For each pair of matrices in this question, the sizes of the matrices are written underneath the matrices, and the middle two numbers are boxed in order to use the method described in the text.

(a) \[
\begin{pmatrix}
  2 & 9 \\
  7 & 1 \\
-6 & 8 \\
\end{pmatrix}
\begin{pmatrix}
  -2 & 3 & 1 \\
  4 & 0 & 5 \\
\end{pmatrix}
\]

The middle numbers are the same, so these matrices can be multiplied together. Their product is

\[
\begin{pmatrix}
  2 & 9 \\
  7 & 1 \\
-6 & 8 \\
\end{pmatrix}
\begin{pmatrix}
  -2 & 3 & 1 \\
  4 & 0 & 5 \\
\end{pmatrix} =
\begin{pmatrix}
  32 & 6 & 47 \\
 -10 & 21 & 12 \\
 44 & 18 & 34 \\
\end{pmatrix}.
\]

(b) \[
\begin{pmatrix}
  1 & 2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
  9 \\
  8 \\
  7 \\
\end{pmatrix}
\]

The middle numbers are the same, so these matrices can be multiplied together. Their product is

\[
\begin{pmatrix}
  1 & 2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
  9 \\
  8 \\
  7 \\
\end{pmatrix} = (46).
\]

(c) \[
\begin{pmatrix}
  2 & 9 \\
  7 & 1 \\
-6 & 8 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 10 \\
  4 & 1 \\
-2 & 7 \\
\end{pmatrix}
\]

The middle numbers are different, so these matrices cannot be multiplied together.

(d) \[
\begin{pmatrix}
  9 \\
  8 \\
  7 \\
\end{pmatrix}
\begin{pmatrix}
  1 & 2 & 3 \\
\end{pmatrix}
\]

The middle numbers are the same, so these matrices can be multiplied together. Their product is

\[
\begin{pmatrix}
  9 \\
  8 \\
  7 \\
\end{pmatrix}
\begin{pmatrix}
  1 & 2 & 3 \\
\end{pmatrix} =
\begin{pmatrix}
  9 & 18 & 27 \\
  8 & 16 & 24 \\
  7 & 14 & 21 \\
\end{pmatrix}.
\]
The middle numbers are the same, so these matrices can be multiplied together. Their product is

\[
\begin{pmatrix}
1 & 2 & 3 \\
7 & 6 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -1 \\
1 & 0 & -1 \\
2 & 1 & 3 \\
\end{pmatrix}
= 
\begin{pmatrix}
9 & 3 & 8 \\
15 & 1 & 4 \\
\end{pmatrix}
\]

Matrix multiplication shares many of the properties of multiplication of numbers, as you’ll see later in this subsection and later in the unit. However, there’s an important difference between the properties for matrices and those for numbers, as you can find out in the next activity.

**Activity 2**  *Investigating matrix multiplication*

(a) Determine whether the products $AB$ and $BA$ are defined for the following pairs of matrices.

(i) $A = \begin{pmatrix}
-1 & 4 \\
2 & 1 \\
\end{pmatrix}$ and $B = \begin{pmatrix}
3 \\
5 \\
\end{pmatrix}$

(ii) $A = \begin{pmatrix}
1 & 1 \\
-1 & 0 \\
\end{pmatrix}$ and $B = \begin{pmatrix}
0 & 1 \\
2 & 3 \\
\end{pmatrix}$

(b) For each of parts (a)(i) and (a)(ii), if the products $AB$ and $BA$ are both defined, evaluate them. What do you notice?

**Solution**

(a) (i) Here $A$ has size $2 \times 2$ and $B$ has size $2 \times 1$. For $AB$, the size check is

\[
2 \times \begin{pmatrix}
2 \\
1 \\
\end{pmatrix} = 2 \times 1.
\]

The middle numbers are the same, so the product $AB$ is defined. For $BA$, the size check is

\[
2 \times \begin{pmatrix}
2 \\
2 \\
\end{pmatrix} = 2 \times 2.
\]

The middle numbers are different, so the product $BA$ is undefined.

(ii) Here both $A$ and $B$ have size $2 \times 2$, so the products $AB$ and $BA$ are both defined.

(b) For the matrices in part (a)(i), the product $BA$ is not defined.

For the matrices in part (a)(ii),

\[
AB = \begin{pmatrix}
1 & 1 \\
-1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
2 & 3 \\
\end{pmatrix}
= \begin{pmatrix}
2 & 4 \\
0 & -1 \\
\end{pmatrix}
\]

and

\[
BA = \begin{pmatrix}
0 & 1 \\
2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-1 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
-1 & 0 \\
-1 & 2 \\
\end{pmatrix}
\]

Therefore $AB \neq BA$. 
Activity 2 illustrates the important fact that matrix multiplication is not commutative. First, there are matrices \(A\) and \(B\) for which the product \(AB\) exists but the product \(BA\) is not defined. Second, even in cases where both the products \(AB\) and \(BA\) are defined, these two products can be different matrices.

In the next activity, you can investigate whether another property that holds for multiplication of numbers also holds for multiplication of matrices.

**Activity 3 Investigating matrix multiplication further**

Let \(A = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}\), \(B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\) and \(C = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix}\).

(a) What are the sizes of the products \(AB\) and \(BC\)? Calculate these matrices.

(b) Do the products \((AB)C\) and \(A(BC)\) exist? If so, calculate them. What do you notice?

**Solution**

(a) The size of \(A\) is \(1 \times 2\) and the size of \(B\) is \(2 \times 2\). The size check for the product \(AB\) is

\[
1 \times 2 \times 2. 
\]

Thus \(AB\) has size \(1 \times 2\).

The size of \(C\) is \(2 \times 3\). The size check for the product \(BC\) is

\[
2 \times 2 \times 3. 
\]

Thus \(BC\) has size \(2 \times 3\).

We have

\[
AB = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix},
\]

and

\[
BC = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix} 
= \begin{pmatrix} 4 & -1 & -1 \\ 5 & 2 & -1 \end{pmatrix}. 
\]

(b) From part (a), the size of \(AB\) is \(1 \times 2\). Since \(C\) has size \(2 \times 3\), the size check for \((AB)C\) is

\[
1 \times 2 \times 3. 
\]

So the product \((AB)C\) exists and has size \(1 \times 3\).

From part (a), the size of \(BC\) is \(2 \times 3\). Since \(A\) has size \(1 \times 2\), the size check for \(A(BC)\) is

\[
1 \times 2 \times 3. 
\]

So the product \(A(BC)\) exists and has size \(1 \times 3\).
Using the answers for the matrices $AB$ and $BC$ found in part (a) we obtain

\[
(AB)C = (-1 \ 1) \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix} = (-4 \ 1 \ 1)
\]

and

\[
A(BC) = (-1 \ 0) \begin{pmatrix} 4 & -1 & -1 \\ 5 & 2 & -1 \end{pmatrix} = (-4 \ 1 \ 1).
\]

We can see that $(AB)C = A(BC)$.

In Activity 3 you should have found that, for the particular three matrices $A, B$ and $C$ in the activity, $(AB)C = A(BC)$. This does not of course tell you that this property holds for matrices in general. However, this property does hold in general; whenever $A, B$ and $C$ are matrices such that the products $AB$ and $BC$ are defined, we have

\[
(AB)C = A(BC).
\]

**Activity 4 Combining matrix multiplication and multiplication by a scalar**

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Show that $A(2B) = 2(AB)$.

**Solution**

We have

\[
2B = 2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix},
\]

so

\[
A(2B) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 22 & 8 \end{pmatrix}.
\]

Also,

\[
AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix},
\]

so

\[
2(AB) = 2 \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 22 & 8 \end{pmatrix}.
\]

Therefore $A(2B) = 2(AB)$, as required.
In Activity 4 you verified a particular instance of a general property of matrices: if the matrix product $AB$ exists, then, for any scalar $k$,

$$A(kB) = (kA)B = k(AB).$$

**Activity 5 Combining matrix addition and multiplication**

Let $A = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

By calculating $A(B + C)$ and $AB + AC$, show that

$$A(B + C) = AB + AC.$$

**Solution**

We have

$$B + C = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

so

$$A(B + C) = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}.$$

Also,

$$AB = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix},$$

$$AC = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

so

$$AB + AC = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}.$$

Thus $A(B + C) = AB + AC$, as required.

**Some properties of matrix multiplication**

The following properties hold for all matrices $A$, $B$ and $C$ for which the products and sums mentioned are defined.

- $(AB)C = A(BC)$
- $k(AB) = (kA)B = A(kB)$, for any scalar $k$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$

Remember also the following important fact.

**Matrix multiplication is not commutative**

- There are matrices $A$, $B$ such that the product $AB$ exists but the product $BA$ does not.
- Even when both products are defined, it can happen that $AB \neq BA.$