**Refresh Integration by substitution**

To allow you to fully exploit all the ideas about integration that you’ve met, it’s important that you’re able to find formulas for the indefinite integrals of as wide a range of functions as possible.

Unfortunately, it’s generally more tricky to integrate functions than it is to differentiate them. That’s because there are no rules for antiderivatives that are similar to the product, quotient and chain rules for derivatives, so you can’t usually integrate functions in the systematic way that you can differentiate them.

However, there are still many useful techniques that you can use to integrate functions. Each of these techniques applies to functions with particular characteristics. You’ll meet some of these techniques in the rest of this unit. You’ll learn how to recognise functions to which each technique can be applied, and how to use the techniques.

The technique that you’ll learn in this section is *integration by substitution*. This technique might seem quite complicated when you first meet it, but you should find it more straightforward once you’ve had some practice with it. Integration by substitution is based on reversing the chain rule for differentiation.

Consider what happens when you differentiate an expression by using the chain rule. Remember that you start by recognising that the expression is a function of ‘something’. You differentiate the expression with respect to the ‘something’, then you multiply the result by the derivative of the ‘something’ with respect to the input variable (usually \(x\)). For example, by the chain rule,

\[
\frac{d}{dx} (\sin(x^2)) = (\cos(x^2)) (2x).
\]

Since differentiation is the reverse of integration, this equation tells you that

\[
\int (\cos(x^2)) (2x) \, dx = \sin(x^2) + c.
\]

Whenever you differentiate an expression by using the chain rule, you always obtain an expression of the form

\[f(\text{something}) \times \text{the derivative of the something},\]

where \(f\) is a function that you can integrate. The essential idea of integration by substitution is to recognise an expression that you want to integrate as having this form, and then perform the integration by reversing the chain rule.

To help you reverse the chain rule, it’s helpful to denote the ‘something’ by an extra variable, in the way that you did when you first learned to use the chain rule in Unit 7. The usual choice of letter for the extra variable is \(u\).

Here’s how you can use this method to find the integral

\[
\int (\cos(x^2)) (2x) \, dx,
\]

if you hadn’t first seen the integrand obtained as an ‘output’ of the chain rule.
The first step is to recognise that the integrand has the form
\[
    f(\text{something}) \times \text{the derivative of the something},
\]
where \( f \) is a function that you can integrate. You can see that it does, since it is
\[
    \cos(\text{something}) \times \text{the derivative of the something},
\]
where the ‘something’ is \( x^2 \).

Then, since the ‘something’ is \( x^2 \), you put \( u = x^2 \). This gives \( du/dx = 2x \).
Hence, in the integral you can replace \( \cos(x^2) \) by \( \cos u \), and \( 2x \) by \( du/dx \),
to give
\[
    \int \cos(u) \frac{du}{dx} \, dx.
\]
Now here’s the crucial step that you need to apply at this stage. Any integral of the form
\[
    \int f(u) \frac{du}{dx} \, dx
\]
is equal to the simpler integral
\[
    \int f(u) \, du.
\]
This is the step that uses the chain rule, and you’ll see an explanation of why it’s correct later in this subsection. Notice that it’s easy to remember, because it looks like we’ve simply cancelled a ‘\( dx \)’ in a denominator with a ‘\( dx \)’ in a numerator. (Of course that’s not what we’ve done, since \( du/dx \) isn’t a fraction.)

So the integral that we’re trying to find here can be expressed in terms of \( u \) as the simpler integral
\[
    \int \cos u \, du,
\]
where \( u = x^2 \). You can now do the integration, which gives
\[
    \sin u + c.
\]
The final step is to express this answer in terms of \( x \), using the fact that \( u = x^2 \). This gives the final answer
\[
    \sin(x^2) + c.
\]
So we’ve now worked out that
\[
    \int (\cos(x^2)) \,(2x) \, dx = \sin(x^2) + c,
\]
as expected.

Here’s a summary of the method used above, which is the method known as **integration by substitution**.
Integration by substitution

1. Recognise that the integrand is of the form
   \[ f(\text{something}) \times \text{the derivative of the something}, \]
   where \( f \) is a function that you can integrate.
2. Set the something equal to \( u \), and find \( du/dx \).
3. Hence write the integral in the form
   \[ \int f(u) \, du, \]
   by using the fact that \( \int f(u) \frac{du}{dx} \, dx = \int f(u) \, du \).
4. Do the integration.
5. Substitute back for \( u \) in terms of \( x \).

Consider once more the integral above:
\[ \int (\cos(x^2)) \, (2x) \, dx. \]
Remember that we put \( u = x^2 \), which gave \( du/dx = 2x \), and this allowed us to write the integral in terms of \( u \) as
\[ \int \cos u \, du. \]
It’s convenient to think of this new form of the integral as being obtained from the original form by making two replacements, as shown below:
\[ \int (\cos(x^2)) \frac{du}{dx} \, dx. \]
It’s straightforward to work out that you can replace \( \cos(x^2) \) by \( \cos u \), as that just comes from the equation \( u = x^2 \).

A helpful way to work out that you can replace \( (2x) \, dx \) by \( du \) is to imagine ‘cross-multiplying’ in the equation
\[ \frac{du}{dx} = 2x, \]
to obtain
\[ du = (2x) \, dx. \]
This isn’t a real equation, of course, since \( du \) and \( dx \) don’t have independent meanings outside the notation for a derivative or integral. But it tells you immediately that you can replace \( (2x) \, dx \) by \( du \).

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Example 1   Integrating by substitution

Find the integral
\[ \int e^{\tan x} \sec^2 x \, dx. \]
Solution

Since the derivative of \( \tan x \) is \( \sec^2 x \), the integrand is of the form \( e^{\text{something}} \times \text{the derivative of the something} \).

So set the something equal to \( u \).

Let \( u = \tan x \); then \( \frac{du}{dx} = \sec^2 x \).

Imagine ‘cross-multiplying’ in the second equation to obtain \( du = (\sec^2 x) \, dx \).

So
\[
\int e^{\tan x} \sec^2 x \, dx = \int e^u \, du
\]

Do the integration.

\[
= e^u + c
\]

Substitute back for \( u \) in terms of \( x \).

\[
= e^{\tan x} + c.
\]

Activity 1  Integrating by substitution

Find the following indefinite integrals.

(a) \( \int e^{\cos x} (\sin x) \, dx \)   (b) \( \int (\sin(x^3)) \, (3x^2) \, dx \)

(c) \( \int \left( \frac{1}{\sin x} \right) \cos x \, dx \)   (d) \( \int \left( \frac{1}{\cos^2 x} \right) (-\sin x) \, dx \)

(e) \( \int \sin^4 x \cos x \, dx \)   (f) \( \int \left( \frac{1}{1 + \sin^2 x} \right) \cos x \, dx \)

Solution

(a) Let \( u = \cos x \); then \( \frac{du}{dx} = -\sin x \). So
\[
\int e^{\cos x} (-\sin x) \, dx = \int e^u \, du = e^u + c = e^{\cos x} + c.
\]

(b) Let \( u = x^3 \); then \( \frac{du}{dx} = 3x^2 \). So
\[
\int (\sin(x^3)) \, (3x^2) \, dx = \int \sin(u) \, du = -\cos u + c = -\cos(x^3) + c.
\]
(c) Let \( u = \sin x \); then \( \frac{du}{dx} = \cos x \). So
\[
\int \left( \frac{1}{\sin x} \right) \cos x \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |\sin x| + c.
\]

(d) Let \( u = \cos x \); then \( \frac{du}{dx} = -\sin x \). So
\[
\int \left( \frac{1}{\cos^2 x} \right) (-\sin x) \, dx = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = -\frac{1}{u} + c = -\frac{1}{\cos x} + c = -\sec x + c.
\]

(A quicker way to find this indefinite integral, without using substitution, is to write the integrand as \(-\sec x \tan x\) and use the table of standard indefinite integrals.)

(e) Let \( u = \sin x \); then \( \frac{du}{dx} = \cos x \). So
\[
\int \sin^4 x \cos x \, dx = \int u^4 \, du = \frac{1}{5}u^5 + c = \frac{1}{5}\sin^5 x + c.
\]

(f) Let \( u = \sin x \); then \( \frac{du}{dx} = \cos x \). So
\[
\int \left( \frac{1}{1 + \sin^2 x} \right) \cos x \, dx = \int \left( \frac{1}{1 + u^2} \right) \, du = \tan^{-1} u + c = \tan^{-1} (\sin x) + c.
\]

In the examples that you’ve seen so far, it’s been fairly straightforward to decide what the ‘something’ should be, so that the integrand is of the form
\[ f(\text{something}) \times \text{the derivative of the something}. \]

However, sometimes you need to think about this more carefully, as illustrated in the next example.
Example 2 Choosing the ‘something’ when you integrate by substitution

Find the integral
\[ \int \left( \frac{1}{2 + \sin x} \right) \cos x \, dx. \]

Solution

 freopen The integrand is of the form
\[ \frac{1}{2 + \text{something}} \times \text{the derivative of the something}, \]
where the something is \( \sin x \).

It’s also of the form
\[ \frac{1}{\text{something}} \times \text{the derivative of the something}, \]
where the something is \( 2 + \sin x \).

Choosing the first option would lead to the first part of the integrand being replaced by \( 1/(2 + u) \), which isn’t straightforward to integrate. Choosing the second option would lead to the first part of the integrand being replaced by \( 1/u \), which you can integrate using a result from the table of standard integrals.

So choose the second option: take the something to be \( 2 + \sin x \), and set it equal to \( u \).

Let \( u = 2 + \sin x \); then \( \frac{du}{dx} = \cos x \).

Imagine ‘cross-multiplying’ to obtain \( du = \cos x \, dx \).

So
\[ \int \left( \frac{1}{2 + \sin x} \right) \cos x \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |2 + \sin x| + c \]

In the particular case here you can remove the modulus signs, since \( 2 + \sin x \) is always positive.

\[ = \ln(2 + \sin x) + c. \]

Activity 2 Choosing the ‘something’ when you integrate by substitution

Find the following indefinite integrals.

(a) \[ \int (4 + \cos x)^7 (-\sin x) \, dx \]
(b) \[ \int \sqrt{1 + x^2} (2x) \, dx \]
(c) \[ \int (x^5 - 8)^{10} (5x^4) \, dx \]
(d) \[ \int \left( \frac{1}{e^x + 5} \right) e^x \, dx \]
(e) \[ \int (\sin(5 + 2x^3)) (6x^2) \, dx \]
Solution

(a) Let $u = 4 + \cos x$; then $\frac{du}{dx} = -\sin x$. So

$$\int (4 + \cos x)^7 (-\sin x) \, dx = \int u^7 \, du = \frac{1}{8}u^8 + c = \frac{1}{8}(4 + \cos x)^8 + c.$$  

(b) Let $u = 1 + x^2$; then $\frac{du}{dx} = 2x$. So

$$\int \sqrt{1 + x^2} (2x) \, dx = \int \sqrt{u} \, du = \int u^{1/2} \, du = \frac{1}{3/2}u^{3/2} + c = \frac{2}{3}(1 + x^2)^{3/2} + c.$$  

(c) Let $u = x^5 - 8$; then $\frac{du}{dx} = 5x^4$. So

$$\int (x^5 - 8)^{10} (5x^4) \, dx = \int u^{10} \, du = \frac{1}{11}u^{11} + c = \frac{1}{11}(x^5 - 8)^{11} + c.$$  

(d) Let $u = e^x + 5$; then $\frac{du}{dx} = e^x$. So

$$\int \left( \frac{1}{e^x + 5} \right) e^x \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |e^x + 5| + c = \ln(e^x + 5) + c.$$  

(e) Let $u = 5 + 2x^3$; then $\frac{du}{dx} = 6x^2$. So

$$\int (\sin(5 + 2x^3)) (6x^2) \, dx = \int \sin u \, du = -\cos u + c = -\cos(5 + 2x^3) + c.$$  

Activity 3  Choosing the ‘something’ when you integrate by substitution

Find the following indefinite integrals.

(a) $\int (\cos(e^x)) e^x \, dx$  
(b) $\int e^{1 + \sin x} (\cos x) \, dx$  
(c) $\int \left( \frac{1}{x^{10} + 6} \right) (10x^9) \, dx$  
(d) $\int (\cos^3 x)(-\sin x) \, dx$
Solution

(a) Let \( u = e^x \); then \( \frac{du}{dx} = e^x \). So

\[
\int (\cos(e^x)) e^x \, dx = \int \cos u \, du
= \sin u + c
= \sin(e^x) + c.
\]

(b) Let \( u = 1 + \sin x \); then \( \frac{du}{dx} = \cos x \). So

\[
\int e^{1+\sin x} (\cos x) \, dx = \int e^u \, du
= e^u + c
= e^{1+\sin x} + c.
\]

(c) Let \( u = x^{10} + 6 \); then \( \frac{du}{dx} = 10x^9 \). So

\[
\int \left( \frac{1}{x^{10} + 6} \right) (10x^9) \, dx = \int \frac{1}{u} \, du
= \ln |u| + c
= \ln |x^{10} + 6| + c
= \ln(x^{10} + 6) + c.
\]

(d) Let \( u = \cos x \); then \( \frac{du}{dx} = -\sin x \). So

\[
\int (\cos^3 x)(-\sin x) \, dx = \int u^3 \, du
= \frac{1}{4} u^4 + c
= \frac{1}{4} \cos^4 x + c.
\]