Handling powers

Raising a number to a power means multiplying the number by itself a specified number of times. For example, raising 2 to the power 3 gives

\[ 2^3 = 2 \times 2 \times 2 = 8. \]

Number 2 is called the base number or just base, and the superscript 3 is called the power, index or exponent. The word ‘power’ also refers to the result of raising a number to a power – for example, we say that \(2^3\), or 8, is a power of 2.

### Index laws for a single base

To multiply two powers with the same base, add the indices:

\[ a^m a^n = a^{m+n}. \]

To divide two powers with the same base, subtract the indices:

\[ \frac{a^m}{a^n} = a^{m-n}. \]

To find a power of a power, multiply the indices:

\[ (a^m)^n = a^{mn}. \]

**Remember** that any non-zero number raised to the power 0 must be 1. For example,

\[ 25^0 = 1. \]

### More index laws for a single base

A number raised to the power 0 is 1:

\[ a^0 = 1. \]

A number raised to a negative power is the reciprocal of the number raised to the corresponding positive power:

\[ a^{-n} = \frac{1}{a^n}. \]

### Index laws for two bases

A power of a product of numbers is the same as the product of the same powers of the numbers, and similarly for a power of a quotient:

\[ (ab)^n = a^n b^n, \]
\[ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \]

For example,

\[ (2a)^2 = 2a \times 2a = 4a^2. \]

**Notice** that both the 2 and the \(a\) need to be squared.
Example 1  
\textit{Simplifying expressions containing indices}

Simplify the following expressions, ensuring that the simplified versions contain no negative indices.

(a) \( \frac{d^2}{d^{-4}} \)  
(b) \( \frac{b^{-3}}{b^2 c^{-4}} \)  
(c) \( (2h^{-3}g)^2 \)

\textbf{Solution}

(a) Use the law \( a^{-n} = 1/a^n \) to change the negative index into a positive index, then use the law \( a^m a^n = a^{m+n} \) to combine the powers.
\[
\frac{d^2}{d^{-4}} = d^2 \times \frac{1}{d^{-4}} = d^2 d^4 = d^6
\]
Alternatively, use the law \( a^m/a^n = a^{m-n} \).
\[
\frac{d^2}{d^{-4}} = d^{2-(-4)} = d^6
\]

(b) Use the law \( a^{-n} = 1/a^n \) to change the negative indices into positive indices, then use the law \( a^m a^n = a^{m+n} \) to combine the powers of \( b \).
\[
\frac{b^{-3}}{b^2 c^{-4}} = \frac{c^4}{b^{2b^3}} = \frac{c^4}{b^5}
\]
Alternatively, use the law \( a^m/a^n = a^{m-n} \) to combine the powers of \( b \), then use the law \( a^{-n} = 1/a^n \) to change the negative indices into positive indices.
\[
\frac{b^{-3}}{b^2 c^{-4}} = \frac{b^{-5}}{c^{-1}} = \frac{c^4}{b^5}
\]

(c) Remove the brackets by using the law \( (ab)^n = a^n b^n \), then the law \( (a^m)^n = a^{mn} \). Then use the law \( a^{-n} = 1/a^n \) to change the negative index into a positive index.
\[
(2h^{-3}g)^2 = 2^2(h^{-3})^2 g^2 = 4h^{-6}g^2 = \frac{4g^2}{h^6}
\]
Alternatively, use the law \( a^{-n} = 1/a^n \) to change the negative index into a positive index, then use the law \( (a/b)^n = a^n/b^n \), then the laws \( (ab)^n = a^n b^n \) and \( (a^m)^n = a^{mn} \).
\[
(2h^{-3}g)^2 = \left(\frac{2g}{h^3}\right)^2 = \frac{(2g)^2}{(h^3)^2} = \frac{2^2 g^2}{h^6} = \frac{4g^2}{h^6}
\]
Converting between fractional indices and roots

\[
\begin{align*}
a^{1/n} &= \sqrt[n]{a} \\
a^{m/n} &= \left(\sqrt[n]{a}\right)^m = \sqrt[m]{a^m}
\end{align*}
\]

Example 2  *Simplifying expressions containing fractional indices*

Simplify the following expressions.

(a) \(\frac{c^{3/4}}{c^{5/4}}\)  
(b) \((2h^{1/6})^2\)

**Solution**

(a) Use the index laws in the same ways as for integer indices.

\[
\frac{c^{3/4}}{c^{5/4}} = c^{(3/4)-(5/4)} = c^{-2/4} = c^{-1/2}
\]

Usually, change the negative index to a positive one, and perhaps use a square root sign instead of the index \(\frac{1}{2}\).

\[
= \frac{1}{c^{1/2}} = \frac{1}{\sqrt{c}}
\]

(b) Proceed as in part (a). Leave the final fractional index as it is, since it’s not \(\frac{1}{2}\) or \(-\frac{1}{2}\).

\[
(2h^{1/6})^2 = 2^2(h^{1/6})^2 = 4h^{(1/6)\times 2} = 4h^{1/3}
\]

A summary of the index laws is shown below:

**Index laws**

\[
\begin{align*}
a^m a^n &= a^{m+n} & \frac{a^m}{a^n} &= a^{m-n} & (a^m)^n &= a^{mn} \\
a^0 &= 1 & a^{-n} &= \frac{1}{a^n} \\
(ab)^n &= a^n b^n & \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \\
a^{1/n} &= \sqrt[n]{a} & a^{m/n} &= \left(\sqrt[n]{a}\right)^m = \sqrt[m]{a^m}
\end{align*}
\]