The exponential function is any function of the form $f(x) = b^x$ where $b > 0$ is any real number. The special number $b = e$, where $e$ is the base of the natural logarithms, is of special importance and occurs often in mathematics and its applications. The approximate value is $2.71828182846$ but there is no exact numerical value known.

Usually by the exponential function we mean $f(x) = e^x$. The expression $e^x$ is sometimes written as $\exp(x)$, or $\exp(x)$.

The natural logarithm of a number $x$, denoted by $\ln x$, is the power to which the base $e$ must be raised to give the number $x$. So the two equations $y = \ln x$ and $x = e^y$ are equivalent. $\ln x$ is the inverse function to $e^x$. This means that $\ln e^x = x$ and that $e^{\ln x} = x$.

Useful values to remember are $\ln(1) = 0$ and $\ln(e) = 1$.

Graphs of $y = e^x$ and $y = \ln x$ are shown in Figure 1.

**Figure 1** The graphs of $y = \ln x$ and $y = e^x$

**Graphs of exponential functions**

The graph of the function $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has the following features.

- The graph lies entirely above the $x$-axis.
- If $b > 1$, then the graph is increasing, and it gets steeper as $x$ increases.
- If $0 < b < 1$, then the graph is decreasing, and it gets less steep as $x$ increases.
- The $x$-axis is an asymptote.
- The $y$-intercept is 1.
- The closer the value of $b$ is to 1, the flatter is the graph.
Graphs of logarithmic functions

The graph of the function \( f(x) = \log_b x \), where \( b > 0 \) and \( b \neq 1 \), has the following features.

- The graph lies entirely to the right of the \( y \)-axis.
- If \( b > 1 \), then the graph is increasing, and it gets less steep as \( x \) increases.
- If \( 0 < b < 1 \), then the graph is decreasing, and it gets less steep as \( x \) increases.
- The \( y \)-axis is an asymptote.
- The \( x \)-intercept is 1.
- The closer the value of \( b \) is to 1, the steeper is the graph.

Rules for using logarithms

For any base \( b \),
\[
\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x.
\]
In particular,
\[
\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x.
\]

Example 1

Simplify \( \ln \left( \frac{1}{e^5} \right) \) as fully as possible.

Solution

As \( 1/e^5 \) is the same as \( e^{-5} \), this means that
\[
\ln \left( \frac{1}{e^5} \right) = \ln(e^{-5}) = -5.
\]

Example 2

Simplify \( e^{5\ln x} \) as fully as possible.

Solution

Use \( 5 \ln x = \ln x^5 \). Therefore
\[
e^{5\ln x} = e^{\ln x^5} = x^5.
\]
Example 3
Solve the equation $e^{\frac{1}{3}\ln x} = y$ for $x$ in terms of $y$.

Solution
Just as in the previous example we can write $e^{\frac{1}{3}\ln x} = e^{\ln x^{1/3}} = x^{1/3}$.

So the equation $e^{\frac{1}{3}\ln x} = y$ becomes $e^{\ln x^{1/3}} = y$, which then gives us $x^{1/3} = y$. We can get $x$ in terms of $y$ by raising each side to the power 3 to find $x = y^3$. This is the answer as it gives $x$ in terms of $y$.

Three logarithm laws
\[
\begin{align*}
\log_b x + \log_b y & = \log_b (xy) \\
\log_b x - \log_b y & = \log_b \left(\frac{x}{y}\right) \\
r \log_b x & = \log_b (x^r)
\end{align*}
\]

Example 4
Simplify the expression $\ln e^{2x} + \ln e^{-2x}$.

Solution
There are two different ways we can tackle this simplification. We could use the logarithm law $\ln (xy) = \ln x + \ln y$ and this will give:

\[
\begin{align*}
\ln e^{2x} + \ln e^{-2x} & = \ln (e^{2x} e^{-2x}) \\
& = \ln 1 \\
& = 0.
\end{align*}
\]

The second method uses the fact that $\ln e^{2x} = 2x$, and that $\ln e^{-2x} = -2x$. These results give $\ln e^{2x} + \ln e^{-2x} = 2x - 2x = 0$.

So by either method the answer is that $\ln e^{2x} + \ln e^{-2x} = 0$.

Example 5
Simplify the expression $\ln e^{2x-3} + 3 \ln e^{x+4}$ as fully as possible.

Solution
Start with $\ln e^{2x-3} = 2x - 3$. Now use $\ln e^{x+4} = x + 4$. This will give us

\[
\begin{align*}
\ln e^{2x-3} + 3 \ln e^{x+4} & = (2x - 3) + 3(x + 4) \\
& = 2x - 3 + 3x + 12 \\
& = 5x + 9.
\end{align*}
\]
Exponential growth

A quantity that changes in a way that can be modelled by a function whose rule is of the form $f(x) = ae^{kx}$, where $a$ and $k$ are non-zero constants, is said to change exponentially. If $a$ and $k$ are both positive, then the graph of $y = f(x)$ looks like Figure 2(a), or a part of it. In this case the quantity is said to grow exponentially, the function is called an exponential growth function, and the graph is called an exponential growth curve.

Similarly, if $a$ is positive as before but $k$ is negative, then the graph of $y = f(x)$ looks like the graph in Figure 2(b), or a part of it. In this case the quantity is said to decay exponentially, the function is called an exponential decay function, and the graph is called an exponential decay curve.

![Figure 2](image-url)  
(a) An exponential growth curve (b) an exponential decay curve
Example 6

Suppose that the number of weeds in a pond is given by the exponential function $f(t) = 2e^{t/5}$ where $t$ is the time in hours. What type of growth does the function $f(t)$ represent? After how many hours will the number of weeds double? Give the exact answer as well as an answer that is accurate to 3 decimal places.

Solution

As $t$ is positive the function $f(t) = 2e^{t/5}$ represents exponential growth. Let $x$ be the time in hours required for the number of weeds to double. Expressed mathematically this means that

$$f(t + x) = 2f(t).$$

Use the form $f(t) = 2e^{t/5}$ given for $f(t)$ in the question. Then $f(t + x) = 2f(t)$ becomes

$$2e^{(t+x)/5} = 2 \times 2e^{t/5}.$$

Divide both sides by the factor of 2 to get

$$e^{(t+x)/5} = 2e^{t/5}.$$

Notice that the left hand side can be expressed as

$$e^{(t+x)/5} = e^{t/5} \times e^{x/5}.$$ We then have

$$e^{t/5} \times e^{x/5} = 2e^{t/5}.$$

We can then divide each side by $e^{t/5}$ to leave

$$e^{x/5} = 2.$$

To solve for $x$ take the natural logarithm of each side and use

$$\ln e^{x/5} = x/5.$$ This gives us

$$\ln e^{x/5} = \frac{x}{5} = \ln 2.$$

Multiply both sides by 5 to find the exact result for $x$ which is $x = 5 \ln 2$. So it takes $5 \ln 2$ hours for the number of weeds to double.

To get the time to 3 decimal places use your calculator. To more accuracy than needed $x = 5 \ln 2 \approx 5 \times 0.69314718056 \approx 3.4657359028$. Rounding this to 3 decimal places gives $x \approx 3.466$ hours. So it takes just a bit less than $3 \frac{1}{2}$ hours for the number of weeds to double.