**Refresh Differentiation of standard functions**

MST 224 expects you to be comfortable with the differentiation of a number of standard functions (listed in the table below) and illustrated in the following examples. The problem sheet gives you practice in using these results.

In each case, \(a\) is a constant.

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
</tr>
<tr>
<td>(x^a)</td>
<td>(ax^{a-1})</td>
</tr>
<tr>
<td>(e^{ax})</td>
<td>(ae^{ax})</td>
</tr>
<tr>
<td>(\ln(ax))</td>
<td>( \frac{1}{x} )</td>
</tr>
<tr>
<td>(\sin(ax))</td>
<td>(a\cos(ax))</td>
</tr>
<tr>
<td>(\cos(ax))</td>
<td>(-a\sin(ax))</td>
</tr>
<tr>
<td>(\tan(ax))</td>
<td>(a\sec^2(ax))</td>
</tr>
<tr>
<td>(\cot(ax))</td>
<td>(-a\cosec^2(ax))</td>
</tr>
<tr>
<td>(\sec(ax))</td>
<td>(a\sec(ax)\tan(ax))</td>
</tr>
<tr>
<td>(\cosec(ax))</td>
<td>(-a\cosec(ax)\cot(ax))</td>
</tr>
<tr>
<td>(\arcsin(ax))</td>
<td>(\frac{a}{\sqrt{1-a^2x^2}})</td>
</tr>
<tr>
<td>(\arccos(ax))</td>
<td>(-\frac{a}{\sqrt{1-a^2x^2}})</td>
</tr>
<tr>
<td>(\arctan(ax))</td>
<td>(\frac{a}{1+a^2x^2})</td>
</tr>
<tr>
<td>(\arccot(ax))</td>
<td>(-\frac{a}{1+a^2x^2})</td>
</tr>
<tr>
<td>(\arccsec(ax))</td>
<td>(\frac{a}{</td>
</tr>
<tr>
<td>(\arccosec(ax))</td>
<td>(-\frac{a}{</td>
</tr>
</tbody>
</table>

**Examples using sin, tan and cos**

**Example 1  Differentiating functions involving sin, cos and tan**

Write down the derivatives of the following functions.

(a) \(f(x) = \sin x + \cos x\)  
(b) \(g(u) = u^2 - \cos u\)  
(c) \(P = 6\tan \theta\)  
(d) \(r = -2(1 + \sin \phi)\)
Solution

(a) \( f(x) = \sin x + \cos x \), so \( f'(x) = \cos x - \sin x \).

(b) \( g(u) = u^2 - \cos u \), so
\[
\begin{align*}
g'(u) &= 2u - (-\sin u) \\
&= 2u + \sin u.
\end{align*}
\]

(c) \( P = 6 \tan \theta \), so
\[
\frac{dP}{d\theta} = 6 \sec^2 \theta.
\]

(d) \( r = -2(1 + \sin \phi) \), so
\[
\frac{dr}{d\phi} = -2(0 + \cos \phi) = -2 \cos \phi.
\]

Example involving exponentials and logs

Example 2 Differentiating functions involving exp and log

Find the derivatives of the following functions.

(a) \( f(x) = e^x + \ln x \) \hspace{1cm} (b) \( h(r) = r - \cos r - 3 \ln r \)

(c) \( v = \frac{1}{t} + \ln t \) \hspace{1cm} (d) \( w = 5 - 3e^u \) \hspace{1cm} (e) \( k = 4(\ln v - \tan v) \)

The formulas for the derivatives of standard functions that you’ve met in this section are all included in the Handbook. However, it’s worth memorising the formulas for the derivatives of \( \sin, \cos, \exp \) and \( \ln \), at least, as they occur frequently.

Solution

(a) \( f(x) = e^x + \ln x \), so
\[
\begin{align*}
f'(x) &= e^x + \frac{1}{x}.
\end{align*}
\]

(b) \( h(r) = r - \cos r - 3 \ln r \), so
\[
\begin{align*}
h'(r) &= 1 - (-\sin r) - 3 \times \frac{1}{r} = 1 + \sin r - \frac{3}{r}.
\end{align*}
\]

(c) \( v = \frac{1}{t} + \ln t \), so
\[
\begin{align*}
\frac{dv}{dt} &= -\frac{1}{t^2} + \frac{1}{t} \\
&= -\frac{1}{t^2} + \frac{t}{t^2} \\
&= \frac{t - 1}{t^2}.
\end{align*}
\]

(d) \( w = 5 - 3e^u \), so
\[
\frac{dw}{du} = -3e^u.
\]
(e) \( k = 4(\ln v - \tan v) \), so
\[
\frac{dk}{dv} = 4 \left( \frac{1}{v} - \sec^2 v \right).
\]

**Examples using cot, sec and cosec**

**Derivatives of inverse functions**

In this subsection you’ll meet a rule, known as the *inverse function rule*, which you can use to work out a formula for the derivative of an inverse function when you know a formula for the derivative of the original function.

**Inverse function rule (Leibniz notation)**

If \( y \) is an invertible function of \( x \), then
\[
\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},
\]
for all values of \( x \) such that \( \frac{dx}{dy} \) exists and is non-zero.

Here’s an informal way to see that the inverse function rule makes sense. Suppose that \( y \) is an invertible function of \( x \), which means that \( x \) is also a function of \( y \). Suppose that, for a particular value of \( x \) and its corresponding value of \( y \), the value of \( x \) is increasing at the rate of 2 units for every unit that \( y \) increases, as illustrated in the figure below. Then, at these values of \( x \) and \( y \), you’d expect the value of \( y \) to be increasing at the rate of \( \frac{1}{2} \) unit for every unit that \( x \) increases. This is what the inverse function rule tells you.

![A point \((x, y)\) on the graph of a function](image-url)
Example 3  Using the inverse function rule again

Use the inverse function rule, and the derivative of the sine function, to differentiate \( y = \sin^{-1} x \).

Solution

Express \( x \) in terms of \( y \).

We have \( y = \sin^{-1} x \), so

\[ x = \sin y. \]

Differentiate \( x \) with respect to \( y \).

Therefore

\[ \frac{dx}{dy} = \cos y. \]

Use the inverse function rule.

By the inverse function rule,

\[ \frac{dy}{dx} = \frac{1}{\cos y}, \]

provided that \( \cos y \neq 0 \).

Use the relationship between \( x \) and \( y \) to express the derivative in terms of \( x \). In this case the relationship is given by \( x = \sin y \) (or \( y = \sin^{-1} x \)), and you need to express \( \cos y \) in terms of \( x \). You could write \( \cos y = \cos(\sin^{-1} x) \), but you can obtain a simpler expression by using the identity \( \sin^2 y + \cos^2 y = 1 \), as follows.

The identity \( \cos^2 y + \sin^2 y = 1 \) gives

\[ \cos y = \pm \sqrt{1 - \sin^2 y} = \pm \sqrt{1 - x^2}. \]

The + sign applies here, because \( y \) takes values only in the interval \([ -\pi/2, \pi/2 ]\) (since \( y = \sin^{-1} x \)) and so \( \cos y \) is always non-negative. Hence

\[ \cos y = \sqrt{1 - x^2}. \]

Therefore

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \]

provided that \( x \neq \pm 1 \).
The inverse function rule in Lagrange notation

Let’s now translate the inverse function rule into Lagrange notation. To do this, we set $y = f^{-1}(x)$. Then

$$\frac{dy}{dx} = (f^{-1})'(x),$$

where $(f^{-1})'$ denotes the derivative of $f^{-1}$, as you’d expect.

Also, the equation $y = f^{-1}(x)$ is equivalent to the equation $x = f(y)$, so

$$\frac{dx}{dy} = f'(y) = f'(f^{-1}(x)).$$

This gives the following form of the inverse function rule.

Inverse function rule (Lagrange notation)

If $f$ is a function with inverse function $f^{-1}$, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

for all values of $x$ such that $f'(f^{-1}(x))$ exists and is non-zero.