**Refresh Definite integrals**

Before you learn more about why signed areas are important, it’s useful for you to learn some terminology and notation that are used when working with them.

If $f$ is a continuous function and $a$ and $b$ are numbers in its domain, then the signed area between the graph of $f$ and the $x$-axis from $x = a$ to $x = b$ is called the **definite integral** of $f$ from $a$ to $b$, and is denoted by

\[ \int_a^b f(x) \, dx. \]

This notation is read as ‘the integral from $a$ to $b$ of $f$ of $x$, $d\,x$’. The numbers $a$ and $b$ are called the **lower** and **upper limits of integration**, respectively.

For example, for the function $f$ in Figure 1,

\[ \int_3^7 f(x) \, dx = -9, \quad \int_7^9 f(x) \, dx = 2, \]
and \[ \int_3^9 f(x) \, dx = -9 + 2 = -7. \]

Similarly, for the same function,

\[ \int_4^4 f(x) \, dx = 0, \quad \text{and} \quad \int_7^3 f(x) \, dx = -(9) = 9. \]

![Figure 1](image_url)  

The graph of a function, and some areas

You’ll see a little later in this subsection where this notation for a definite integral comes from.

The symbol $\int$ is called the **integral sign**. It sometimes appears in a smaller form, $\int$, when it’s in a line of typed text.

As with Leibniz notation for derivatives, the ‘$d$’ in the notation for definite integrals has no independent meaning. In many texts, including this one, it’s printed in upright type, rather than italic type, to emphasise that it’s not a variable. You don’t need to do anything special when you handwrite it.
When you use the notation $\int_{a}^{b} f(x) \, dx$, remember that you must include not only the $\int_{a}^{b}$ at the beginning, but also the $dx$ at the end. Try not to forget the $dx$!

The box below lists some standard properties of definite integrals, which come from their definition as signed areas. These properties hold for all values of $a$, $b$ and $c$ in the domain of the continuous function $f$. The second property comes from the extended definition of signed area, to cases where $b$ is less than $a$. The third property expresses the fact that the signed area from $a$ to $c$ is equal to the signed area from $a$ to $b$ plus the signed area from $b$ to $c$. Figure 2 illustrates this property in a case where $a < b < c$, but the extended definition of signed area ensures that the property holds even when $a$, $b$ and $c$ aren’t in this order. This is one reason why the definition of signed area is extended in the way that you’ve seen.

**Standard properties of definite integrals**

\[
\int_{a}^{a} f(x) \, dx = 0
\]
\[
\int_{b}^{a} f(x) \, dx = - \int_{a}^{b} f(x) \, dx
\]
\[
\int_{c}^{a} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx
\]

**Figure 2**  Two adjacent signed areas

**Activity 1  Understanding definite integrals**

Consider the function $f$ whose graph is shown below. The areas of some regions are marked. By using these areas, write down the values of the following definite integrals.

Hint: notice that in some of these definite integrals the upper limit of integration is less than the lower limit of integration.

(a) $\int_{-5}^{-3} f(x) \, dx$  (b) $\int_{-5}^{2} f(x) \, dx$  (c) $\int_{2}^{7} f(x) \, dx$

(d) $\int_{-5}^{2} f(x) \, dx$  (e) $\int_{-3}^{7} f(x) \, dx$  (f) $\int_{-5}^{7} f(x) \, dx$

(g) $\int_{5}^{7} f(x) \, dx$  (h) $\int_{2}^{-3} f(x) \, dx$  (i) $\int_{7}^{2} f(x) \, dx$

(j) $\int_{-3}^{-5} f(x) \, dx$  (k) $\int_{7}^{-3} f(x) \, dx$  (l) $\int_{7}^{-5} f(x) \, dx$
Solution

(a) \( \int_{-5}^{-3} f(x) \, dx = -5 \).

(b) \( \int_{-3}^{2} f(x) \, dx = 6 \).

(c) \( \int_{2}^{7} f(x) \, dx = -11 \).

(d) \( \int_{-5}^{2} f(x) \, dx = -5 + 6 = 1 \).

(e) \( \int_{-3}^{7} f(x) \, dx = 6 - 11 = -5 \).

(f) \( \int_{-5}^{7} f(x) \, dx = -5 + 6 - 11 = -10 \).

(g) \( \int_{5}^{5} f(x) \, dx = 0 \).

(h) \( \int_{2}^{-3} f(x) \, dx = - \int_{-3}^{2} f(x) \, dx = -6 \).

(i) \( \int_{2}^{7} f(x) \, dx = - \int_{7}^{2} f(x) \, dx = -(-11) = 11 \).

(j) \( \int_{-3}^{-5} f(x) \, dx = - \int_{-5}^{-3} f(x) \, dx = -(-5) = 5 \).

(k) \( \int_{7}^{3} f(x) \, dx = - \int_{3}^{7} f(x) \, dx \)
   \[ = -(-5) = 5, \]
   by part (e).

(l) \( \int_{7}^{5} f(x) \, dx = - \int_{5}^{7} f(x) \, dx \)
   \[ = -(-10) = 10, \]
   by part (f).
Activity 2  Using a standard property of definite integrals

Consider again the graph in Activity 1.

Given that \( \int_2^9 f(x) \, dx = -15 \), find \( \int_7^9 f(x) \, dx \).

**Solution**

\[
\int_2^9 f(x) \, dx = \int_2^7 f(x) \, dx + \int_7^9 f(x) \, dx
\]

so

\[
-15 = -11 + \int_7^9 f(x) \, dx
\]

and hence

\[
\int_7^9 f(x) \, dx = -15 - (-11) = -4.
\]

As you’d expect, you can replace the expression \( f(x) \) in the notation for a definite integral by the formula for a particular function. For example, the signed areas in Figure 3 are denoted by

\[
\int_{-1}^1 x^2 \, dx \quad \text{and} \quad \int_0^1 (x^3 - 1) \, dx,
\]

respectively.

**Figure 3**  Signed areas between the graphs of particular functions and the \( x \)-axis

As always with algebraic notation, the notation for a definite integral can be used with letters other than the standard ones. For example, if \( g \) is a continuous function whose domain contains the numbers \( p \) and \( q \), and you use \( t \) to denote the input variable of \( g \), then the definite integral of \( g \) from \( t = p \) to \( t = q \) is denoted by

\[
\int_p^q g(t) \, dt.
\]

In fact, the input variable of the function in a definite integral is what’s known as a **dummy variable** – you can change its name to any other variable name that you like, without affecting the value of the definite integral. For example, if \( f \) is a continuous function whose domain contains the numbers \( a \) and \( b \), then

\[
\int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(u) \, du,
\]
and so on. As a particular example,
\[ \int_{-1}^{1} x^2 \, dx = \int_{-1}^{1} t^2 \, dt = \int_{-1}^{1} u^2 \, du, \]
since all these definite integrals denote the signed area shown in Figure 3(a).

If \( f \) is any continuous function, and \( a \) and \( b \) are numbers in its domain, then you can find an approximate value for the definite integral \( \int_{a}^{b} f(x) \, dx \), as accurately as you like, by using the method that you met in the last subsection. Here’s the method expressed algebraically.

Suppose that you want to find \( \int_{a}^{b} f(x) \, dx \), as illustrated in Figure 4(a). You divide the interval between \( a \) and \( b \) into \( n \) subintervals, each of width \( (b - a)/n \), as illustrated in Figure 4(b). We’ll denote \( (b - a)/n \) by \( w \) here, for conciseness.

**Figure 4**  
(a) A definite integral \( \int_{a}^{b} f(x) \, dx \)  
(b) A collection of \( n \) rectangles whose total signed area is approximately this definite integral

**Algebraic definition of a definite integral**

Suppose that \( f \) is a continuous function and \( a \) and \( b \) are numbers in its domain. Then the **definite integral** of \( f \) from \( x = a \) to \( x = b \) is given by the equation
\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \left( f(a + 0w) + f(a + 1w) + f(a + 2w) + \cdots + f(a + (n-1)w) \right) w
\]
where \( w = (b - a)/n \).

**Figure 5**  
(a) The subinterval method applied to a function \( f \) from \( x = a \) to \( x = b \)  
(b) One of the subintervals and its corresponding rectangle
With this notation, the signed area of the rectangle corresponding to the subinterval, from $x$ to $x + \delta x$, is

$$f(x) \, \delta x.$$  \hspace{1cm} (1)

So the total signed area of all the rectangles is the sum of a number of terms, each of form (1). As the number of subintervals gets larger and larger, the size of $\delta x$ gets smaller and smaller, and the total signed area of the rectangles gets closer and closer to the definite integral of $f$ from $a$ to $b$. So, loosely, you can think of this definite integral as the sum of infinitely many terms of form (1), where the quantity $\delta x$ is infinitely small. Historically, the quantity $\delta x$ was denoted by $dx$ when it becomes infinitely small, so the sum described above was denoted by

$$\int_a^b f(x) \, dx,$$

where the symbol $\int$ is an elongated ‘S’, which stands for ‘sum’.

To see where the term ‘integral’ comes from, remember that the verb ‘to integrate’ means ‘to join together’. Loosely, you can think of a definite integral as being obtained by joining together infinitely many signed areas, each infinitely narrow, into a single signed area.

### Definite integrals

Suppose that $f$ is a continuous function, and $a$ and $b$ are numbers in its domain. The signed area between the graph of $f$ and the $x$-axis from $x = a$ to $x = b$ is called the **definite integral** of $f$ from $a$ to $b$, and is denoted by

$$\int_a^b f(x) \, dx.$$