Refresh Basic integration by parts

Suppose that $f$ and $g$ are functions, and that the function $G$ is an antiderivative of the function $g$. Consider the equation that you obtain when you use the product rule to differentiate the product $f(x)G(x)$:
\[
\frac{d}{dx} (f(x)G(x)) = f(x) \left( \frac{d}{dx} G(x) \right) + G(x) \left( \frac{d}{dx} f(x) \right),
\]
that is,
\[
\frac{d}{dx} (f(x)G(x)) = f(x)g(x) + G(x)f'(x).
\]
If you swap the order of $G(x)$ and $f'(x)$ in the final term, and subtract this term from both sides of the equation, then you obtain
\[
\frac{d}{dx} (f(x)G(x)) - f'(x)G(x) = f(x)g(x).
\]
Swapping the sides of this equation gives
\[
f(x)g(x) = \frac{d}{dx} (f(x)G(x)) - f'(x)G(x).
\]
If you now integrate both sides of this equation with respect to $x$, then you obtain
\[
\int f(x)g(x) \, dx = f(x)G(x) - \int f'(x)G(x) \, dx.
\]
There’s no need to add an explicit arbitrary constant here, because it’s included in the integral $\int f(x)g(x) \, dx$ on the left-hand side, and in the integral $\int f'(x)G(x) \, dx$ on the right-hand side.

The formula above is known as the integration by parts formula. It’s stated again in the box below, for easy reference. If you’ve met it before, then you may have seen it stated in a different form. This is explained at the end of this section.

**Integration by parts formula (Lagrange notation)**
\[
\int f(x)g(x) \, dx = f(x)G(x) - \int f'(x)G(x) \, dx.
\]
Here $G$ is an antiderivative of $g$.

The integration by parts formula is useful for integrating some products of the form $f(x)g(x)$. You can see that it changes the problem of integrating $f(x)g(x)$ into the problem of integrating $f'(x)G(x)$, where $G$ is an antiderivative of $g$.

So it’s useful for integrating products of the form $f(x)g(x)$ that have the following characteristics.

- You can find an antiderivative $G$ of $g$; in other words, you can integrate the expression $g(x)$.
- The product $f'(x)G(x)$ is easier to integrate than the product $f(x)g(x)$. For example, this may be the case if $f'(x)$ is simpler than $f(x)$.

When we integrate by using the integration by parts formula, we say that we’re integrating by parts. Here’s an example.
Example 1  Integrating by parts

Find the integral \( \int x \sin x \, dx \).

Solution

The integrand is a product of two expressions, \( x \) and \( \sin x \). You can integrate the second expression, \( \sin x \), and differentiating the first expression, \( x \), makes it simpler. So try integration by parts. ◇

Let \( f(x) = x \) and \( g(x) = \sin x \).

Then \( f'(x) = 1 \), and an antiderivative of \( g(x) \) is \( G(x) = -\cos x \).

So the integration by parts formula gives

\[
\int x \sin x \, dx = \int f(x)g(x) \, dx
= f(x)G(x) - \int f'(x)G(x) \, dx
= x \times (-\cos x) - \int 1 \times (-\cos x) \, dx
= -x \cos x + \int \cos x \, dx
\]

Find the integral in this expression. ◇

\[
= -x \cos x + \sin x + c.
\]

Activity 1  Integrating by parts

Find the following integrals.

(a) \( \int x \cos x \, dx \)  (b) \( \int xe^x \, dx \)

Solution

(a) Let \( f(x) = x \) and \( g(x) = \cos x \). Then \( f'(x) = 1 \) and an antiderivative of \( g(x) \) is \( G(x) = \sin x \). Hence

\[
\int x \cos x \, dx = \int f(x)g(x) \, dx
= f(x)G(x) - \int f'(x)G(x) \, dx
= x \sin x - \int \sin x \, dx
= x \sin x - (-\cos x) + c
= x \sin x + \cos x + c.
\]
(b) Let \( f(x) = x \) and \( g(x) = e^x \). Then \( f'(x) = 1 \) and an antiderivative of 
\( g(x) \) is \( G(x) = e^x \). Hence
\[
\int xe^x \, dx = \int f(x)g(x) \, dx \\
= f(x)G(x) - \int f'(x)G(x) \, dx \\
= xe^x - \int e^x \, dx \\
= xe^x - e^x + c \\
= e^x(x - 1) + c.
\]

As you become more familiar with integration by parts, you’ll probably find that you can carry it out more quickly and conveniently if you remember the informal version of the formula stated below. You can recite this version in your head as you apply integration by parts, in a similar way to the informal versions of the product and quotient rules for differentiation. It’s useful for applying integration by parts in fairly simple cases, like the ones that you’ve seen so far, which is usually all that you need to do.

**Integration by parts formula (informal)**

\[
\int \text{integral of product} = (\text{first}) \times (\text{antiderivative of second}) \\
- \int \text{integral of } (\text{derivative of first}) \times (\text{antiderivative of second})
\]

**Example 2  Integrating by parts, using the informal formula**

Find the integral \( \int x \sin x \, dx \).

**Solution**

The integrand is a product of two expressions, \( x \) and \( \sin x \). You can integrate the second expression, \( \sin x \), and differentiating the first expression, \( x \), makes it simpler. So try integration by parts.

Recite the informal version of the formula in your head. As you think ‘first’, write down the first expression, then as you think ‘antiderivative of second’, write down an antiderivative of the second expression, and so on.

Integrating by parts gives
\[
\int x \sin x \, dx = x \times (-\cos x) - \int 1 \times (-\cos x) \, dx \\
= -x \cos x + \int \cos x \, dx \\
= -x \cos x + \sin x + c.
\]
You can practise using the informal version of the integration by parts formula in the next activity. Alternatively, you might prefer to continue using the formal version for a little longer, and move to the informal version when you feel more confident with it.

**Activity 2 Integrating by parts**

Find the following integrals.

(a) \( \int x \sin(5x) \, dx \)  
(b) \( \int x e^{2x} \, dx \)  
(c) \( \int x e^{-x} \, dx \)

**Solution**

(a) \( \int x \sin(5x) \, dx = x \left( -\frac{1}{5} \cos(5x) \right) \)

\[ \begin{align*}
&- \int 1 \times \left( -\frac{1}{5} \cos(5x) \right) \, dx \\
&= -\frac{1}{5} x \cos(5x) + \frac{1}{5} \int \cos(5x) \, dx \\
&= -\frac{1}{5} x \cos(5x) + \frac{1}{5} \cdot \frac{1}{5} \sin(5x) + c \\
&= -\frac{1}{25} \sin(5x) - \frac{1}{5} x \cos(5x) + c.
\end{align*} \]

(b) \( \int x e^{2x} \, dx = x \left( \frac{1}{2} e^{2x} \right) - \int 1 \times \left( \frac{1}{2} e^{2x} \right) \, dx \)

\[ \begin{align*}
&= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} \, dx \\
&= \frac{1}{2} x e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + c \\
&= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c \\
&= \frac{1}{4} e^{2x} (2x - 1) + c.
\end{align*} \]

(c) \( \int x e^{-x} \, dx = x \left( -e^{-x} \right) - \int 1 \times (-e^{-x}) \, dx \)

\[ \begin{align*}
&= -xe^{-x} + \int e^{-x} \, dx \\
&= -xe^{-x} + (-e^{-x}) + c \\
&= -xe^{-x} - e^{-x} + c \\
&= -e^{-x}(x + 1) + c.
\end{align*} \]

When you use integration by parts, remember that it matters which expression in the integrand you take to be the ‘first expression’, and which you take to be the ‘second expression’. If you’re trying to use integration by parts, but you find that it leads to an integral that’s more complicated than the original integral, then try swapping the two expressions in the original integrand.