Refresh Areas and signed areas

Sometimes it’s useful to calculate the area of a flat shape with a curved boundary. For example, suppose that an architect is designing a large building shaped as shown in Figure 1(a). The curve of the roof is given by the graph of the function
\[ f(x) = 9 - \frac{1}{36}x^2 \quad (-18 \leq x \leq 18), \]
where \( x \) and \( f(x) \) are measured in metres. This graph is shown in Figure 1(b).

![Figure 1](image)

Figure 1  (a) The shape of a building (b) the graph that gives the curve of the roof

Suppose that the architect wants to calculate the area shaded in Figure 1(b), so that she can determine the amounts of materials needed to build the end wall of the building.

Here’s a way of calculating an approximate value for this area, which you can make as accurate as you wish. You start by dividing the interval \([-18, 18]\), which is the interval of \( x \)-values that corresponds to the curve, into a number of subintervals of equal width.

For example, in Figure 2 the interval has been divided into twelve subintervals. The right endpoint of the first subinterval is equal to the left endpoint of the next subinterval, and so on.

For each subinterval, you approximate the shape of the curve on that subinterval by a horizontal line segment whose height is the value of the function at the left endpoint of the subinterval, as shown in Figure 2. You calculate the areas of the rectangles between these line segments and the \( x \)-axis (simply by multiplying their heights by their widths in the usual way), and add up all these areas to give you an approximate value for the required area. The more subintervals you use, the better will be the approximation.

![Figure 2](image)

Figure 2  A collection of rectangles whose total area is approximately the area in Figure 1(b)
In the next example this method is used, with four subintervals, to find an approximate value for the area discussed above. Notice that, in both the example and in Figure 2 above, the rectangle corresponding to the first subinterval has zero height and therefore zero area – this happens because the value of the function at the left endpoint of this subinterval is zero.

Example 1 Calculating an approximate value for an area

Use the method described above, with four subintervals as shown below, to find an approximate value for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the x-axis.

Solution

We divide the interval $[-18, 18]$ into four subintervals of equal width. The whole interval has width 36, so each subinterval has width $36/4 = 9$.

The left endpoints of the four subintervals are

$-18, \quad -18 + 9, \quad -18 + 2 \times 9, \quad -18 + 3 \times 9,$

that is

$-18, \quad -9, \quad 0, \quad 9.$

So the heights of the rectangles are

$f(-18), \quad f(-9), \quad f(0), \quad f(9),$

that is,

$9 - \frac{1}{36} \times (-18)^2, \quad 9 - \frac{1}{36} \times (-9)^2, \quad 9 - \frac{1}{36} \times 0^2, \quad 9 - \frac{1}{36} \times 9^2,$

which evaluate to

$0, \quad \frac{27}{4}, \quad 9, \quad \frac{27}{4}.$

So we obtain the following approximate value for the area:

$(0 \times 9) + (\frac{27}{4} \times 9) + (9 \times 9) + (\frac{27}{4} \times 9) = \frac{405}{2} = 202.5.$

The calculation in Example 1 shows that a (rather crude) approximate value for the area of the cross-section of the roof discussed at the beginning of this subsection is $202.5 \text{ m}^2$. 
**Activity 1  Calculating an approximate value for an area**

Use the method described above, with six subintervals as shown below, to find an approximate value for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the $x$-axis.

![Graph of the function $f(x) = 9 - \frac{1}{36}x^2$]

**Solution**

If we divide the interval $[-18, 18]$ into six subintervals of equal width, then each subinterval has width $36/6 = 6$.

The left endpoints of the six subintervals are

- $-18$, $-18 + 6$, $-18 + 2 \times 6$,
- $-18 + 3 \times 6$, $-18 + 4 \times 6$, $-18 + 5 \times 6$

that is

- $-18$, $-12$, $-6$, $0$, $6$, $12$.

So the heights of the rectangles are:

- $f(-18) = 9 - \frac{1}{36} \times (-18)^2 = 0$
- $f(-12) = 9 - \frac{1}{36} \times (-12)^2 = 5$
- $f(-6) = 9 - \frac{1}{36} \times (-6)^2 = 8$
- $f(0) = 9 - \frac{1}{36} \times 0^2 = 9$
- $f(6) = 9 - \frac{1}{36} \times 6^2 = 8$
- $f(12) = 9 - \frac{1}{36} \times 12^2 = 5$.

So we obtain the following approximate value for the area:

$$(0 \times 6) + (5 \times 6) + (8 \times 6) + (9 \times 6) + (8 \times 6) + (5 \times 6) = 210.$$  

Table 1 shows the approximate values for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the $x$-axis that were found in Example 1 and in the solution to Activity 1. It also shows, to three decimal places, some further approximate values that were found in the same way, but using larger numbers of subintervals. The calculations were carried out using a computer.

**Table 1  Approximate values for the area in Figure 1(b)**

<table>
<thead>
<tr>
<th>Number of subintervals</th>
<th>4</th>
<th>6</th>
<th>12</th>
<th>50</th>
<th>100</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximation obtained</td>
<td>202.5</td>
<td>210</td>
<td>214.5</td>
<td>215.914</td>
<td>215.978</td>
<td>215.999</td>
</tr>
</tbody>
</table>
You can see that as the number of subintervals gets larger and larger, the
approximation obtained seems to be getting closer and closer to a
particular number. We say that this number is the limit of the
approximations as the number of subintervals tends to infinity. It’s the
exact value of the area between the graph of the function \( f(x) = 9 - \frac{1}{36}x^2 \)
and the x-axis. From Table 1, it looks as if the exact value of this area is
216, or perhaps a number very close to 216.

In general, if you have a continuous function \( f \) whose graph lies on or
above the x-axis throughout an interval \([a, b]\), then you can use the method
demonstrated above to find, as accurately as you want, the area between
the graph of \( f \) and the x-axis, from \( x = a \) to \( x = b \). Figure 3(a) illustrates
an area of this type, and Figure 3(b) illustrates the approximation to the
area that’s obtained by using 8 subintervals.

**Figure 3**  (a) The area between a graph and the x-axis, from \( x = a \) to
\( x = b \)  (b) A collection of rectangles whose total area is approximately
this area

Here’s a summary of the method that you’ve met for finding an
approximate value for an area of this type. You start by dividing the
interval \([a, b]\) into a number, say \( n \), of subintervals of equal width. The
width of each subinterval is then \((b - a)/n\). For each subinterval you
calculate the product

\[
 f \left( \text{left endpoint of subinterval} \right) \times \frac{b - a}{n},
\]

and you add up all these products. In general, the larger the number \( n \) of
subintervals, the closer your answer will be to the area between the graph
of \( f \) and the x-axis, from \( x = a \) to \( x = b \).

Now suppose that you have a continuous function \( f \) whose graph lies on or
below the x-axis throughout an interval \([a, b]\), and you want to calculate
the area between the graph of \( f \) and the x-axis, from \( x = a \) to \( x = b \), as
illustrated in Figure 4(a).

You can use the method above, with a small adjustment at the end, to
calculate an approximate value for an area like this. Consider what
happens when you apply the method to a graph like the one in
Figure 4(a). You start by dividing the interval \([a, b]\) into \( n \) subintervals of
equal width, then you calculate all the products of form (1) and add them
all up. For a graph like this, because the value of \( f \) at the left endpoint of
each subinterval is negative (or possibly zero), each product of form (1)
will also be negative (or zero). In fact, each product of form (1) will be the
negative of the area between the line segment that approximates the curve
and the x-axis, as illustrated in Figure 4(b). So when you add up all the
products of form (1), you’ll obtain an approximate value for the negative of
the area between the curve and the x-axis, from \( x = a \) to \( x = b \).
Figure 4  (a) The area between another graph and the $x$-axis, from $x = a$ to $x = b$ (b) A collection of rectangles whose total area is approximately this area

That’s not a problem, because you can simply remove the minus sign to obtain the approximate value for the area that you want. However, to help us deal with situations like this, it’s useful to make the following definitions. These definitions will be important throughout the unit.

Consider any region on a graph that lies either entirely above or entirely below the $x$-axis. The **signed area** of the region is its area with a plus or minus sign according to whether it lies above or below the $x$-axis, respectively. For example, in Figure 5 the two shaded regions above the $x$-axis have signed areas $+4$ and $+6$, respectively, which you can write simply as 4 and 6, and the shaded region below the $x$-axis has signed area $-3$.

Figure 5  Regions on a graph

If you have a collection of regions on a graph, where each region in the collection lies either entirely above or entirely below the $x$-axis, then the total signed area of the collection is the sum of the signed areas of the individual regions. For example, the total signed area of the collection of three regions in Figure 5 is $4 + (-3) + 6 = 7$.

The units for signed area on a graph are, as you’d expect, the same as the units for area on the graph. That is, they’re the units on the vertical axis times the units on the horizontal axis. If there are no specific units on the axes, then we don’t use any specific units for signed area.
**Example 2  Understanding signed areas**

The areas of some regions on a graph are marked below.

In each of parts (a)–(c), use these areas to find the signed area between the graph and the $x$-axis, from the first value of $x$ to the second value of $x$.

(a) From $x = -3$ to $x = -2$.  
(b) From $x = -2$ to $x = 1$.  
(c) From $x = -3$ to $x = 1$.

**Solution**

(a) The signed area from $x = -3$ to $x = -2$ is $-2.2$.  
(b) The signed area from $x = -2$ to $x = 1$ is $+4.8 = 4.8$.  
(c) The signed area from $x = -3$ to $x = 1$ is $-2.2 + 4.8 = 2.6$.

Of course, the signed area value found in Example 2(c) doesn’t correspond to any actual area on the graph, as it’s the sum of a positive signed area and a negative signed area.

**Activity 2  Understanding signed areas**

Consider again the graph in Example 2. In each of parts (a)–(d) below, use the given areas to find the signed area between the graph and the $x$-axis, from the first value of $x$ to the second value of $x$.

(a) From $x = 1$ to $x = 5$.  
(b) From $x = 5$ to $x = 7$.  
(c) From $x = 1$ to $x = 7$.  
(d) From $x = 1$ to $x = 9$.  

Positive signed area

Negative signed area
Solution

(a) The signed area from $x = 1$ to $x = 5$ is $-6.3$.
(b) The signed area from $x = 5$ to $x = 7$ is $1.3$.
(c) The signed area from $x = 1$ to $x = 7$ is $-6.3 + 1.3 = -5$.
(d) The signed area from $x = 1$ to $x = 9$ is $-6.3 + 1.3 - 5.9 = -10.9$.

With the definition of signed area that you’ve now seen, the method that you’ve met in this subsection can be described concisely as follows.

**Strategy:**
To find an approximate value for the signed area between the graph of a continuous function $f$ and the $x$-axis, from $x = a$ to $x = b$

Divide the interval between $a$ and $b$ into $n$ subintervals, each of width $(b - a)/n$. For each subinterval, calculate the product

$$f \left( \text{endpoint of subinterval nearest } a \right) \times \frac{b - a}{n},$$

and add up all these products.

In general, the larger the number $n$ of subintervals, the closer your answer will be to the required signed area.

Notice that the box above uses the phrase ‘endpoint of subinterval nearest $a$’, rather than ‘left endpoint of subinterval’, which means the same thing. This will be convenient later in this subsection. (Note that, in this phrase, it’s the endpoint of the subinterval, not the subinterval itself, that’s nearest $a$!)

Figures 6 and 7 illustrate the strategy above. If you add up the signed areas of all the rectangles in Figure 6(a), then you’ll obtain an approximate value for the signed area in Figure 6(b).

![Figure 6](image_url)

**Figure 6** (a) A collection of rectangles whose total signed area is approximately the signed area shown in (b)

Similarly, if you add up the signed areas of all the rectangles in Figure 7(a), then you’ll obtain an approximate value for the signed area in Figure 7(b).
Activity 3  Calculating an approximate value for a signed area

Use the method described in the box above, with six subintervals as shown on the left below, to find an approximate value for the signed area between the graph of the function \( f(x) = 3 - x^2 \) and the \( x \)-axis from \( x = -3 \) to \( x = 3 \), as shown on the right below.

Solution

If we divide the interval \([-3, 3]\) into six subintervals of equal width, then each subinterval has width \( 6/6 = 1 \).

The left endpoints of the six subintervals are 
\(-3, -2, -1, 0, 1, 2,\)

So the heights of the rectangles are:
\[
\begin{align*}
  f(-3) &= 3 - (-3)^2 = -6 \\
  f(-2) &= 3 - (-2)^2 = -1 \\
  f(-1) &= 3 - (-1)^2 = 2 \\
  f(0) &= 3 - 0^2 = 3 \\
  f(1) &= 3 - 1^2 = 2 \\
  f(2) &= 3 - 2^2 = -1.
\end{align*}
\]

So we obtain the following approximate value for the signed area:
\[
(-6 \times 1) + (-1 \times 1) + (2 \times 1) \\
+ (3 \times 1) + (2 \times 1) + (-1 \times 1) = -1.
\]