Refresh Antiderivatives and indefinite integrals

In integral calculus you usually start with a function $f$, and you want to find another function, say $F$, whose derivative is $f$. Such a function $F$ is called an antiderivative of the original function $f$. For example, an antiderivative of the function $f(x) = 2x$ is the function $F(x) = x^2$, because the derivative of $x^2$ is $2x$.

The process of finding an antiderivative of a function is called antidifferentiation, or, more commonly, integration. So integration is the reverse of differentiation.

The first thing to realise is that a function can have more than one antiderivative. To see this, try the following activity.

### Activity 1 Differentiating functions whose formulas are the same apart from a constant term

Differentiate the following functions.

(a) $F(x) = x^2$  
(b) $F(x) = x^2 + 3$  
(c) $F(x) = x^2 - \frac{5}{7}$

**Solution**

(a) $F(x) = x^2$, so $F'(x) = 2x$.

(b) $F(x) = x^2 + 3$, so $F'(x) = 2x$.

(c) $F(x) = x^2 - \frac{5}{7}$, so $F'(x) = 2x$.

You should have found that all three functions in Activity 1 have derivative $F'(x) = 2x$. In fact, you can see that any function of the form

$$F(x) = x^2 + c,$$

where $c$ is a constant, has derivative $F'(x) = 2x$. This is because the derivative of a constant term is zero. So each function of the form $F(x) = x^2 + c$ is an antiderivative of the function $f(x) = 2x$.

Another way to think about the fact that all the functions of the form $F(x) = x^2 + c$ have the same derivative is to remember that adding a constant to the formula of a function has the effect of translating its graph vertically. So the graphs of all the functions of the form $F(x) = x^2 + c$ are vertical translations of each other, as illustrated in Figure 1(b). Translating a graph vertically doesn’t change its gradient at each $x$-value, of course. In other words, it doesn’t change the derivative of the function that the graph represents. So each of the functions in Figure 1(b) has the same derivative, namely $f(x) = 2x$, whose graph is shown in Figure 1(a).

Another thing to appreciate about all the functions of the form $F(x) = x^2 + c$, where $c$ is a constant, is that these functions are the only antiderivatives of the function $f(x) = 2x$. That’s because any antiderivative of $f(x) = 2x$ has the same gradient at each $x$-value as the function $F(x) = x^2$, and the functions of the form $F(x) = x^2 + c$ are the only functions with this property.
So the formula $F(x) = x^2 + c$ describes the complete family of antiderivatives of the function $f(x) = 2x$. We call the general function $F(x) = x^2 + c$ the indefinite integral of the function $f(x) = 2x$. The word ‘indefinite’ refers to the fact that the constant $c$ can take any value.

In general, consider any function $f$ that has an antiderivative. You can see that you can add any constant that you like to the formula for the antiderivative, and you’ll get the formula for another antiderivative of $f$. Equivalently, you can translate the graph of the antiderivative vertically by any amount, and you’ll get the graph of another antiderivative of $f$.

These antiderivatives are the only antiderivatives of the function $f$, provided that $f$ is a continuous function.

So, if $f$ is a function that has an antiderivative, and $f$ is continuous, then the general function

$$F(x) = \text{formula for any particular antiderivative of } f + c,$$

where $c$ represents any constant, and whose domain is the same as the domain of $f$, describes the complete family of antiderivatives of $f$. We call this general function $F$ the indefinite integral of the function $f$.

The constant $c$ in an indefinite integral is called an arbitrary constant, or the constant of integration. It’s usually denoted by $c$, but you can use any letter. Like the word ‘indefinite’ in ‘indefinite integral’, the word ‘arbitrary’ in ‘arbitrary constant’ refers to the fact that the constant $c$ can take any value.

You don’t need to be concerned about what happens when the function $f$ isn’t continuous, because normally when we’re solving problems in integral calculus we work only with continuous functions. These are usually the only functions that we need for the sorts of calculations that we want to perform. However, if you’d like to know what the problem is with functions that aren’t continuous, then read the explanation at the end of this subsection, when you reach it.

Before you go on, it’s important to make sure that you fully understand the difference between an antiderivative and an indefinite integral of a function. The definitions that you’ve seen are summarised below.
Antiderivatives and indefinite integrals

Suppose that \( f \) is a function.

An antiderivative of \( f \) is any specific function whose derivative is \( f \).

If \( f \) has an antiderivative, and \( f \) is continuous, then the indefinite integral of \( f \) is the general function obtained by adding an arbitrary constant \( c \) to the formula for an antiderivative of \( f \). It describes the complete family of antiderivatives of \( f \).

Example 1 Understanding antiderivatives and indefinite integrals

(a) Show that the function \( F(x) = x^3 \) is an antiderivative of the function \( f(x) = 3x^2 \).

(b) What is the indefinite integral of the function \( f(x) = 3x^2 \)?

Solution

(a) Since
\[
\frac{d}{dx} (x^3) = 3x^2,
\]
it follows that \( F(x) = x^3 \) is an antiderivative of \( f(x) = 3x^2 \).

(b) The indefinite integral of \( f(x) = 3x^2 \) is \( F(x) = x^3 + c \).

Activity 2 Understanding antiderivatives and indefinite integrals

(a) Show that the function \( F(x) = \frac{1}{2} \sin(2x) \) is an antiderivative of the function \( f(x) = \cos(2x) \).

(b) What is the indefinite integral of the function \( f(x) = \cos(2x) \)?

(c) Write down an antiderivative of the function \( f(x) = \cos(2x) \) other than the antiderivative in part (a).

Solution

(a) Since
\[
\frac{d}{dx} \left( \frac{1}{2} \sin(2x) \right) = \frac{1}{2} \times 2 \cos(2x) = \cos(2x),
\]
it follows that \( F(x) = \frac{1}{2} \sin(2x) \) is an antiderivative of the function \( f(x) = \cos(2x) \).

(b) The indefinite integral of \( f(x) = \cos(2x) \) is \( F(x) = \frac{1}{2} \sin(2x) + c \).

(c) An antiderivative of \( f(x) = \cos(2x) \), other than the antiderivative in part (a), is \( F(x) = \frac{1}{2} \sin(2x) + 1 \).

(You probably chose a different antiderivative. Any function obtained by setting the constant \( c \) in the indefinite integral \( F(x) = \frac{1}{2} \sin(2x) + c \) equal to a particular number will do.)
Even though the idea of an indefinite integral really applies only to continuous functions, it’s convenient in practice to state indefinite integrals of other functions. For example, consider again the function $f(x) = 1/x^2$, which isn’t continuous. Its graph is repeated in Figure 2. An antiderivative of this function is the function $F(x) = -1/x$, as you can check by differentiating this function $F$. Even though $f$ isn’t continuous, we do still say

the function $f(x) = 1/x^2$ has indefinite integral $F(x) = -1/x + c$.

This is shorthand for

any continuous function $f$ with rule $f(x) = 1/x^2$ has indefinite integral $F$ with rule $F(x) = -1/x + c$.

For example, the continuous function $f(x) = 1/x^2 \ (x > 0)$, whose graph is shown in Figure 3, has indefinite integral $F(x) = -1/x + c \ (x > 0)$.

We use this shorthand for any function that isn’t continuous.