Converting from Cartesian to polar (or exponential) form

Given a complex number \( z = a + bi \), proceed as follows.

1. Find the modulus \( r \), using \( r = |z| = \sqrt{a^2 + b^2} \).

2. Find the principal argument \( \theta \):
   - (a) Mark \( z \) in roughly the right position in the complex plane, draw the line from 0 to \( z \), and mark and label the principal argument \( \theta \).
   - (b) If \( z \) lies on one of the axes, then use your diagram to find the principal argument \( \theta \). Otherwise carry out steps (c) to (e).
   - (c) Label the acute angle between the real axis and the line from 0 to \( z \) as \( \phi \).
   - (d) Draw a line from \( z \) perpendicular to the real axis to form a right-angled triangle, and mark the lengths of the horizontal and vertical sides.
   - (e) Use the triangle to work out the angle \( \phi \) and hence the principal argument \( \theta \).

3. Write \( z \) in polar form \( z = r(\cos \theta + i \sin \theta) \) or exponential form \( z = re^{i\theta} \), as required.

**Example 1  Converting from Cartesian form to polar form**

Write the complex number \(-2 - 2\sqrt{3}i\) in polar form.

**Solution**

First find the modulus.

The modulus is

\[
r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4.
\]

To find the principal argument, sketch \( z = -2 - 2\sqrt{3}i \) in the complex plane. The important thing is to get it in the correct quadrant. Label the principal argument \( \theta \), and label by \( \phi \) the acute angle between the real axis and the line from the origin to \( z \).
Draw a line from $z$ to the real axis that is perpendicular to the real axis, to form a right-angled triangle. Mark the lengths of the horizontal and vertical sides of the triangle.

Use the triangle to work out the acute angle $\phi$, and hence work out the principal argument $\theta$.

From the diagram,
\[ \tan \phi = \frac{2\sqrt{3}}{2} = \sqrt{3}. \]

Therefore $\phi = \pi/3$. So
\[ \theta = - (\pi - \phi) = - \left( \pi - \frac{\pi}{3} \right) = - \frac{2\pi}{3}. \]

Write $z$ in polar form.

Hence
\[ z = 4 \left( \cos \left( - \frac{2\pi}{3} \right) + i \sin \left( - \frac{2\pi}{3} \right) \right). \]