**Refresh Solving quadratic equations**

In general there are two values of $x$ which satisfy the general quadratic equation $ax^2 + bx + c = 0$, represented by where the graph of the parabola $y = ax^2 + bx + c$ crosses the $x$-axis (i.e. when $y = 0$). For example, it is easy to see that the equation $x^2 - 4 = 0$ has solutions $x = -2$ and $x = 2$. These values are the $x$-intercepts of the graph of the parabola $y = x^2 - 4$, shown in Figure 1.

![Figure 1](image)

**Figure 1**

One way to solve a quadratic equation with real roots is to rearrange it in the form $ax^2 + bx + c = 0$ and then factorise the expression on the left-hand side. This will produce the product of two brackets, say $(px - q)(rx - s)$. If the value of either factor is 0, then the product will be 0. So putting each factor equal to 0 in turn, will produce two values of $x$.  

<table>
<thead>
<tr>
<th>If a quadratic equation can be written in the form $(px - q)(rx - s) = 0$ then either $px - q = 0$ or $rx - s = 0$</th>
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<tbody>
<tr>
<td>If $px - q = 0$, then $px = q$; so $x = q/p$.</td>
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<tr>
<td>If $rx - s = 0$, then $rx = s$; so $x = s/r$. Hence the solutions to the quadratic equation $(px - q)(rx - s) = 0$ are $x = q/p$ and $x = s/r$.</td>
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**Example 1**

Use factorisation to solve each of the following quadratic equations.

(a) $x^2 - 5x + 6 = 0$

(b) $x^2 + 6x + 9 = 0$

(c) $2y^2 - 9y - 5 = 0$
Solution

(a) \( x^2 - 5x + 6 = 0 \) factorises to give \((x - 2)(x - 3) = 0\) and so has the solutions \(x = 2\) and \(x = 3\).

(b) \( x^2 + 6x + 9 = 0 \) factorises to give \((x + 3)(x + 3) = 0\) and, because the two brackets are identical, \(x = -3\) is the only solution.

(c) \( 2y^2 - 9y - 5 = 0 \) factorises to give \((2y + 1)(y - 5) = 0\) and so has the solutions \(y = -\frac{1}{2}\) and \(y = 5\).

Factorising is often the simplest way to solve a quadratic equation, but when this proves difficult there is a formula (the quadratic formula) to use. For the general equation \(ax^2 + bx + c = 0\), the formula for the two solutions is

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Note: It is the \(\pm\) which produces the two solutions and \(\sqrt{\text{ }}\) indicates the positive square root. If \(b^2 - 4ac\) is negative, there are no real solutions because no negative number has a real square root.

Example 2

Use the quadratic formula to solve \(6x^2 + 7x - 5 = 0\).

Solution

Comparing \(6x^2 + 7x - 5 = 0\) to the general form \(ax^2 + bx + c = 0\), gives \(a = 6\), \(b = 7\), \(c = -5\).

Then substitute these values into the quadratic formula:

\[
x = \frac{-7 \pm \sqrt{7^2 - 4 \times 6 \times (-5)}}{2 \times 6}
\]

\[
= \frac{-7 \pm \sqrt{49 + 120}}{12}
\]

\[
= \frac{-7 \pm \sqrt{169}}{12}
\]

\[
= \frac{-7 \pm 13}{12}
\]

So, taking the + sign, we have

\[
x = \frac{-7 + 13}{12} = \frac{6}{12} = \frac{1}{2};
\]

and taking the - sign, we have

\[
x = \frac{-7 - 13}{12} = \frac{-20}{12} = \frac{-5}{3}
\]

Thus the two solutions are \(x = \frac{1}{2}\) and \(x = -\frac{5}{3}\).

(So the original equation \(6x^2 + 7x - 5 = 0\) could have been factorised to \(6(x - \frac{1}{2})(x + \frac{5}{3}) = 0\), or written another way, to \((2x - 1)(3x + 5) = 0\).)

Sometimes an equation for a variable may involve algebraic fractions. In such a case, the first step is to simplify the algebraic fractions before solving the equation that results.
Example 3

Solve the equation \( \frac{1}{x+1} - \frac{1}{2x+1} = \frac{1}{6} \).

Solution

First we bring all the terms onto the left-hand side, rewriting the equation as \( \frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{6} = 0 \).

This time the common denominator is \( 6(x+1)(2x+1) \).

\[
\frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{6} = \frac{6(2x+1)}{6(x+1)(2x+1)} - \frac{6(x+1)}{6(x+1)(2x+1)} - \frac{(2x + 1)(x + 1)}{6(x+1)(2x+1)}
\]

\[
= \frac{6(2x + 1) - 6(x + 1) - (2x + 1)(x + 1)}{6(x+1)(2x+1)}
\]

\[
= \frac{12x + 6 - 6x - 6 - 2x^2 - 3x - 1}{6(x+1)(2x+1)}
\]

\[
= \frac{-2x^2 + 3x - 1}{6(x+1)(2x+1)}
\]

\[
= \frac{-2x^2 - 3x + 1}{6(x+1)(2x+1)}
\]

\[
= \frac{(2x - 1)(x - 1)}{6(x+1)(2x+1)}
\]

Hence \( \frac{1}{x+1} - \frac{1}{2x+1} = \frac{1}{6} \) is equivalent to \( \frac{(2x - 1)(x - 1)}{6(x+1)(2x+1)} = 0 \).

That is, \((2x - 1)(x - 1) = 0\), so \(x = 1\) or \(x = \frac{1}{2}\).