Inequalities and intervals

Inequalities are more easily understood and manipulated when you are fluent with the symbols, and reference is made to the number line (either actually or mentally).

The following example explains how inequalities may be represented using intervals of the number line.

Example 1

(a) The inequality \( \frac{1}{2} < x \) is equivalent to \( x > \frac{1}{2} \), which is read as ‘\( x \) is greater than a half’.

This means that \( x \) can take any value strictly greater than \( \frac{1}{2} \), that is, any point on the number line to the right of \( \frac{1}{2} \). This range of values of \( x \) is shown as an interval on the number line in Figure 1. The open dot shows that \( \frac{1}{2} \) is excluded.

![Figure 1](image1.png)

(b) The inequality \( -4 < x \leq 23 \) can be read as ‘\( x \) is greater than minus 4 and less than or equal to 23’.

This means that \( x \) can take any of the values between \( -4 \) and 23, including 23 but excluding \( -4 \). The set of values of \( x \) which satisfies both inequalities can be represented as an interval on the number line shown in Figure 2, where the solid dot shows that 23 is included, and the open one that \( -4 \) is excluded.

![Figure 2](image2.png)

(c) The inequality illustrated in Figure 3 is ‘\( x \) is either less than or equal to \( -2 \), or greater than 50’ and is expressed symbolically as \( x \leq -2 \) or \( x > 50 \).

![Figure 3](image3.png)

In case (c) above the description refers to two separate parts of the number line, and so two separate inequalities are needed. The first states that \( x \) can take the value \( -2 \) or any value to the left of \( -2 \), the second that \( x \) can take any value to the right of 50.

The fact that there is no point on the line which marks a boundary below \( -2 \) or above 50 could have been indicated by using the symbol \( \infty \), for infinity, as follows:

\[-\infty < x \leq -2 \quad \text{or} \quad 50 < x < \infty.\]

Note: \( \leq \) and \( \geq \) are never used with \( \infty \).
**Rearranging inequalities**

Inequalities can be rearranged and manipulated in similar ways to equations provided care is taken with the direction of the inequality signs. It helps to consider an algebraic and numerical example of each rule.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Example</th>
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<tbody>
<tr>
<td><strong>1</strong> Interchanging the sides of an inequality, reverses the direction of the inequality</td>
<td>If ( a &lt; b ), then ( b &gt; a )</td>
</tr>
<tr>
<td><strong>2</strong> Adding the same number to, or subtracting the same number from both sides, preserves the direction of the inequality</td>
<td>If ( a &gt; b ), then ( a - c &gt; b - c ) and ( a + c &gt; b + c )</td>
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<tr>
<td><strong>3</strong> Multiplying or dividing both sides by the same positive number preserves the direction of the inequality</td>
<td>If ( a &gt; b ) and ( c &gt; 0 ), then ( ac &gt; bc ) and ( a/c &gt; b/c )</td>
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<tr>
<td><strong>4</strong> Multiplying or dividing both sides by the same negative number reverses the direction of the inequality</td>
<td>If ( a &lt; b ) and ( c &lt; 0 ), then ( ac &gt; bc ) and ( a/c &gt; b/c )</td>
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</tbody>
</table>

Using these rules it is possible to rearrange and simplify algebraic inequalities.

**Example 2**

Rewrite each of the following inequalities in the form: \( x \) ‘inequality symbol’ number, e.g. \( x \geq 4 \). This process is called ‘solving the inequality’.

(a) \( 3 < x + 1 \)  \hspace{1cm} (b) \( x - 5 > -7 \)  \hspace{1cm} (c) \( 15 > 3x \)

(d) \( -4x < -12 \)  \hspace{1cm} (e) \( 2(7 - x) \geq 22 \)
Solution

(a) \[ 3 < x + 1 \]
\[ x + 1 > 3 \quad \text{(to get } x \text{ on the L.H.S.)} \quad \text{(Rule 1)} \]
\[ x > 3 - 1 \quad \text{(to get } x \text{ on its own)} \quad \text{(Rule 2)} \]
\[ x > 2 \quad \text{L.H.S. denotes left-hand side.} \]

(b) \[ x - 5 > -7 \]
\[ x > -7 + 5 \quad \text{(adding 5)} \quad \text{(Rule 2)} \]
\[ x > -2 \]

(c) \[ 15 > 3x \]
\[ 3x < 15 \quad \text{(to get } x \text{ on the L.H.S.)} \quad \text{(Rule 1)} \]
\[ x < 5 \quad \text{(dividing by +3)} \quad \text{(Rule 3)} \]

(d) \[ -4x < -12 \]
\[ x > 3 \quad \text{(dividing by } -4) \quad \text{(Rule 4)} \]

(e) \[ 2(7 - x) \geq 22 \]
\[ 14 - 2x \geq 22 \quad \text{(expanding)} \]
\[ -2x \geq 22 - 14 \quad \text{(subtracting 14)} \quad \text{(Rule 2)} \]
\[ -2x \geq 8 \]
\[ x \leq -4 \quad \text{(dividing by } -2) \quad \text{(Rule 4)} \]