Contents

Introduction

1 Second-order differential equations 7
  1.1 Notes on set book: sections 1.1 to 1.6 ................................. 7
  1.2 Solutions for exercises from chapter 1 .............................. 20

2 Plane autonomous systems and linearisation 39
  2.1 Introduction .......................................................... 39
  2.2 Notes on set book: sections 2.1 to 2.5 .............................. 39
  2.3 An alternative proof of existence and uniqueness .................. 41
  2.4 Notes on set book: sections 2.2 to 2.5 .............................. 41
  2.5 Classification of the fixed points of a nonlinear system .......... 42
    2.5.1 Summary of Classifications .................................. 46
  2.6 Notes on set book: sections 2.7 and 2.8 ............................ 48
  2.7 Solutions for exercises from chapter 2 ............................ 51

3 Geometrical aspects 69
  3.1 Introduction .......................................................... 69
  3.2 Notes on set book .................................................... 69
  3.3 Solutions for exercises from chapter 3 ............................ 74

4 Periodic solutions; averaging methods 87
  4.1 Introduction .......................................................... 87
  4.2 Notes on set book: sections 4.1 to 4.3 .............................. 89
  4.3 An Averaging Theorem .............................................. 91
  4.4 Notes on set book: section 4.4 .................................... 94
  4.5 Solutions for exercises from chapter 4 ............................ 97

5 Perturbation methods 109
  5.1 Introduction .......................................................... 109
  5.2 Perturbations of Algebraic equations ............................... 110
    5.2.1 Regular perturbations ........................................ 110
    5.2.2 Singular perturbations ....................................... 112
  5.3 Perturbation Theory for some differential equations ............. 115
  5.4 Notes on the set book .............................................. 119
  5.5 Solutions of exercises from chapter 5 ............................ 130
6 Singular perturbations ............................................. 149
   6.1 Introduction .................................................................. 149
   6.2 Notes on set book: sections 6.1, 6.2 and 6.4 ....................... 151
   6.3 A Multiple Scales method for boundary
        value problems .................................................................. 159
   6.4 Solutions for exercises from chapter 6 ............................ 163

7 Forced Oscillations ...................................................... 181
   7.1 Introduction .................................................................. 181
   7.2 Notes on set book .......................................................... 181
   7.3 Solutions for exercises from chapter 7 ............................ 187

8 Stability ........................................................................ 199
   8.1 Introduction .................................................................. 199
   8.2 Notes on set book: sections 8.1 to 8.4 .............................. 199
   8.3 Linear systems: a summary ............................................ 200
   8.4 Notes on set book: sections 8.5 to 8.10 ............................ 203
   8.5 Solutions for exercises from chapter 8 ............................ 206

9 Determination of stability ............................................... 215
   9.1 Introduction .................................................................. 215
   9.2 Notes on set book .......................................................... 218
   9.3 Solutions for exercises from chapter 9 ............................ 223

References ........................................................................ 227
Introduction

Nonlinear differential equations arise in almost all branches of science. Studies as diverse as population dynamics, chemical reactions, the effects of winds on large structures, particle acceleration, electrical circuits, planetary and galactic motion and the internal motion of molecules all give rise to nonlinear differential equations: this, alone, is sufficient excuse for their study. Moreover, the solutions of these equations display such a rich variety of phenomena that no further motivation for their study is usually necessary.

Until the widespread use of computers the study of such equations was limited: some important existence theorems were known, a few of which are given in the later parts of this course, and there was a variety of methods designed to find approximations to particular types of equations. Usually these methods are based on some form of perturbation theory, so did not reveal the different types of behaviour manifest in nonlinear systems. The work of Poincaré (1899), however, was different and it was here that chaotic motion was first discovered, though this name was not used until the last third of the twentieth century. In recent years the use of computers and modern mathematical techniques have added very considerably to our knowledge. These studies have changed our understanding of the solutions of nonlinear differential equations. Previously, most knowledge of such systems came, of necessity, from approximate solutions which could be written in a closed form; this severely restricted the class of solutions accessible so coloured our view of how nonlinear systems behave. Modern computers, and especially small microcomputers with good graphical displays, have removed this barrier and have allowed numerical experiments to be performed on otherwise intractable systems, so in the last few decades of the twentieth century scientists and mathematicians became aware that most nonlinear systems behave in a very complicated manner. This new knowledge has had important effects on many disciplines, including atomic physics, studies of the Solar System, studies of galactic motion, weather forecasting, population models and the study of epidemic dynamics. A very readable, semi-technical, discussion of the history of dynamics, from the time of Laplace to the present day, is provided by Diacu and Holmes (1996) This book charts the history of technical developments and provides some interesting sociological background and any student interested in dynamics should read it. Another enjoyable, but journalistic, account of recent developments in our knowledge of nonlinear systems, which emphasises the contributions from many disciplines, is provided by Gleick (1987): this non-technical account is good scientific journalism, even though it is heavily biased towards North American science; the details of these and other books are given on page 227.

Computers can also help in the study of nonlinear systems by assisting with the algebra. Many of the methods used to find approximate solutions involve quite complicated algebraic manipulations: in many circumstances a computer, with the relevant software, can make this
This use of a computer is not taught in M821, though some of the methods are; the automation of these methods using Maple is dealt with in the set book for M833 (Richards, 2002) though this part of the text is not included in M833.

This course, being an introduction, reflects the earlier developments of the subject and much of it is concerned with obtaining approximate solutions and trying to understand how solutions behave when the nonlinear terms have only a relatively small effect. Surprisingly there are few accessible, alternative books that teach these methods and fewer that present modern work in a form suitable for an introductory course. The book *Ordinary Differential Equations* by Arrowsmith and Place (1982), however, is an excellent modern introduction to the earlier parts of this course. Other good books which teach the theory and approximations methods dealt with in M821 are listed below, with the most relevant given first.

- *Ordinary Differential Equations* by D K Arrowsmith and C M Place.
- *An Introduction to Dynamical Systems* by D K Arrowsmith and C M Place.
- *Perturbation Methods* by E J Hinch.
- *Introduction to Perturbation Methods* by M H Holmes.
- *Nonlinear systems* by P G Drazin.
- *Nonlinear Differential Equations and Dynamical Systems* by F Verhulst.
- *Stability, instability and chaos* by P Glendinning.
- *Chaos in Dynamical Systems* by E Ott.
- *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* by J Guckenheimer and P Holmes.
- *Regular and Chaotic Motion* by A J Lichtenberg and M A Lieberman.
- *Perturbation Methods* by A H Nayfeh.
- *Nonlinear Oscillations* by A H Nayfeh and D T Mook.

You may find the book by Devaney (1986) worth studying, despite the many errors, if you are interested in the mathematical aspects of the subject: at the other extreme the book by Thompson and Stewart (1986) provides an interesting overview of some modern ideas described with as little mathematics as possible. In between these two is the book by Beltrami (1987) which introduces some methods of nonlinear analysis through a variety of applications. Finally, there is an interesting introductory account of some aspects of chaos by Baker and Gollub (1990). The details of these and other books can be found in the reference list at the end of the notes.

This course is based on the third edition of *Nonlinear Ordinary Differential Equations* by D.W. Jordan and P. Smith. You are guided through this book by these Course Notes, which give the order in which to read the book, but also supplement the book with other material which either expands on important topics or provides an alternative derivation of results. Throughout the Course Notes many exercises are set and it is essential that you do

---

a substantial number of these exercises because it is only by doing such problems that you can understand and remember the ideas and methods taught. Solutions to many exercises have been provided, but you should not feel it necessary to attempt all of these: for this reason, many are given as revision exercises, with the idea that they be used towards the end of the year as revision for the exam.

You should start at the first page of the notes and work your way sequentially through the notes and the book. Besides the exercises from Jordan and Smith we have also included additional exercises where necessary. In the solution sections we treat these extra exercises first, in numerical order, followed by the exercises set from Jordan and Smith, also in numerical order. In other words, the order in which the solutions appear is not necessarily the same as the order in which they are set.

The different chapters are of varying degrees of difficulty; you should not be discouraged to find Chapters 4–7 harder than the previous chapters as in our view these are the most difficult parts; subsequent chapters are easier. In order to pace you through the course we expect you to complete four Tutor Marked Assignments (TMAs) at given times during the year. The dates at which these are due vary slightly from year to year, but the approximate cut-off dates are listed in the table below: we cannot accept TMAs after the cut-off date, except in extraordinary circumstances. The TMAs are compulsory and will test you on various parts of the book as shown below.

<table>
<thead>
<tr>
<th>TMA</th>
<th>Chapters</th>
<th>Approximate cut-off dates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 2</td>
<td>3rd week of March</td>
</tr>
<tr>
<td>2</td>
<td>3, 4</td>
<td>mid May</td>
</tr>
<tr>
<td>3</td>
<td>5, 6</td>
<td>2nd week of July</td>
</tr>
<tr>
<td>4</td>
<td>7, 8, 9</td>
<td>2nd week of September</td>
</tr>
</tbody>
</table>

Although assessing more material the final TMA will contain roughly the same amount of work as the first three. It is important that you attempt all four TMAs because they are designed to test and consolidate all the important ideas in the course. However the final grade is computed in such a manner that if, for any reason, you do not submit one TMA you will not be penalised. This is achieved by using a substitution rule: if $T_k, k = 1, 2, \ldots, 4$ are the TMA scores and $E$ the exam mark, we first compute the average $A = \frac{1}{2}(E + \frac{1}{4} \sum_{k=1}^{4} T_k)$ and then replace the smallest TMA grade by $A$ to give a new average TMA score, which is used, in conjunction with the exam mark, to determine the final grade. It is not clear why this iterative process is not continued until a converged result is obtained.

In these course notes the book is referred to as JS and all references to the equations and figures in these notes are prefixed by the letter N.

Doubtless there are errors in the notes, and we have probably not found all the errors in the book. I should be grateful if you would inform either your Tutor or me of any errors you find and of any improvement that you can suggest.

D. Richards

(October 2005)
Chapter 1

Second-order differential equations

1.1 Notes on set book: sections 1.1 to 1.6

JS Sections 1.1 to 1.6 only are used in M821.
Read the introductory text and section 1.1, pages 1–5.

Comments on text:

(i) Page 1: The first sentence of the introduction is wrong. All the equations considered in JS have exact solutions: the problem is that they cannot usually be expressed in terms of ‘standard functions’ or as finite combinations of these functions.

Even if such solutions exist they are not always useful. For instance, suppose we require the function $x(t)$ defined by the first-order differential equation

$$
\frac{dx}{dt} = x^2(1 - x), \quad x(0) = A, \quad 0 < A < 1.
$$

Separating variables and integrating gives

$$
t = \int_A^x \frac{1}{x^2(1 - x)} \, dx = \int_A^x \frac{1}{1 - x} + \frac{1}{x} + \frac{1}{x^2} \, dx
$$

$$
= -\ln \left( \frac{1 - x}{x} \cdot \frac{A}{1 - A} \right) - \frac{1}{x} + \frac{1}{A}
$$

which may be rearranged as follows

$$
\frac{1}{A} - t = \frac{1}{x} + \ln \left( \frac{1 - x}{x} \right) - \ln \left( \frac{1 - A}{A} \right).
$$

From this expression it is not easy to see how $x(t)$ behaves. Yet from the original equation we see that if $0 < x(0) < 1$, $\dot{x} > 0$, so $x(t)$ is increasing, but the rate of increase slows as $x(t) \to 1$. Near $x = 1$ we set $x = 1 - u$ and expand to $O(u)$, to
obtain \( \dot{u} = -u \) to give \( x(t) = 1 - \alpha e^{-t} \) for some \( \alpha \). In this case a simple qualitative analysis shows that if \( 0 < x(0) < 1 \) then \( x(t) \to 1 \) as \( t \to \infty \): sometimes this type of qualitative description is more useful than the exact solution.

(ii) Page 1, equation (1.1): A significant feature of this equation is that it is independent of the mass, \( m \), of \( P \).

(iii) Page 1: The term **energy transformation** is not common usage.

(iv) Page 2: In the equation

\[
\frac{1}{2}ma^2\dot{x}^2 - mga\cos x = E,
\]

after equation 1.3, the first term, \( \frac{1}{2}ma^2\dot{x}^2 \) is the kinetic energy and the second term, \( -mga\cos x \) is the potential energy. These quantities are often denoted by \( T \) and \( V \) respectively, so the total energy \( E \) is given by \( E = T + V \).

The kinetic energy of a particle of mass \( m \) is \( T = \frac{1}{2}mv^2 \) where \( v \) is its speed.

The potential energy of the particle near the surface of the Earth is \( V = mgh \), \( h \) being the height above a specified point: the potential energy is defined only to within an additive constant, because only differences in potential energy are observable. For the pendulum the reference point is taken to be the lowest point, \( x = 0 \), so

\[
V = mga(1 - \cos x) = -mga\cos x + \text{constant}.
\]

(v) Page 2, after 1.4: The elementary functions mentioned are the Jacobi elliptic functions. These are defined in the glossary, but are not needed for this course.

(vi) Page 3, figure 1.2: This figure shows some representative phase curves of the vertical pendulum. In this diagram three types of motion are seen: because these occur frequently it is helpful to name them. The equation of motion is \( 2\pi \)-periodic in \( x \), consequently two types of periodic motion exist.

The first type of motion is represented by the closed phase curves surrounding the origin. These phase curves describe the normal pendulum oscillations about the downward vertical, in which the angle \( x \) oscillates about \( x = 0 \): on these phase curves \( y = \dot{x} \) changes sign. Such motion is named **librational motion**.

The second type of motion is represented by the phase curves that do not cross the \( x \)-axis, so are always either above or below it: on these phase curves \( y = \dot{x} \) does not change sign and they represent the motion of the pendulum that has sufficient energy to rotate about the pivot. This is named **rotational motion**.

Librational and rotational motion are distinct. The regions of phase space containing each type are examples of **invariant regions**: this means that an orbit with initial conditions in an invariant region remains in that region for all time.

Librational and rotational phase curves are topologically distinct in the sense that a phase curve of one type cannot be deformed into a phase curve of the other type without breaking it.
The boundary between invariant regions is clearly important and is named the \textit{separatrix}. This phase curve represents the motion of the pendulum that approaches the upward vertical in either the infinite past or future. In \textbf{JS} figure 1.2 the curve joining $A$ to $B$ is the separatrix and it has two branches corresponding to the motion that reaches the upward vertical in the future, the upper branch, or in the past, the lower branch.

These three types of motion are important features of many dynamical systems, not just the vertical pendulum. This is because for many systems the phase curves in local regions of phase space look exactly like those in figure 1.2. The separatrix is of special significance because it is very susceptible to perturbations, which can destroy it to cause chaos.

(vii) Page 4, line 6: $x = -\pi$, $x = 0$ should be $x = -\pi$, $\dot{x} = 0$.

\textbf{Read} section 1.2, pages 5-14.

Comments on text:

(i) Page 5: The initial condition $(x(t_0), \dot{x}(t_0))$ normally determine past and future motion. The magnitude of the interval $|t - t_0|$ for which the solutions exists may be finite or infinite. Thus the simple linear, one-dimensional system

$$\dot{x} = x, \quad x(t_0) = x_0,$$

has the solution $x(t) = x_0 e^{t-t_0}$ which exists for all finite intervals. But for the nonlinear system

$$\dot{x} = x^2, \quad x(t_0) = x_0,$$

the solution $x(t) = \frac{x_0}{1 - (t - t_0)x_0}$ is undefined at $t = t_\infty$ where $t_\infty = t_0 + 1/x_0$ depends upon the initial condition. For $t > t_\infty$ the function $x(t)$ is bounded but is not a solution of the original equation.

(ii) Page 5: \textbf{Non-autonomous} and \textbf{forced} equations are not the same: not all non-autonomous equations are forced and often have different physical origins. The first example given,

$$\ddot{x} + k \dot{x} + \omega_0^2 x = F \cos \omega t$$

is an example of a forced equation because it represents the damped linear oscillator, $\ddot{x} + k \dot{x} + \omega_0^2 x$, being driven by a periodic force $F \cos \omega t$ which is external to the system and is independent of the dynamical variables of the system.

The second example

$$\ddot{x} + (\alpha + \beta \cos t)x = 0$$

does not normally describe a forced system. Typically this type of equation represents a linear oscillator $\ddot{x} + \omega(t)x = 0$, with periodically varying parameter, for instance a pendulum with a point of support that is moving or with changing length. This type of system is usually described as \textbf{parametrically} driven, and occurs when the external force acts to periodically change the system parameters.

The distinction between these two types of systems is important because the behaviour of their solutions is different.
A set of non-autonomous equations usually represents a dynamical system acted upon by some external agency that is not affected by the motion of the system. For instance, the motion of a vertical pendulum hanging in a moving train is affected by the movement of the carriage but the effect of the pendulum on the train is normally negligible; the effect of the motion of the train on the pendulum support is accounted for by time-dependent terms in the equations of motion.

(iii) Page 7: The term *singular point* is discussed on page 40 of the course notes.

(iv) Page 7: The text between equations 1.11 and 1.13 is confusing and illogical.

- If a system is at $A$ and $B$ at times $t_A$ and $t_B$ respectively then by definition the time taken to move from $A$ to $B$ is

$$T_{AB} = t_B - t_A$$

regardless of whether the system is autonomous or not.

- If the system is autonomous, so the velocity $(y, f(x, y))$ depends only on the phase point, but not the time, then if $(\xi(t), \eta(t))$ is a solution of JS equation 1.7 so is $(u(t), v(t)) = (\xi(t + \tau), \eta(t + \tau))$ for any constant $\tau$. This follows because

$$\frac{du}{dt} = \frac{d}{dt} \xi(t + \tau) = \frac{d\xi(t + \tau)}{d(t + \tau)} = \eta(t + \tau) = v(t)$$

$$\frac{dv}{dt} = \frac{d}{dt} \eta(t + \tau) = \frac{d\eta(t + \tau)}{d(t + \tau)} = f(\xi(t + \tau), \eta(t + \tau)) = f(u, v).$$

- If the solution $(\xi(t), \eta(t))$ passes through $A$ and $B$ at the times $t_A$ and $t_B$ respectively, then the solution $(u(t), v(t))$ passes through these points at times $t_A - \tau$ and $t_B - \tau$. Hence the time taken to move from $A$ to $B$ is the same for all solutions and is independent of the starting time $t_A$. This is not true of non-autonomous systems.

(v) Page 8, equation 1.13: The time $T_{AB}$ defined in equation 1.13 need not exist. It does exist if the orbit between $A$ and $B$ is finite and if $y(x) \neq 0$ on this orbit. If $y(x) = 0$ at some point then the function $1/y(x)$ needs to be integrable. Consider two different orbits of the vertical pendulum, figure 1.2.

The time along the separatrix from $x = 0$ to $\pi$ is infinite, because near $A$, $y = \pm \omega(x - \pi) + O(x - \pi)^2$ and, for instance,

$$T_{0A} = \lim_{\chi \to \pi} \int_0^\chi \frac{dx}{y(x)} \to \infty.$$  

On a librational phase curve, however, the behaviour of the integral as $y \to 0$ is quite different. If the energy equation is $\frac{1}{2}y^2 + V(x) = E$ and the phase curve crosses the $x$-axis at $x = a$ and $x = b > a$ then $y(a) = y(b) = 0$ and for $a \leq x \leq b$ the function $y(x)$ has the form $y(x) = g(x)\sqrt{(x - a)(b - x)}$ provided $V'(a) \neq 0$. 
and \( V'(b) \neq 0 \); if \( y(x) \) has no other zeros in this interval \( g(x) \) has no zeros for \( a \leq x \leq b \). The time of passage between \( a \) and \( b \) is

\[
T = \int_a^b \frac{dx}{g(x)\sqrt{(x-a)(b-x)}},
\]

This integral is finite, as may be seen by changing variables to

\[
T = 2 \int_0^{\pi/2} d\theta \frac{1}{g(x(\theta))}.
\]

This integrand is finite for all \( \theta \).

Near a turning point, \( x = a \) for instance, we have

\[
\frac{dy}{dx} \sim \frac{g(a)\sqrt{b-a}}{2\sqrt{x-a}} \to \infty \quad \text{as} \quad x \to a,
\]

which shows that if \( V'(a) \neq 0 \) the phase curves cut the \( x \)-axis at right angles.

(vi) Page 10, points (ii) and (v) after equation 1.15: This description of the flow direction is true only for a system with the equation of motion \( \dot{x} = f(x, t) \) and where \( y = \dot{x} \). Similarly, equation 1.16 is valid only for this system.

(vii) Examples 1.2 and 1.3: The equation of motion \( \ddot{x} + \omega^2 x = 0 \) approximates the motion of a vertical pendulum, JS equation 1.1, near \( x = 0 \) because for \( |x| \ll 1, \sin x = x + O(x^3) \).

Similarly near \( x = \pi \) on putting \( x = \pi - \xi \) and using the relation \( \sin(\pi - \xi) = \sin \xi \) the equation of motion becomes

\[
-\frac{d^2\xi}{dt^2} + \omega^2 \sin \xi = 0 \quad \text{or, for small} \quad \xi, \quad \ddot{\xi} - \omega^2 \xi = 0.
\]

(viii) Page 11: Another commonly used name for a centre is elliptic fixed point.

(ix) Page 12, equations 1.18 and 1.19: The constants \( A \) and \( B \) are determined uniquely by the initial conditions. The amplitude \( \kappa \) is positive and determined uniquely by \( A \) and \( B \): the phase \( \phi \), however, is not unique for we could write

\[
x(t) = \kappa \sin(\omega t + \phi'), \quad A = \kappa \sin \phi', \quad B = \kappa \cos \phi'.
\]

(x) Page 13, line −8: The condition \( C \geq 1 \) should be \( C \geq -1 \).

(xi) Page 13, line 8 of example 1.4: \( (n = 0, \pm \pi, \pm 2\pi, \cdots) \) should be \( (n = 0, \pm 1, \pm 2, \cdots) \).

(xii) Page 13, example 1.4: The table at bottom of this page is confusing. The value of the constant \( C \) depends upon the initial conditions and does not determine the nature of the fixed points, as implied by the first line. If \( C = -1 \) the only permissible values of \( x \) are \( x = 0 \) (or equivalently \( x = 2\pi n \)) and here \( y = 0 \). These are the stable fixed points.
If \( C = 1 \) we have
\[
y^2 = 2(1 + \cos x) = 4\cos^2(x/2) \implies y = \pm 2\cos(x/2)
\]
which is the equation of the separatrix.
If \(-1 \leq C < 1\) the motion is librational and if \( C > 1 \) it is rotational.

Read section 1.3, pages 14-22.

Comments on text:

(i) The most important parts of this section are equations 1.31, (JS page 18), and the qualitative analysis of this equation that leads to figures 1.11 to 1.14. It is important that you understand how these diagrams are formed and that you know that at maxima and minima of the potential and \( y = 0 \) there are respectively saddles and centres.

The system considered in this section
\[
\dot{x} = y, \quad \dot{y} = f(x)
\]
is special because equation 1.15, (JS page 10), is separable,
\[
\frac{dy}{dx} = \frac{f(x)}{y} \implies \int dy = \int dx f(x) \implies \frac{1}{2}y^2 - \int dx f(x) = \text{constant},
\]
which is equation 1.31. Because of this relation between \( y \) and \( x \) the behaviour of the phase curves is constrained. This type of system is special because it possesses an integral of the motion.

An integral of the motion of the system
\[
\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y),
\]
is a single valued, differentiable function \( F(r) \) which is constant along all solutions of the equations of motion; that is, given a solution \( r(t) \), \( F(r(t)) \) is constant. Using the chain rule we obtain
\[
0 = \frac{d}{dt} F(r(t)) = \dot{x} \frac{\partial F}{\partial x} + \dot{y} \frac{\partial F}{\partial y} = X \frac{\partial F}{\partial x} + Y \frac{\partial F}{\partial y}.
\]
This equation shows that \( \text{grad } F \) is perpendicular to the velocity function. The equation \( F(r) = \text{constant} \) defines a contour, or level curve, of the function \( F(r) \) and these contours must be solutions of the equations of motion. Integrals of the motion are often called constants of the motion, particularly in mechanics where the energy and angular momentum are constants of the motion for conservative and spherically symmetric systems, respectively. For the vertical pendulum the function \( H(x,y) = y^2/2 - \omega^2 \cos x \) is an integral of the motion.

Integrals of the motion are often obtained by integrating the equation
\[
\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}
\]
and hence are sometimes called first integrals. Such integration is always possible when coordinates \((u, v)\) can be found such that component of the velocity function can be expressed as products of functions of \(u\) only and of \(v\) only. Thus if \(X = f_1(u)g_1(v)\) and \(Y = f_2(u)g_2(v)\) the differential equation becomes

\[
\frac{dv}{du} = \frac{f_2(u)g_2(v)}{f_1(u)g_1(v)} \implies F(u, v) = \int \frac{g_1(v)}{g_2(v)} dv - \int \frac{f_2(u)}{f_1(u)} du.
\]

The function \(F(u, v)\) constructed in this manner need not be single valued and then it is not an integral of the motion; examples of this type of system are given in exercise N1.5.

An important case for which integrals exist is when a function \(H(x, y)\) exists such that the equations of motion can be written in the form

\[
\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}.
\]  (1.1)

Then \(H(x, y)\) is an integral of the motion, exercise N1.1. This type of system occurs frequently in physics and is named a Hamiltonian system and the function \(H(x, y)\) is the Hamiltonian function or just the Hamiltonian named after the Irish mathematician Hamilton who developed the theory of Hamiltonian dynamics; the function \(H(x, y)\) defined in exercise N1.1 is the Hamiltonian function for the vertical pendulum. The Hamiltonian may be a function of time, in which case it is not constant along a solution; Hamiltonian systems of higher dimensions exist and are also very important.

(ii) Page 18, equation 1.31: The value of \(C\) depends upon the initial conditions, not \(f(x)\). The range of allowed values may depend upon \(f(x)\), for instance if \(x^2 + x^2 = C\) then \(C \geq 0\) whereas if \(x^2 - x^2 = C\) all values are allowed.

(iii) Page 21, line 2: “If \(x^2\)” should be “If \(f(x)\)”.

(iv) Page 21, line 2: “saddle points at \((-1,1)\)” should be “saddle points at \((-1,0)\)”.

(v) Page 22, figure 1.14: In this case the separatrix, the phase curve for \(C = 1\), goes to infinity because there is an equilibrium point at \(y = 0\) and \(x = \infty\). The separatrix divides phase space into the region containing bound, periodic motion, where \(0 < C < 1\), and the region where the motion is unbounded, \(C > 1\).

Exercise N1.1

Using equation N1.1 show that the Hamiltonian function

\[
H(x, y) = \frac{1}{2}y^2 - \omega^2 \cos x
\]

gives the equation of motion for the vertical pendulum, JS equation 1.3 (page 2). Show also that \(H(x, y)\) is an integral of the motion.
Exercise N1.2
If \( H(x, y) \) is a Hamiltonian function show that \( K(x, y) = H(x, y) + A(t) \), where \( A(t) \) is a function of \( t \), but not \( x \) or \( y \), gives the same equations of motion.

Exercise N1.3
Show that the two Hamiltonian functions \( H(x, y) \) and \( K(x, y) = AH(x, y) \), where \( A \) is a constant, have the same phase curves, but that phase point traverses the equivalent phase curves at different rates.

Exercise N1.4
Show that the two Hamiltonian functions \( H(x, y) \) and \( K(x, y) = F(H(x, y)) \), where \( F(z) \) is a differentiable function of \( z \), have the same phase curves. What distinguishes the motion of these Hamiltonians.

Exercise N1.5
(i) Show that the function \( Ax^2 + 2Bxy + Cy^2 = D \), where \( A, B, C \) and \( D \) are constants, is a solution of the equation
\[
\frac{dy}{dx} = -\frac{Ax + By}{Bx + Cy}.
\]
(ii) Show that the function \( F(r) = a \tan^{-1}(y/x) + \ln \sqrt{x^2 + y^2} \), where \( a \) is a constant, is a solution of the equation
\[
\frac{dy}{dx} = \frac{ay - x}{y + ax}.
\]
(iii) Which, if any, of the functions defined parts (i) and (ii) is an integral of the motion?

Exercise N1.6
Show that the equation
\[
\frac{dy}{dx} = -\frac{x + axy}{y + bx^2 + cy^2}
\]
has the solution \( x^2 = Ay^2 + By + C \) if
\[
A = -\frac{c}{a + b}, \quad B = \frac{2(c - a - b)}{(a + b)(a + 2b)}, \quad C = \frac{a + b - c}{b(a + b)(a + 2b)}.
\]
Is the function \( F(r) = Ay^2 - x^2 + By \) an integral of the motion for the system \( \dot{x} = y + bx^2 + cy^2 \), \( \dot{y} = -x - axy \)? Hint: a function \( F(r) \) is an integral of the motion if \( F(r) = \text{constant on all solutions in a region of phase space.} \)
Exercises for Section 1.3: 1.3 (hint use equation 1.15), 1.6, 1.15

Read section 1.4, pages 22-26.

Comments on text:

(i) Page 22, after equation 1.35: The most significant difference between the system \( \ddot{x} = f(x) \), considered in section 1.3, and the system \( \ddot{x} = f(x, \dot{x}) \) of section 1.4 is that the former possesses an integral of the motion whilst the latter usually does not.

Conservative systems with many degrees of freedom can also behave in qualitatively different ways if the energy is the only integral of the motion.

(ii) Page 24, after equation 1.43: \textit{is cut most} should be \textit{is cut at most}.

(iii) Page 24, strong damping:

- The two straight lines in figure 1.16b are obtained from equation 1.46 by choosing \( A = 0 \), to give the phase curve \( y = p_2 x \), and \( B = 0 \), to give the phase curve \( y = p_1 x \).

- As is often the case it is instructive to investigate limiting cases which can help understand how systems behave. One limiting case for the damped oscillator is when the damping is very strong, \( k \gg c \): then since \( \sqrt{k^2 - 4c} = k - 2c/k + O(k^{-2}) \) we have

\[
p_2 = -k, \quad p_1 = -\frac{c}{k}, \quad \text{so} \quad |p_1| \gg |p_2|,
\]

and hence

\[
x(t) = A \exp(-ct/k) + B \exp(-kt), \quad k \gg c.
\]

For most initial conditions \( A \neq 0 \) and for times \( t > 1/k \) the second term is negligible and \( y = \dot{x} = -cx/k \). That is most phase curves approach the origin along the line \( ky + cx = 0 \). This approximates one of the straight lines in figure 1.16b.

If \( A = 0 \), then the phase curve is \( y + kx = 0 \), which is the other straight line. The angle between the lines is \( \tan^{-1} k \), if \( k \gg c \).

(iv) Page 25, line definition of \( p_k \): The definition of \( p_1 \) and \( p_2 \) should be

\[
p_1, p_2 = -\frac{1}{2} k \pm i \sqrt{-\Delta}.
\]

And on line 15, the equation for \( A \) should be \( A = \frac{1}{2} Ce^{i\alpha} \).

Read section 1.5, pages 26–33.

Comments on text:

(i) Page 27, equation 1.53: This is the \textit{definition} of the energy for this system.
(ii) Page 29: The combination of circumstances that leads to the equation \( yh(x, y) = 0 \) defining a solution is rare and usually limited to text book exercises, as in example 1.9, rather than real systems.

(iii) Page 30: Equation \( \ddot{x} = f(x) \) does not have a limit cycle solution, because it has an integral of the motion, \( F(x, y) = C \), and its periodic solutions are not isolated: this follows because the contours defined by \( F(x, y) = C \) depend continuously on \( C \).

(iv) Page 32–33: Generalised coordinates are associated with Hamiltonian systems. The configuration of a dynamical system with one degree of freedom is represented by a point in a one-dimensional configuration space with a single coordinate, often denoted by \( q \). Examples are the height of a particle constrained to move vertically, or the angle of rotation of a pendulum whose motion is confined to a vertical plane. The configuration of a dynamical system with two degrees of freedom is represented by a point in a two dimensional configuration space: examples are a particle confined to any two-dimensional surface, for instance the end of a vertical pendulum not confined to move in a plane such as the Foucault pendulum. Generalised coordinates are any set of independent coordinates required to define the configuration of a system.

The generalised coordinates do not determine the motion: the related momenta are also required. For a Hamiltonian for each set of generalised coordinates infinitely many sets of generalised momenta can be defined: given initial values of the generalised coordinates and momenta the motion is determined uniquely. There is not a unique choice of generalised coordinates and momenta and most problems are simplified a great deal with a good choice.

**Exercise N1.7**

Find the limit cycles of the system

\[
\begin{align*}
\dot{x} &= y + x \sin(br^2), \\
\dot{y} &= -x + y \sin(br^2), \\
r^2 &= x^2 + y^2, \\
\end{align*}
\]

where \( b \) is a positive constant, and determine which are stable.

**Read** section 1.6, pages 33–39.

Comments on text:

(i) Page 34: The \( \text{sgn} \) function is closely related to the Heaviside function defined by

\[
H(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0.
\end{cases}
\]

The Heaviside function is discontinuous at \( x = 0 \) and is not defined at \( x = 0 \). We have \( \text{sgn}(x) = 2H(x) - 1 \) and normally \( \text{sgn}(x) \) is not defined at \( x = 0 \).

(ii) Page 35: A better way of understanding the phase diagram in figure 1.22 is to replace the friction term \( F(\dot{x}) = F_0\text{sgn}(v_0 - \dot{x}) \) by the continuous function,

\[
F(\dot{x}) = \begin{cases} 
mF_0, & \dot{x} < v_0 - \sigma, \\
\frac{mF_0(v_0 - \dot{x})}{\sigma}, & v_0 - \sigma \leq \dot{x} \leq v_0 + \sigma, \\
-mF_0, & \dot{x} > v_0 + \sigma.
\end{cases}
\]
If $\sigma$ is a small speed $0 < \sigma \ll v_0$, this function is similar to that shown in JS figure 1.21a and, because it is linear in $\dot{x}$, the equations of motion can be solved. Note that we have introduced the mass into the definition of $F(\dot{x})$ so if we also write $c = m\omega^2$ the equation of motion becomes

$$\ddot{x} + \omega^2 x = \begin{cases} F_0, & \dot{x} < v_0 - \sigma, \\ F_0(v_0 - \dot{x})/\sigma, & v_0 - \sigma \leq \dot{x} \leq v_0 + \sigma, \\ -F_0, & \dot{x} > v_0 + \sigma. \end{cases} \tag{1.2}$$

If $y = \dot{x} < v_0 - \sigma$ or $y > v_0 + \sigma$ then $F(\dot{x})$ is constant and the phase curves are the ellipses

$$\begin{align*}
y^2 + \left(x + F_0/\omega^2\right)^2 &= 2E, & y > v_0 + \sigma, \\
y^2 + \left(x - F_0/\omega^2\right)^2 &= 2E, & y < v_0 - \sigma. \tag{1.3}
\end{align*}$$

These are just equations given on page 35 of JS.

For $v_0 - \sigma \leq y \leq v_0 + \sigma$ the equation of motion can be written in the form

$$\ddot{x} + \Omega \dot{x} + \omega^2 x = \frac{F_0v_0}{\sigma}, \quad \Omega = \frac{F_0}{\sigma} \gg \omega. \tag{1.4}$$

Before analysing the solutions of this equation we show graphs of some phase curves in the case $F_0 = v_0 = \omega = 1, \sigma = 1/10$.

![Phase curves of equation N1.4.](image)

For the phase curves starting on the line $y = v_0 - \sigma = 0.9$, if $x < -F_0/\omega^2 = -1$ the phase point moves upwards to the line $y = v_0 + \sigma = 1.1$ and for $|x| < F_0/\omega^2$ it starts moving upwards, but turns round and moves on the straight line to the point $x = F_0/\omega^2 = 1, y = v_0 - \sigma$.

For the phase curves starting on the line $y = v_0 + \sigma = 1.1$, if $x > F_0/\omega^2 = 1$ the solution moves almost vertically downwards to $y = v_0 - \sigma$ but if $|x| < F_0/\omega^2$ it moves downwards initially to the line joining the point $(-F_0/\omega^2, v_0 + \sigma)$ to the point $(F_0/\omega^2, v_0 - \sigma)$. This is consistent with the phase curves shown in JS figure 1.22, if $\sigma$ is very small.

In order to understand these phase curves we observe that for small $\sigma$, $\Omega \gg \omega$; the frequencies $\omega$ and $\Omega$ define the time-scales that produce the phase curves in
Consider the homogeneous equation $\ddot{x} + \Omega \dot{x} + \omega^2 x = 0$, on putting $x = e^{-\lambda t}$ we obtain the equation

$$\lambda^2 - \lambda \Omega + \omega^2 = 0$$

which has the solutions

$$\lambda_1 = \frac{1}{2} \left( \Omega + \sqrt{\Omega - 4\omega^2} \right) \simeq \Omega \quad \text{and} \quad \lambda_2 = \frac{\omega}{\lambda_1} \simeq \frac{\omega}{\Omega}.$$ 

Thus $\lambda_1 \gg \lambda_2$. The general solution of equation N1.4 is

$$x = \frac{F_0 v_0}{\omega^2 \sigma} + \alpha e^{-\lambda_1 t} + \beta e^{-\lambda_2 t},$$

for some constants $\alpha$ and $\beta$. Since $\lambda_2 \ll \lambda_1 \simeq \Omega$ for $\Omega t > 1$ we have

$$x \simeq \frac{F_0 v_0}{\omega^2 \sigma} + \beta e^{-\lambda_2 t}.$$ 

This is the solution of the equation derived from N1.4 by ignoring the $\ddot{x}$ term, that is

$$\Omega \dot{x} + \omega^2 x = \frac{F_0 v_0}{\sigma}, \quad \Omega = \frac{F_0}{\sigma} \gg \omega. \quad (1.5)$$ 

This equation defines the line

$$\Omega y + \omega^2 x = \frac{F_0 v_0}{\sigma} \quad (1.6)$$

joining the points $(\pm F_0/\omega^2, v_0 \pm \sigma)$, which approximates the straight line seen in figure N1.1. The actual solution of N1.5 with $x(0) = -F_0/\omega^2$ is

$$x(t) = \frac{F_0 v_0}{\omega^2 \sigma} - \frac{F_0}{\omega^2 \sigma} (v_0 + \sigma) \exp \left(-\frac{\omega^2 t}{\Omega}\right),$$

$$y(t) = (v_0 + \sigma) \exp \left(\frac{-\omega^2 t}{\Omega}\right).$$

Now return to the original equation

$$\dot{x} = y, \quad \dot{y} = \frac{F_0 v_0}{\sigma} - (\omega^2 x + \Omega y).$$

For a phase point below the line N1.6 $\dot{x} \simeq v_0$, because $v_0 - \sigma < y < v_0 + \sigma$, and $\dot{y}$ is large and positive, because $y$ is multiplied by the large quantity $\Omega$. Thus $\dot{y} \gg \dot{x}$ and the phase point rapidly moves upwards to the line N1.6. Similarly for a phase point above the line N1.6 $\dot{y}$ is large and negative so the phase point moves rapidly down towards the line. This explains the phase curves shown in figure N1.1.

(iii) Page 36, figure 1.23: The definition of $\theta$ implied in the figure is wrong. The angle $\theta$ is the angle turned by a point fixed on the wheel, not the angle between the brake and the vertical.
(iv) The ratchets on the escape wheel seem to point in the wrong direction for this clock to work. Alternatively, the weight should hang from the other side of the axle.

**Do Exercises:** 1.1, 1.2 (ignore last sentence), 1.7 (assume $\theta < \pi$), 1.11, 1.32, 1.35, 1.37.

**Revision Exercises:** 1.4, 1.18, 1.20, 1.38, 1.42.
CHAPTER 1. SECOND-ORDER DIFFERENTIAL EQUATIONS

1.2 Solutions for exercises from chapter 1

Solutions for exercises set in Course Notes

Solution to Exercise N1.1
From equation N1.1 the equation of motion are
\[ \dot{x} = \frac{\partial H}{\partial y} = y, \quad \dot{y} = -\frac{\partial H}{\partial x} = -\omega^2 \sin x. \]

On differentiating the first of these we obtain \( \ddot{x} + \omega^2 \sin x = 0 \).
Differentiating the Hamiltonian gives
\[
\frac{dH}{dt} = \frac{\partial H}{\partial x} \ddot{x} + \frac{\partial H}{\partial y} \ddot{y} = \omega^2 \dot{x} \sin x + y \ddot{y},
\]
\[ = -\omega^2 y \sin x + \omega^2 y \sin x = 0. \quad \text{Therefore } H(x, y) = \text{constant}. \]

Solution to Exercise N1.2
Since \( \frac{\partial A}{\partial x} = \frac{\partial A}{\partial y} = 0 \) the result follows.

Solution to Exercise N1.3
The equations of motion for \( H \) and \( K \) are, respectively
\[ \dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} \quad \text{and} \quad \dot{x} = A \frac{\partial H}{\partial y}, \quad \dot{y} = -A \frac{\partial H}{\partial x}. \]

In the equations for \( K \) define a new time \( \tau = \frac{t}{A} \), so these equations become
\[ \frac{dx}{d\tau} = \frac{\partial H}{\partial y}, \quad \frac{dy}{d\tau} = -\frac{\partial H}{\partial x}, \]
which are the same as the equations for \( H \) but with \( t \) replaced by \( \tau \).

The phase curves for each system are therefore the same but are traversed at different rates: if \( A > 1 \) the \( K \)-motion is faster.

Solution to Exercise N1.4
A contour of \( H(x, y) \) is also a contour of \( F(H(x, y)) \), so the phase curves of the two systems are identical. The equations of motion for \( K \) are
\[ \dot{x} = \frac{\partial K}{\partial y} = \frac{\partial H}{\partial y} F'(H), \quad \dot{y} = -\frac{\partial K}{\partial x} = -\frac{\partial H}{\partial x} F'(H). \]

Since \( H \) is a constant on each phase curve these equations are similar to those treated in exercise N1.3, the only difference is that the value of \( F'(H) \) is different on each phase curve, so the difference in the rate of traversal of phase curves varies between phase curves, unlike in exercise N1.3.
Solution to Exercise N1.5

(i) The equation is homogeneous so we define a function \( v(x) \) by the equation \( y = xv(x) \) to give

\[
v + x \frac{dv}{dx} = \frac{-A + Bv}{B + Cv} \quad \text{or} \quad \int dv \frac{B + Cv}{A + 2Bv + Cv^2} + \int \frac{dx}{x} = \text{constant}.
\]

Integration then gives \( \ln(A + 2Bv + Cv^2) + \ln(x^2) = \text{constant} \). Replacing \( v \) by \( y/x \) then gives

\[
Ax^2 + 2Bxy + Cy^2 = \text{constant}.
\]

(ii) The same transformation, \( y = xv(x) \), gives

\[
x \frac{dv}{dx} = -\frac{1 + v^2}{a + v} \quad \text{or} \quad \int dv \frac{a + v}{1 + v^2} + \int \frac{dx}{x} = \text{constant}.
\]

Integration gives

\[
a \tan^{-1} \left( \frac{y}{x} \right) + \frac{1}{2} \ln(x^2 + y^2) = C,
\]

where \( C \) is a constant.

(iii) The function defined in part (ii) is multi-valued — because \( \tan^{-1}(z) \) has infinitely many values for each \( z \) — so is not an integral of the motion. This is because the fixed point of the system

\[
\dot{x} = y + ax, \quad \dot{y} = ay - x
\]

is a spiral, stable if \( a < 0 \) and unstable if \( a > 0 \). So each phase curve crosses the line \( y = kx \) at infinitely many places, each corresponding to a branch of the \( \tan^{-1}(y/x) \) function.

In the first case the function \( Ax^2 + 2Bxy + Cy^2 = \text{constant} \) is single valued so is an integral of the motion. The equation defines ellipses or hyperbolae according to the relative values of \( A, B \) and \( C \). The equation

\[
\dot{x} = Bx + Cy, \quad \dot{y} = -Ax - By
\]

has the linearised matrix, see chapter 2, \( \begin{pmatrix} B & C \\ -A & -B \end{pmatrix} \) with eigenvalues given by \( \lambda^2 = B^2 - AC \), which are real with opposite sign if \( B^2 > AC \) and purely complex if \( B^2 < AC \).

Solution to Exercise N1.6

Write the equations of motion in the form

\[
y + bx^2 + cy^2 = -x(1 + ay) \frac{dx}{dy}
\]

On differentiating the given solution we obtain

\[
x \frac{dx}{dy} = Ay + \frac{1}{2} B
\]

so the first equation becomes

\[
y + cy^2 + b \left( Ay^2 + By + C \right) + (1 + ay) \left( Ay + \frac{1}{2} B \right) = 0.
\]
Since the coefficient of each power of \( y \) must be zero this gives

\[
C = -\frac{B}{2b}, \quad A = -\frac{c}{a+b}, \quad B = -\frac{1+A}{b+2a},
\]

which gives the required result.

The function \( F(r) = Ay^2 - x^2 + By \) is \textit{not} an integral of the motion because it defines only one solution.

**Solution to Exercise N1.7**

The equation for \( r \) is

\[
r\ddot{r} = x\ddot{x} + y\ddot{y} = r^2 \sin br^2 \quad \implies \quad \dot{r} = r \sin br^2.
\]

Thus \( \dot{r} = 0 \) when \( r^2 = n\pi/b, \ n = 0, 1, 2, \ldots \), so there is a fixed point at the origin and infinitely many limit cycles at \( r = r_n = \sqrt{n\pi/b}, \ n = 1, 2, \ldots \).

Near the origin \( \dot{r} = br^3 > 0 \), so this fixed point is unstable; therefore we should expect the \( r_1 \) limit cycle to be stable and since the stable and unstable limit cycles must alternate we expect the \( r_{2n} \) limit cycles to be unstable and the \( r_{2n+1} \) limit cycle to be unstable.

This may be seen geometrically by plotting the graph of \( f(r) = r \sin br^2 \) and noting that near \( r_{2n+1} \), \( f(r) \) decreases through the zero, so \( \dot{r} > 0 \) for \( r < r_{2n+1} \) and \( \dot{r} < 0 \) for \( r > r_{2n+1} \).

![](image.png)

**Figure N1.2** Graph of \( f(r) = r \sin r^2 \).

Alternatively, set \( r = \sqrt{n\pi/b} + s \) and expand to first-order in \( s \):

\[
\dot{s} = \left( \sqrt{\frac{n\pi}{b}} + s \right) \left( n\pi + 2\sqrt{n\pi bs} + O(s^2) \right)
\]

\[
= 2n\pi(-1)^ns.
\]

Thus if \( n \) is even \( s \) increases so the \( r_{2n} \) limit cycles are unstable and similarly the \( r_{2n+1} \) limit cycles are stable.
1.2. SOLUTIONS FOR EXERCISES FROM CHAPTER 1

Solutions of exercises from set book

Solution to Exercise JS 1.1
In this question we put \( y = \dot{x} \) throughout.

(i) \( \dot{y} = ky \): the equilibrium points are \( y = 0 \), all \( x \), i.e. the whole \( x \)-axis. The phase paths satisfy \( dy/dx = k \) which gives \( y = kx + c \). This does not give the direction of motion along the phase paths which can only be obtained from the equations of motion, as discussed on page 3 of JS, and is shown in figure N1.3 below.

(ii) \( \dot{y} = 8xy \): the equilibrium points are the whole \( x \)-axis and the phase curves satisfy \( dy/dx = 8x \) or \( y = 4x^2 + c \) which are a set of parabolas. As in (i) care must be taken with the direction of motion, figure N1.4.

(iii) The equations of motion are

\[ \begin{align*}
\dot{x} &= y & \text{if } |x| < 1, \\
\dot{y} &= 0 \\
\dot{x} &= y & \text{if } |x| \geq 1, \\
\dot{y} &= k 
\end{align*} \]

For \( |x| < 1 \) the phase curves are straight lines parallel to the \( x \)-axis. For \( |x| > 1 \) we have \( y^2 = 2kx - D \), so for \( k > 0 \) the phase diagram looks like that shown in figure N1.5. The equilibrium points are all points on the line segment \( y = 0, |x| < 1 \).

For \( |x| < 1 \) the phase curves are straight lines parallel to the \( x \)-axis. For \( |x| > 1 \) we have \( y^2 = 2kx - D \), so for \( k > 0 \) the phase diagram looks like that shown in figure N1.5. The equilibrium points are all points on the line segment \( y = 0, |x| < 1 \).
(iv) The discriminant, JS equation (1.42), is $\Delta = 1$ giving strong damping, so the origin is a stable node; this is the only fixed point so the phase diagram is as shown on JS figure 1.16b, (page 24).

(v) The auxiliary equation has roots $m = -2 \pm 6i$ giving a stable spiral, JS figure 1.17b, (page 26).

(vi) The solution of this problem is messy: we provide the details only for the upper half of the phase plane, $\dot{x} > 0$. For $\dot{x} < 0$ if we put $\tau = -t$ the equations of motion become the same as in the upper half plane, so the phase curves are the same but the direction of flow is reversed. Here the auxiliary equation gives $m = -1, -2$. The two special solutions $x = Ae^{-t}, x = Be^{-2t}, (A,B > 0)$ give the two faint lines shown in figure N1.6. The solution with $\dot{x}(0) = 0, x(0) = \alpha$ is $x = \alpha(2e^{-t} - e^{-2t})$, is valid for $t < 0$ if $\alpha > 0$ and vice versa, provides the phase curves crossing the $x$-axis. Other solutions, not of this form, fill in the space between the two faint lines.

![Figure N1.6 Phase flow for exercise 1.1vi](attachment:figure.png)

Because $|\dot{x}| = |y|$ is not differentiable at the origin the equilibrium point here does not fit into the classification scheme discussed in detail in JS chapter 2.

(vii) Each quadrant of the phase plane must be treated separately and we shall assume $c > k > 0$:

\[
\begin{array}{ccc}
 & \dot{x} & \\
\ddot{x} & (c-k) & -(c+k) \\
\dddot{x} & (c+k) & -(c-k) \\
& \dddot{x} & \\
\end{array}
\]

For a phase point at $x = x_0 > 0, y = \dot{x} = 0$ when $t = 0$ we have:

fourth quadrant : $x = x_0 - (c-k)t^2/2$
third quadrant: \[ x = (c + k)t^2/2 - \sqrt{2/(c - k)x_0} t \]
second quadrant: \[ x = (c - k)t^2/2 - x_0(c - k)/(c + k) \]
first quadrant: \[ x = (c - k)(c + k)^{-1/2}\sqrt{2x_0} t - (c + k)t^2/2 \]

where in each case the time origin is that time at which the phase point enters the quadrant. The desired relation follows from the last equation by finding the time at which \( \dot{x} = 0 \) and substituting back into \( x(t) \).

The period of the cycle is
\[
T_1 = \sqrt{2x_0} \frac{2c}{c + k} \left[ \sqrt{\frac{1}{c - k}} + \sqrt{\frac{1}{c + k}} \right] = A\sqrt{x_0},
\]
the last equality defining \( A \). The phase curves encircle the origin, getting closer as time increases. The origin however is not a spiral point. First, the term ‘spiral point’ is defined to be a particular type of fixed, or singular point, see JS chapter 2, (page 73), of a differentiable system. Second, the phase point starting at \( x_0 \) when \( t = 0 \) reaches the origin after a finite time, which is quite different from the examples discussed in JS page 26, in particular figure 1.17b. To see this let \( x_k \) be the successive points where the phase curve crosses the positive \( x \)-axis, so that \( x_k = d^2 x_{k-1} \), \( k = 1, 2, \ldots \) and \( d = (c - k)/(c + k) < 1 \). The period of the \( k \)th cycle is \( T_k = A\sqrt{x_{k-1}} \), so the total time to reach \( x_N \) is
\[
\tau_N = \sum_{k=1}^{N} T_k = A\sqrt{x_0} \sum_{k=0}^{N-1} d^k = A\sqrt{x_0} \left( \frac{1 - d^N}{1 - d} \right).
\]

As \( N \to \infty \), \( \tau_n \to A\sqrt{x_0}/(1 - d) \) which is finite. That is, the motion terminates at the origin after a finite time.

Solution to Exercise JS 1.2

This is a conservative system, section 1.3, with \( f(x) = -x - \alpha x^3 \) and potential energy \( V(x) = x^2/2 + \alpha x^4/4 \). The phase curves are given by the energy equation
\[
\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{4} \alpha x^4 = E = \text{constant},
\]
but are usually easiest to draw by first sketching the potential \( V(x) \), as shown in the upper panel of figure N1.7 for the case \( \alpha < 0 \).
For the case $\alpha > 0$ the potential has only one stationary point, a minimum at the origin. Thus there is a centre at the phase space origin and no other fixed points, as shown in figure N1.8.

Figure N1.8 Graphs of $V(x) = x^2/2 + \alpha x^4/4$ for $\alpha = 1$, above, and the contours of the flow, below. In this case there is no separatrix.
Solution to Exercise JS 1.3
Put \( y = \dot{x} \) so that the phase curves are solutions of \( \frac{dy}{dx} = -y - x/y \). Put \( y^2 = v \) so \( \frac{dv}{dx} + 2v = -2x \), or \( e^{-2x}d(ve^{2x})/dx = -2x \). Thus the equation
\[
v = y^2 = (1 - 2x + Ae^{-2x})/2,
\]
where \( A \) is a constant, describes the phase curves, which are symmetric about the \( x \)-axis.
Consider the points where the phase curve crosses the \( x \)-axis, \( y = 0 \) that is the roots of \( Ae^{-2x} = 2x - 1 \). If \( A \geq 0 \) this equation always has only one solution, \( x_0(A) > 1/2 \), which increases with \( A \), and when \( A = 0 \) we have is \( x_0(0) = 1/2 \) (to see this sketch the graphs of \( w = 2x - 1 \) and \( w = Ae^{-2x}, A \geq 0 \)). If \( -1 < A < 0 \) there are two roots, one positive and one negative, which degenerate to a single root at \( x = 0 \) when \( A = -1 \). When \( A < -1 \), \( y(x) \) is not real for any \( x \). The phase curves are shown below: when \( A > 0 \) the motion is unbounded; for \( A < 0 \) it is bounded and periodic and the separatrix, \( A = 0 \) shown by the thick phase curve, separates these two types of motion:

\[\text{Figure N1.9 Graphs of } 2y^2 = 1 - 2x + A e^{-2x} \text{ for } A = -1/2, 0 \text{ and } 1; \text{ the thicker line, } A = 0, \text{ is the separatrix.}\]

Solution to Exercise JS 1.4
This is a conservative system with the potential \( V(x) = e^x - ax \) and energy equation
\[
\frac{1}{2}y^2 + e^x - ax = E, \quad y = \dot{x}.
\]
If \( a = 0 \) the potential, \( V = e^x \), is monotonic increasing and \( V(x) \to 0 \) as \( x \to -\infty \) (so \( y \simeq \sqrt{2E} \) as \( x \to -\infty \)) and \( V(x) \to \infty \) as \( x \to \infty \), so there are no centres or saddles (see JS page 19, figure 1.12). The potential and some representative phase curves are shown in the figure below.
CHAPTER 1. SECOND-ORDER DIFFERENTIAL EQUATIONS

If $a = -b < 0$ the potential $V(x) = e^x + bx$ is also monotonic, but now $V(x) \to -\infty$ as $x \to -\infty$ and $V(x) \to \infty$ as $x \to \infty$. Again the potential has no stationary points, so there are centres or saddles. The potential and some representative phase curves for the case $a = -1$ are shown in the figure below.

If $a > 0$ the potential $V(x) = e^x - ax$ has a single minimum at $x = \ln a$ and $V(x) \to \infty$ as $|x| \to \infty$. At the minimum the system has a centre (JS page 19, left-hand panel). The potential and some representative phase curves for the case $a = 1$ are shown in the figure below.
1.2. SOLUTIONS FOR EXERCISES FROM CHAPTER 1

Solution to Exercise JS 1.6
The shape of the potential is as shown in figure N1.13. The energy equation is

\[ \frac{1}{2}y^2 + V(x) = E, \quad y = \dot{x}. \]

Hence there is a line of fixed points \(|x| \leq 1, y = 0\). For \(|x| \leq 1, V(x) = 0\) so \(y = \dot{x} = \sqrt{2E};\) for \(|x| > 1\) the phase curves are given by \(y = \pm \sqrt{2(E - V(x))}\). In the case

\[ V(x) = \begin{cases} \frac{1}{2}(1 - x^3), & |x| > 1, \\ 0, & |x| \leq 1, \end{cases} \tag{1.7} \]

and the graph of this potential and some representative phase curves are shown below.

Figure N1.12 Graphs of \(V(x) = e^x - x\), upper panel and the contours of the flow: there is single fixed point at the minimum of the potential.

Figure N1.13 Graph of the potential N1.7.

Figure N1.14 Some representative phase curves of the potential N1.7.
Solution to Exercise JS 1.7
The phase curves can be constructed from those shown in JS figure 1.2, (page 3). In the first case \( \dot{\theta} \) changes sign when \( \theta = \alpha \), so there are different types of motion according as \( \alpha > 0 \) or \( \alpha < 0 \), as shown in figures N1.15 and N1.16, below. If \( \alpha < 0 \) ‘ordinary’ pendulum motion of amplitude less than \(|\alpha|\) is possible.

In case (ii) if \( \alpha > 0 \), \( \theta(t) \to \alpha \) as \( t \to \infty \), but if \( \alpha < 0 \) the motion settles down to periodic motion of amplitude \(|\alpha|\).

Solution to Exercise JS 1.11
In this approximation to the pendulum motion \( \ddot{x} = f(x) = -\omega^2 x (1 - x^2 / 6) \) and the potential is \( V(x) = \omega^2 x^2 (1 - x^2 / 12) / 2 \), which is an approximation to \( \omega^2 (1 - \cos x) \). The approximation is good for small \( x \) but useless for large \( x \). The potential and consequent phase curves are shown in figure N1.17. The separatrix, the heavy line, has energy \( E = E_s = 3 \omega^2 / 2 \); that of the exact equation has energy \( \omega^2 \). For \( E < E_s \) the motion is close to the exact. For \( E > E_s \) the type of motion is completely different; in the approximation it is unbounded, whilst the exact motion is bounded (see JS figure 1.2, page 3).
1.2. SOLUTIONS FOR EXERCISES FROM CHAPTER 1

Solution to Exercise JS 1.15

The potential is $V(x) = x^3/3 - x^5/5$ is atypical because it has a point of inflection at $x = 0$, where both its first and second derivatives are zero. The potential has a maximum at $x = 1$, $V(1) = 2/15$ and a minimum at $x = -1$ with $V(-1) = -2/15$. These features are shown in the upper panel of figure N1.18.

From this graph it follows that:

a) there is a saddle at the point $x = 1$, $y = 0$, through which passes a separatrix with energy $E = 2/15$;
b) there is a centre at the point $x = -1$, $y = 0$;
c) there is bound, periodic motion for $-2/15 \leq E \leq 2/15$;
d) there is unbounded motion for all energies, but for $E > 2/15$ and $E < -2/15$ there is only unbounded motion.

These features are shown in the lower panel of figure N1.18.
The energy of the path given is $17/480$ which is less than $E_s$. The motion is therefore periodic because $0 < x(0) < \frac{1}{2}$ lies in the potential well.

**Solution to Exercise JS 1.18**

This is a conservative system with potential given by $V'(x) = -(x-\lambda)(x^2 - \lambda)$, so the fixed points are either centres or saddles.

If $\lambda = 0$ then $V'(x) = -x^3$ so $V(x) = -\frac{1}{4}x^4$ and the only fixed point is at the origin and is unstable, with phase curves being similar to those of a saddle. If $\lambda \neq 0$, but $|\lambda|$ is small the potential must be similar to this limiting case if $|x| \gg 1$.

If $0 < \lambda \ll 1$ there are three fixed points at $x = \lambda$, $\sqrt{\lambda}$ and $-\sqrt{\lambda}$. From the known shape of the potential the outer two must be maxima of the potential and consequently the middle point $x = \sqrt{\lambda}$ must be a minimum of the potential. Hence there are saddles at $(\lambda,0)$ and $(-\sqrt{\lambda},0)$ and a centre at $(\sqrt{\lambda},0)$.

If $\lambda < 0$ the only fixed point is at $(-\lambda,0)$ and since this is at the only stationary point of the potential which must be a maximum this must be a saddle.

Note, for this analysis it is not necessary to determine $V(x)$ explicitly.

**Solution to Exercise JS 1.20**

This is a conservative system with potential given by

$$
\frac{dV}{dx} = \begin{cases} 
0, & |x| < a, \\
-x - a \text{sgn}(x), & |x| \geq a,
\end{cases}
\quad \Rightarrow \quad V(x) = \begin{cases} 
\frac{1}{2}(x-a)^2, & x \geq a, \\
0, & |x| < a, \\
\frac{1}{2}(x+a)^2, & x \leq -a.
\end{cases}
$$

If $y = \dot{x}$ there is a line of fixed points $y = 0$, $|x| \leq a$; the phase curves comprise lines parallel to the $x$-axis for $|x| \leq a$ and for $|x| > a$.

$$
y^2 = \begin{cases} 
2E - (x-a)^2, & x > a, \\
2E - (x+a)^2, & x < -a.
\end{cases}
$$
that is segments of circles centred at \( x = \pm a \) with radius \( \sqrt{2E} \).

**Solution to Exercise JS 1.32**

Let \( x \) be the distance between the mass and the magnet, so using the cosine rule

\[
x(\theta)^2 = a^2 + h^2 - 2ah \cos \theta, \quad a - h \leq x \leq a + h,
\]

and the sine rule

\[
\sin \phi = \frac{a}{x} \sin \theta.
\]

The equation of motion can be written in the form

\[
\ddot{\theta} = -\omega^2 \left(1 - |k/x(\theta)|^3\right) \sin \theta,
\]

where \( \omega = \sqrt{g/a} \) and \( k = (hc/mg)^{1/3} \). The constant \( k \) has the dimensions of length and is a measure of the magnetic strength relative to the gravitational force.

This equation has the usual form for a conservative system: clearly there are equilibrium points at \( \theta = 0, \pi, \dot{\theta} = 0 \), and also at any real roots of \( x(\theta) = k \). If \( k > h + a \), strong magnet, or \( k < h - a \), weak magnet \( x(\theta) = k \) has no real roots so \( \theta = 0, \pi, \dot{\theta} = 0 \) are the only equilibrium points; consider motion near these equilibrium points first. For small \( \theta \), \( \sin \theta \simeq \theta \), and \( x^2 = (h - a)^2 + O(\theta^2) \), so

\[
\ddot{\theta} = -\omega^2 \left[1 - \left(\frac{k}{h - a}\right)^3\right] \theta + O(\theta^2).
\]

For weak magnets, \( k < h - a \), this equation is like that of the simple harmonic oscillator **JS** example 1.2, (page 10), so \( \theta = \dot{\theta} = 0 \) is a centre i.e. a stable equilibrium point. But if \( k > h - a \) the equation is like that treated in **JS** example 1.3, (page 11), so \( \theta = \dot{\theta} = 0 \) is an unstable saddle point.

For \( \theta \) near \( \pi \) put \( \theta = \pi - \chi \) to obtain

\[
\ddot{\chi} = \omega^2 \left[1 - \left(\frac{k}{h + a}\right)^3\right] \chi.
\]

Thus for strong magnets, \( k > h + a \), the point \( \dot{\theta} = 0, \theta = \pi \) is a centre, whilst if \( k < h + a \) it is an unstable saddle point.

If \( h - a < k < h + a \), magnets of intermediate strength, \( x(\theta) = k \) has two real roots, \( \pm \theta_0 \) with \( 0 < \theta_0 < \pi \), so there are two other equilibrium points at \( \theta = \pm \theta_0 \) and \( \dot{\theta} = 0 \). By symmetry both these points are of the same type. Put \( \theta = \theta_0 + \chi \) and expand in powers of \( \chi \) to obtain:

\[
\ddot{\chi} = -3ah \left(\frac{\omega}{k}\right)^2 \chi \sin^2 \theta_0.
\]

When this equilibrium point exists it is a centre.

Thus there are three possible types of motion corresponding to weak, intermediate and strong fields.

**Solution to Exercise JS 1.35**

Consider the equation

\[
\frac{d}{dt} \left(p(x) \frac{dx}{dt}\right) + h_1(x) = 0, \quad p(x) \neq 0,
\]
which may be re-written as
\[ \frac{d^2x}{dt^2} + \frac{p'(x)}{p(x)} \left( \frac{dx}{dt} \right)^2 + \frac{h_1(x)}{p(x)} = 0 \]

which is the same as the given equation if
\[ p(x) = \exp \left( \int dx g(x) \right) \quad \text{and} \quad h_1(x) = h(x)p(x). \]

Thus if we define a new variable \( z(x) \) by the equation
\[ \frac{dz}{dx} = p(x) \quad \text{so} \quad \frac{dz}{dt} = p(x) \frac{dx}{dt} \]

the equation of motion becomes
\[ \frac{d^2z}{dt^2} + h_1(x(z)) = 0, \]

which is a conservative system.

**Solution to Exercise JS 1.37**

One way to solve this equation is to write it in the form
\[ \frac{d}{dt} \left( \frac{dx}{dt} - \frac{1}{2} \epsilon x^2 \right) + x = 0. \]

Now define \( y = \dot{x} - \epsilon x^2/2 \) to give the equations
\[ \frac{dx}{dt} = y + \frac{1}{2} \epsilon x^2, \quad \frac{dy}{dt} = -x \quad \implies \quad \frac{dx}{dy} = -\frac{y}{x} - \frac{1}{2} \epsilon x. \]

Note that the first set of equations shows that there is a fixed point at the origin. The second equation can be written in the form
\[ e^{-\epsilon y/2} \frac{d}{dy} \left( xe^{\epsilon y/2} \right) = -\frac{y}{x} \]

so putting \( z = xe^{\epsilon y/2} \) and integrating gives \( z^2 + y^2 = c \), where \( c \) is a constant. In the \((z,y)\)-plane this solution defines a set of circles, and in the \((x,y)\)-plane these circles are transformed into closed curves. Hence the fixed point at the origin is a centre.

**Solution to Exercise JS 1.38**

This is a potential system with the potential given by
\[ \frac{dV}{dx} = x + \epsilon x^3 \quad \implies \quad V(x) = \frac{1}{2} x^2 + \frac{1}{4} \epsilon x^4. \]

The energy equation is, if \( y = \dot{x} \),
\[ \frac{1}{2} y^2 + \frac{1}{2} x^2 + \frac{1}{4} \epsilon x^4 = E. \]
For the contour passing through \((x, y) = (a, 0)\) we have
\[ E = \frac{1}{2}a^2 + \frac{1}{4}\epsilon a^4 = \frac{1}{2}a^2 \left( 1 + \frac{1}{2}\epsilon a^2 \right), \]
so the equation for the phase curve becomes
\[ y^2 + x^2 + \frac{1}{2}\epsilon x^4 = \frac{1}{2}a^2 \left( 1 + \frac{1}{2}\epsilon a^2 \right). \]

This potential has just one minimum at \(x = 0\) and no maxima (since \(\epsilon > 0\)); hence the origin is the only fixed point and it is a centre. Further the potential tends to infinity as \(|x| \to \infty\) so there is periodic motion for any \(E \geq 0\).

**Solution to Exercise JS 1.42**

If the satellite remains on the \(z\)-axis the only forces on it are due to the massive stars which, in this case may be considered stationary. Suppose that the stars are on the \(x\)-axis at the points \((\pm a, 0, 0)\) then the potential due each star is the same so the combined potential is
\[ V(z) = -\frac{2m\mu G}{\sqrt{z^2 + a^2}}, \]
where \(G\) is the Gravitational constant. This potential has a minimum at \(z = 0\). Hence the equation of motion is
\[ \frac{d^2 z}{dt^2} = -\frac{\partial V}{\partial z} = -\frac{2\mu G}{(a^2 + z^2)^{3/2}} \]
and the energy equation is
\[ \frac{1}{2}\dot{z}^2 - \frac{2m\mu G}{\sqrt{z^2 + a^2}} = E. \]

Expanding about \(z = 0\) gives
\[ \frac{1}{2}\dot{z}^2 + \frac{\mu G z^2}{a^4} = E - \frac{2\mu G}{a^2}. \]

So the origin is a centre of the nonlinear system, not just the linearised system.

This analysis suggests that the system is stable, but has ignored the effect of the satellite being slightly off-axis which, in practice, it would be. In addition the two massive stars would not be moving in circular orbits but ellipses. Thus we need to understand the effects of these two changes. The first is relatively easy to deal with but the second requires the Floquet analysis that is dealt with in **JS** chapter 9.

The analysis that follows uses methods that are not part of this course, but these can be found in any book on analytic dynamics or in MS323.

Let \(S\) be the coordinate system \(Oxyz\) that rotates with the massive stars and with the positions of the stars at \((\pm a, 0, 0)\). In a coordinate system \(S'\) with axes \(Ox'y'z'\) fixed in space with \(Oz\) along \(Oz'\) the stars are moving in a circle with a constant angular velocity \(\Omega z\). Coordinates in the two reference frames are connected by the linear relation
\[ x = R(t)x', \quad R = \begin{pmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
From the definition of \( R \) it follows that \( R(t)^{-1} = R(-t) \). In \( S \) the stars are at \((\pm a,0,0)\) in \( S' \) they are at \( R(t) = \pm a(\cos \Omega t, \sin \Omega t, 0) \).

The Hamiltonian of the satellite in \( S \) is

\[
H = \frac{1}{2m}p'(t)^2 + V(r', t)
\]

where the potential is

\[
V(r', t) = -\frac{m\mu G}{|r' - R(t)|} - \frac{m\mu G}{|r' + R(t)|}.
\]

In \( S \) the massive stars are stationary, so is likely that the equations of motion are simpler in this reference frame. The generating function for the transformation between coordinates is

\[
F_2(p, r') = px(x' \cos \Omega t - y' \sin \Omega t) + py(x' \sin \Omega t + y' \cos \Omega t) + pz',
\]

so that

\[
p_x' = \frac{\partial F_2}{\partial x} = px \cos \Omega t + py \sin \Omega t, \quad p_y' = \frac{\partial F_2}{\partial y} = -px \sin \Omega t + py \cos \Omega t,
\]

and \( p_z' = p_z \), or \( p = R(t)p' \). Also

\[
\frac{\partial F_2}{\partial t} = \Omega(xp_y - yp_x).
\]

The potential in \( S \) does not depend upon the time,

\[
V(x) = -m\mu G \left( \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right)
\]

and this reduces to the original potential if \( y = x = 0 \).

Now assume that \( x \) and \( y \) are small by comparison with \( a \) and expand the potential in powers of \( x \) and \( y \),

\[
V(x) = m\mu G \left( \frac{2}{\sqrt{a^2 + z^2}} - \frac{x^2 + y^2}{(a^2 + z^2)^{3/2}} \right).
\]

The Hamiltonian in the rotating reference frame is \( K = H + \partial F_2/\partial t \) and this becomes

\[
K = \frac{1}{2m}p_z^2 - \frac{2m\mu G}{\sqrt{a^2 + z^2}} + \frac{1}{2m}p_x^2 + \frac{m\mu G x^2}{(a^2 + z^2)^{3/2}} + \frac{1}{2m}p_y^2 + \frac{m\mu G y^2}{(a^2 + z^2)^{3/2}} + \Omega(xp_y - yp_x).
\]

Suppose \( z \) is also small, so we may set \( z = 0 \) in the second line of this Hamiltonian to de-couple the \( z \)-motion from the \((x, y)\)-motion the equations of motion which are

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial K}{\partial p_x} = \frac{1}{m}p_x - \Omega y, \\
\frac{dy}{dt} &= \frac{\partial K}{\partial p_y} = \frac{1}{m}p_y + \Omega x,
\end{align*}
\]

\[
\begin{align*}
\frac{dp_x}{dt} &= -\frac{\partial K}{\partial x} = -\frac{2m\mu G x}{a^3} - \Omega p_y, \\
\frac{dp_y}{dt} &= -\frac{\partial K}{\partial y} = -\frac{2m\mu G y}{a^3} + \Omega p_x.
\end{align*}
\]
Differentiating the first and second of these equations and eliminating the momenta and their derivatives we obtain identical equations for the $x$- and $y$-motion,

\[ \frac{d^2 w}{dt^2} + \left( \Omega^2 + \frac{\mu G}{a^3} \right) w = 0 \]

where $w$ is either $x$ or $y$. The $x$- and $y$-motion are identical because of symmetry.

Thus if the satellite is disturbed slightly from the $z$-axis the subsequent motion simply oscillates about the axis with frequency $\sqrt{\Omega^2 + \mu G a^{-3}}$, and remains close to the axis: that is motion close to the origin is stable.

If the massive stars are orbiting in elliptical orbits the analysis is more complicated because their rotation is not uniform. This means that in the reference frame rotating with the stars, or with their mean angular velocity, the Hamiltonian depends periodically on the time and a different type of analysis is required. For small eccentricities we should expect the motion to be stable, but clearly it is impossible for the satellite to stay on the $z$-axis.
Chapter 2

Plane autonomous systems and linearisation

2.1 Introduction

This chapter is mainly about the classification of the fixed points of two-dimensional autonomous systems. This theory is important and is covered in JS sections 2.3 to 2.5. An alternative development is provided in the course notes where it is shown that the classification depends only upon the nature of the eigenvalues of a $2 \times 2$ matrix. This alternative development is, in my view, simpler and easier to remember, but it is optional. However, it is important that you understand the linearisation theorem quoted on page 47 of the course notes.

JS Sections 2.1–2.5, 2.6 and 2.7 are used in M821.

2.2 Notes on set book: sections 2.1 to 2.5

Read the introductory text and section 2.1, pages 51–55.

Comments on text:

(i) Pages 51–52: There is some confusion at the top of page 52, caused largely by no distinction being made between the solutions of the original equation

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)$$  \hspace{1cm} (2.1)

and the derived equations

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)}.$$  \hspace{1cm} (2.2)

The solutions of equation N2.2 define curves in phase space and contain no information about either the direction or the rate of flow along a given curve. These lines in phase space are named phase curves or phase paths. Through each point $(x_0, y_0)$ provided at most one of $X(x_0, y_0)$ or $Y(x_0, y_0)$ is zero there is a unique

39
phase curve. The direction of flow along a phase curve can only be determined by reference to equations N2.1.

Solutions of equation N2.1 are functions of \( t \) and contain information about the direction and rate of flow, so have more information than the solutions of equation N2.2. It is therefore helpful to give these solutions different names so they are called *orbits* or *trajectories*. Of course, the solutions of equation N2.2 may be obtained from the solutions of N2.1 by eliminating \( t \).

For instance if

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -1
\]

the orbit starting at \((x_0, y_0)\) at \( t = t_0 \) is

\[
y = y_0 - (t - t_0), \quad x = x_0 + (t - t_0)y_0 - \frac{1}{2}(t - t_0)^2.
\]

These equations may also be written in the form

\[
\frac{dy}{dx} = \frac{1}{y}
\]

so the phase curve through \((x_0, y_0)\) is \( \frac{1}{2}(y^2 - y_0^2) = -(x - x_0) \). This relation may also be derived from equation N2.3 by eliminating \( t - t_0 \).

(ii) Page 52, singular point: The text uses this term in an unfortunate and potentially confusing manner. In the theory of differential equations there are at least two meanings associated with this term; these are described next.

In the theory of linear differential equations, for example

\[
p_2(t)\frac{d^2x}{dt^2} + p_1(t)\frac{dx}{dt} + p_0(t)x = 0,
\]

which may be written as a pair non-autonomous first-order equations, a singular point is defined to be at a zero of \( p_2(t) \). At such singular points the series solution of the equation takes a different form than at an ordinary point (where \( p_2(t) \neq 0 \)).

In the theory of autonomous first-order equations considered in this course,

\[
\frac{dx}{dt} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)
\]

singular points are defined to be the points where the vector field \( \mathbf{v}(\mathbf{x}) \) vanishes and in this context the term singular point is a synonym for fixed point: an *ordinary point* is where \( \mathbf{v}(\mathbf{x}) \neq 0 \). With this definition the components of \( \mathbf{v} \) have no singularities at a singular point and are generally differentiable. The term *singular point* stems from the fact that the *direction* of the velocity field changes near such a point, in general discontinuously.

(iii) Page 55, after equation (2.9): It is stated that the *time-scale is implicit in the phase diagram*. This is not true, as may be seen by noting that the equations

\[
\dot{x} = X(x, y), \quad \dot{y} = Y(x, y), \quad \text{and} \quad \dot{x} = f(t)X(x, y), \quad \dot{y} = f(t)Y(x, y),
\]

have identical phase curves yet move along them at different rates.

**Do exercise 2.11**
2.3 An alternative proof of existence and uniqueness

Here we present a simpler proof of the existence and uniqueness theorem stated in the appendix of JS (page 530). Consider the autonomous system \( \dot{x} = v(x) \): in the neighbourhood of an ordinary point \( x_0 \) it can be proved that there exists a differentiable change of coordinates \( u = x(u) \) such that the equations of motion become \( \dot{u} = (1, 0) \), provided \( v(x) \) is smooth and has an inverse which is also smooth in the neighbourhood; in particular this means that the neighbourhood must not contain any fixed points. Thus local phase portraits at ordinary points are all qualitatively equivalent.

The existence of solutions in the neighbourhood of ordinary points follows from this result: the solution of the equation \( \dot{u} = (1, 0) \) is \( (t) = (a_1 + t, a_2) \) for \( |t| \) sufficiently small. Uniqueness also follows, for if \( \phi_1 \) and \( \phi_2 \) are two solutions satisfying the same initial conditions then \( \phi_1 - \phi_2 = 0 \). At a fixed point \( v(x_0) = 0 \) the solution through \( x_0 \) is \( \phi(t) = x_0 \). Existence and uniqueness follows. Note that these results are also true for systems of order \( n \) where \( x = (x_1, x_2, \cdots, x_n) \).

Counter examples are always helpful in understanding the value of such theorems. The first-order system \( \dot{x} = |x|^{1/2}, x(0) = 0 \), has two solutions \( x(t) = 0 \) and \( x(t) = t^2/4 \) because the velocity function is not differentiable at the origin.

The reader will have noted that the above results are valid only for short, or at most finite, times and in many applications we are interested in the long-time behaviour of solutions. If the velocity function \( v(x) \) is defined on a compact manifold then the system has a unique solution for all real times. Examples of compact manifolds are a circle, a sphere, and a torus and in these cases there can be no terminating motion, that is there are no solutions \( x(t) \) such that \( |x(t)| \to \infty \) for some finite \( t \).

2.4 Notes on set book: sections 2.2 to 2.5

Read section 2.2, pages 55–59.

Do exercise 2.16.

Read section 2.3–2.5, pages 59–75.

Comment on text:

(i) Page 60: Equations (2.22) are often called the linearised equations of the system.

(ii) Page 60: Equations (2.23) should be

\[
\dot{x} = ax + by, \quad \dot{y} = cx + dy.
\]

(iii) Page 62, line 11: \( x_i \) and \( y_i \) should be \( x_1 \) and \( x_2 \), respectively.

(iv) Page 64: Equations (2.31) should be

\[
(a - \lambda)r + bs = 0, \quad cr + (d - \lambda)s = 0.
\]

The analysis in JS section 2.3–2.5 deals only with linear systems. In the following section of the course notes we show how this analysis is applied to nonlinear systems and provide an alternative more unified approach to this linear analysis.
2.5 Classification of the fixed points of a nonlinear system

In order to understand the behaviour of a system it is necessary to understand how it behaves in the neighbourhood of the fixed points because the nature and distribution of these points play a significant role in controlling the overall behaviour of the system. This type of analysis was first performed by Poincaré (1881) in his early work which was a prelude to his profound analysis of the three body problem in which chaotic motion was first discovered.

In order to understand and classify the motion in the vicinity of fixed points it is necessary to make some assumptions about the nature of the velocity function, otherwise there are too many possibilities. We assume that its Taylor series expansion about any fixed point of interest exists and that all second-order terms may be ignored. The first assumption is satisfied by many physical system that occur in practice; we shall discuss the implications of the second assumption below.

Consider the autonomous system

\[
\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)
\]

and assume that there is fixed point at \( x = r_f \), so \( X(r_f) = Y(r_f) = 0 \). Expanding the velocity function, \( v = (X, Y) \), about the fixed point gives

\[
X(x - r_f) = (x - x_f) \frac{\partial X}{\partial x} + (y - y_f) \frac{\partial X}{\partial y} + O(|x - r_f|^2),
\]

\[
Y(x - r_f) = (x - x_f) \frac{\partial Y}{\partial x} + (y - y_f) \frac{\partial Y}{\partial y} + O(|x - r_f|^2),
\]

where all derivatives are evaluated at the fixed point. We obtain the linearised equations of motion of the nonlinear equations \( \dot{x} = v(x) \) by ignoring the second order terms:

\[
\frac{dz}{dt} = Az, \quad z = x - r_f, \quad A = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix},
\]

(2.4)

the elements of \( A \) being evaluated at the fixed point. It is convenient to write

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

(2.5)

where all the elements are real. The matrix \( A \) is named the linearisation matrix; this is the linearised system given in J9 equation 2.22 (page 60).

A simple fixed point is one for which the determinant of the matrix \( A \) is nonzero. In the neighbourhood of a simple fixed point we shall show that the shape of the phase curves (of the linear system) depends only upon the nature of the eigenvalues of \( A \) which, in turn, depend only upon the trace and determinant of \( A \). This is seen by transforming to a new coordinate system \( u = Mz \) where \( M \) is a constant, non-singular \( 2 \times 2 \) matrix; if \( \det(M) > 0 \) the
transformation is orientation preserving. In this coordinate system the linearised equations of motion, equation N2.4, become
\[ \frac{du}{dt} = Bu, \quad B = MAM^{-1}. \] (2.6)
The matrix \( M \) can always be chosen to cast \( B \) into one of three distinct types, the actual form being determined only by the eigenvalues of \( A \),
\[ \lambda = \frac{1}{2} \left[ \text{Tr} \left( A \right) \pm \sqrt{\text{Tr} \left( A \right)^2 - 4 \text{det} \left( A \right)} \right], \] (2.7)
where the trace and determinant are
\[ \text{Tr} \left( A \right) = a + d, \quad \text{det} \left( A \right) = ad - bc. \]
Each of these three types produces distinctly different shaped phase curves.

**Type 1: Eigenvalues of \( A \), \( \lambda_1 \) and \( \lambda_2 \), real and distinct**

In this case it is convenient to label the eigenvalues so that \( \lambda_2 > \lambda_1 \) and to choose \( M \) to be
\[ M = \begin{pmatrix} c & \lambda_1 - a \\ |c| & (\lambda_2 - a) \text{sgn}(c) \end{pmatrix} \quad \text{giving} \quad B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \] (2.8)
With this choice \( \det(M) > 0 \). The linearised equations of motion in the \( u \)-representation are
\[ \dot{u}_1 = \lambda_1 u_1, \quad \dot{u}_2 = \lambda_2 u_2, \]
with solutions \( u_1 = \alpha e^{\lambda_1 t}, \quad u_2 = \beta e^{\lambda_2 t} \), where \( \alpha \) and \( \beta \) are constants, from which \( t \) may be eliminated to give the equation \((u_1/\alpha)^{\lambda_2} = (u_2/\beta)^{\lambda_1}\) for the phase curves. There are now three situations to consider depending upon the signs of the eigenvalues.

- If both eigenvalues are positive, \( \lambda_2 > \lambda_1 > 0 \) both \( u_1(t) \) and \( u_2(t) \) increase with increasing \( t \) and the fixed point is unstable; this type of fixed point is named an *unstable node* and some typical phase curves are shown in *JS* figure 2.5b (page 68).
- If both eigenvalues are negative the shape of the phase curves are similar but the flow is in the opposite direction, so the fixed point is strongly stable and is named a *stable node*. An example of this type of flow, for the case \( \lambda_1 < \lambda_2 < 0 \), is shown in *JS* figure 2.5a.
- If the eigenvalues have opposite sign, \( \lambda_1 < 0 < \lambda_2 \), the \( u_1 \) sub-system is stable, the \( u_2 \) sub-system is unstable hence the fixed point is unstable but not strongly unstable. Now the phase curves are given by an equation of the form \( u_1^{\lambda_2} u_2^{\lambda_1} = \) constant and look like generalised hyperbolae, as shown in *JS* figures 2.6 and 2.7 (pages 69-70). This type of fixed point is named a *saddle point* or, in some texts, a *hyperbolic fixed point*.

The eigenvalues determine the nature of the fixed point, but for saddles and nodes it is often useful to compute the eigenvectors as these show the direction of the motion that flows directly away or towards the fixed point; further the orbits that leave the fixed point along these eigenvectors can determine important boundaries in phase space.
CHAPTER 2. PLANE AUTONOMOUS SYSTEMS AND LINEARISATION

Type 2: Complex eigenvalues, \( \lambda = \alpha + i\omega, \lambda^* = \alpha - i\omega, \sgn(\omega) = \sgn(c) \)

In this case we take the transformation matrix \( M \), equation N2.6, to be

\[
M = \begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix}
\]
giving \( B = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} = |\lambda| \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \)

(2.9)

where \( \phi \) is the phase of the eigenvalue \( \alpha + i\omega = \sqrt{\alpha^2 + \omega^2} e^{i\phi} \). The matrix \( B \) represents a rotation through the angle \( \phi \) and a uniform scaling which is a contraction if \( |\lambda| < 1 \) and an expansion if \( |\lambda| > 1 \).

This geometric interpretation of \( B \) suggests that the equations of motion \( \dot{u} = Bu \) would be simpler if expressed in terms of the polar coordinates, \( u = (r \cos \theta, r \sin \theta) \). In this representation the equations of motion are,

\[
\frac{dr}{dt} = \alpha r, \quad \frac{d\theta}{dt} = \omega,
\]

(2.10)

and have the solution.

\[
r(t) = r_0 e^{\alpha t}, \quad \theta(t) = \omega t + \theta_0.
\]

Thus if the real part of the eigenvalue, \( \alpha \), is negative the fixed point is strongly stable. If, in addition, \( \omega > 0 \) the phase point rotates anticlockwise as shown in JS figure 2.9a. This type of fixed point is named a stable spiral, or in some texts a focus.

Changing the sign of \( \omega \), that is \( c \), changes the direction of motion, but not the stability of the fixed point.

On the other hand changing the sign of \( \alpha \) changes stability of the fixed point and if \( \alpha > 0 \) it is an unstable spiral.

If \( \alpha = 0 \), so that \( r(t) = \text{constant} \) then the phase curves in the \( u \)-representation are circles, with the motion clockwise if \( \omega < 0 \): these circles become ellipses in the original \( r \)-representation, as seen in JS figure 2.8 (page 72). This type of fixed point is named a centre or an elliptic fixed point.

Type 3: Eigenvalues real and equal

The eigenvalues are real and equal if

\[
\Tr(A)^2 = 4 \det(A) \quad \text{that is} \quad (a - d)^2 + 4bc = 0,
\]

(2.11)

and this equation can be satisfied in two ways.

First, if \( b = c = 0 \) and \( a = d \), \( A \) is already diagonal and the linearised equations of motion, N2.4, are

\[
\dot{z}_1 = az_1, \quad \dot{z}_2 = az_2, \quad \text{with solutions} \quad z = z_0 e^{at}.
\]

Thus both \( z_1 \) and \( z_2 \) increase, \( (a > 0) \), or decrease, \( (a < 0) \), at the same rate. All phase curves are therefore the straight lines \( z_1 = \alpha z_2 \) as shown in the figure:
2.5. CLASSIFICATION OF THE FIXED POINTS OF A NONLINEAR SYSTEM

This type of fixed point is named a *stable star* if \( a = d < 0 \) and an *unstable star* if \( a = d > 0 \): in the latter case the phase curves are the same as shown in the figure but the direction of the arrows is reversed.

The second way in which equation N2.11 can be satisfied is if at least one of \( b \) or \( c \) is nonzero, then \( A \) has only one linearly independent eigenvector. The matrix \( M \) may be chosen to be

\[
M = \begin{pmatrix} a - d & 2b \\ 2c & 0 \end{pmatrix}
\]

\[
giving \quad B = \begin{pmatrix} \tilde{a} & 0 \\ c & \tilde{a} \end{pmatrix}, \quad \tilde{a} = \frac{1}{2}(a + d)
\]

when \( c \neq 0 \). The equations of motion, N2.4, are now

\[
\dot{z}_1 = \tilde{a}z_1, \quad \dot{z}_2 = cz_1 + \tilde{a}z_2,
\]

and have the solution

\[
z_1 = \alpha e^{\tilde{a}t}, \quad z_2 = (\beta + \alpha t)e^{\tilde{a}t}
\]

for some constants \( \alpha = z_1(0) \) and \( \beta = z_2(0) \).

If \( \text{Tr}(A) = 2\tilde{a} < 0 \) the fixed point is stable and for sufficiently large times the \( \beta \) term may be neglected to give

\[
z_1 = \alpha e^{\tilde{a}t}, \quad z_2 = c\alpha e^{\tilde{a}t} \quad \text{or} \quad z_2 = \frac{c}{|\tilde{a}|}z_1 \ln(\alpha/z_1), \quad \tilde{a} < 0, \quad \frac{z_1}{\alpha} > 0.
\]

If \( c > 0 \) the phase curve crosses the \( z_1 \)-axis at \( z_1 = \alpha \) and approaches the origin along the positive \( z_2 \)-axis as \( t \to \infty \). As \( t \to -\infty \), \( z_2 \) becomes negative so the phase curves go to infinity in the fourth quadrant, as shown in figure N2.2.
If $c > 0$ and $a = z_1(0) < 0$ the phase curve approaches the origin along the positive $z_2$-axis as $t \to \infty$. As $t \to -\infty$, the phase curves go to infinity in the second quadrant. If $a < 0$ and $c < 0$ the fixed point remains stable and the phase curves look like those shown in figure N2.3. This type of fixed point is named a stable improper node if $\text{Tr}(A) < 0$ or an unstable improper node if $\text{Tr}(A) > 0$.

If $a > 0$ the fixed point is unstable but the phase curves have the same shape as those shown in the above figures, though the direction of the arrows is reversed.

### 2.5.1 Summary of Classifications

The nature of the phase curves in the neighbourhood of a simple fixed point depends only upon the eigenvalues of the linearisation matrix $A$, defined in the course notes, equation N2.4, (page 42). We have seen that there are ten distinctly different types of flow.

**Eigenvalues, real and different:** $\text{Tr}(A)^2 > 4 \det(A)$

- **Stable Node** Eigenvalues real and negative.
- **Unstable Node** Eigenvalues real and positive.
- **Saddle** Eigenvalues real and of different sign.

**Eigenvalues complex:** $\text{Tr}(A)^2 < 4 \det(A)$

- **Stable spiral** Real part of eigenvalues negative.
- **Unstable spiral** Real part of eigenvalues positive.
- **Centre** Eigenvalues imaginary.

**Eigenvalues the same:** $\text{Tr}(A)^2 = 4 \det(A)$

- **Stable star** $A$ is diagonal, $b = c = 0$ and $\text{Tr}(A) < 0$.
- **Unstable star** $A$ is diagonal, $b = c = 0$ and $\text{Tr}(A) > 0$.
- **Stable improper node** $A$ is not diagonal and $\text{Tr}(A) < 0$.
- **Unstable improper node** $A$ is not diagonal and $\text{Tr}(A) > 0$. 
2.5. CLASSIFICATION OF THE FIXED POINTS OF A NONLINEAR SYSTEM

This completes the classification of simple fixed points. The most important point of this analysis is that the nature of the linearised flow depends only upon the values of the eigenvalues of the linearisation matrix \( A \), defined in equation N2.4, which in turn depend only upon the trace and determinant of \( A \).

The classification scheme described above used the linear system, \( \dot{z} = Az \), equation N2.4, (page 42), obtained by ignoring all terms \( O(|x - r_f|^2) \). We need to know whether this classification is valid for the original nonlinear equations \( \dot{x} = v(x) \). The linearisation theorem provides the crucial link between the flows of these two related systems.

**Linearisation Theorem** If the nonlinear system

\[
\dot{x} = v(x)
\]

has a simple fixed point at \( r_f \) then there is a neighbourhood of \( r_f \) in which the phase portrait of the system and its linearisation are qualitatively equivalent provided that the fixed point of the linearised system is not a centre.

This theorem is important: its proof may be found in Hartman (1964, chapter 8).

This theorem provides the justification for ignoring the second-order terms in equation N2.4 and it means that a fixed point of the nonlinear system is stable whenever the linearised system is strongly stable and unstable whenever the linearised system is unstable.

If the fixed point is a centre for the linearised system then \( \text{Tr}(A) = 0 \) and any small perturbation, such as the ignored nonlinear terms, could make \( \text{Tr}(A) \) nonzero and hence change the centre into a spiral. For centres more investigation is needed to determine the true nature of the fixed point. Thus when stating that a fixed point is a centre it is essential to also state whether the classification is for the original system or its linearisation.

**Exercise N2.1**

Show that the origin is a centre of the linearisation of the system

\[
\dot{x} = y - x(x^2 + y^2), \quad \dot{y} = -x - y(x^2 + y^2).
\]

Show that in the polar coordinates, \( x = r \cos \theta \), \( y = r \sin \theta \), the equations of motion are

\[
\dot{r} = -r^3, \quad \dot{\theta} = -1,
\]

and hence deduce that the fixed point at the origin is not a centre of the nonlinear system but is stable.

**Exercise N2.2**

Explain why a linear system cannot have limit cycle.

Do exercises: 2.1 (only the classification is required), 2.3\{i,ii\} (you are not required to sketch the phase curves).
2.6 Notes on set book: sections 2.7 and 2.8

Read section 2.7, pages 76–78.

Comments on text:

(i) The text suggests that once the nature of the fixed points has been ascertained then it is relatively easy to deduce the global behaviour of the phase flow. For simple systems this may be true, but in most cases it is not and usually the use of a computer helps considerably.

If the system has several fixed points it is usually more efficient to compute the linearisation matrix $A$ as a function of $(x, y)$,

$$A(x, y) = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix},$$

and then substitute the values of $(x, y)$ at the fixed point, rather than expand about each fixed point separately.

Read section 2.8, pages 78–81.

Comments on text:

(i) The Hamiltonian may also be a function of $t$ and the equations of motion are the same. In this case, however, the Hamiltonian is not constant along a phase curve:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \left( \frac{\partial H}{\partial y} \right) + \frac{\partial H}{\partial y} \left( -\frac{\partial H}{\partial x} \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

When the Hamiltonian is independent of $t$, it is an integral of the motion.

(ii) The fact that the fixed points of an autonomous Hamiltonian system,

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x},$$

are either centres or saddles is because

$a)$ the fixed points are at the points where the Hamiltonian, $H(x, y)$, is stationary, and

$b)$ because the contours of a function of two variables near a stationary point are either ellipses or hyperbolae.

(iii) An autonomous (two-dimensional) Hamiltonian system is always integrable. However, not all integrable systems are Hamiltonian. For instance the Volterra equation (JS page 55)

$$\frac{dx}{dt} = (1 - ay)x, \quad \frac{dy}{dt} = -(1 - x)y$$

is integrable, with the integral

$$F(x, y) = \ln(xy) - ay - x$$
but it is not Hamiltonian. This is easily shown by noting that if it were Hamiltonian then there would exist a function $H(x,y)$ such that

$$(1 - ay)x = \frac{\partial H}{\partial y} \quad \text{and} \quad (1 - y)x = \frac{\partial H}{\partial x}.$$ 

Differentiating the first of these equations with respect to $x$ and the second with respect to $y$ gives

$$\frac{\partial^2 H}{\partial x \partial y} = -ay \quad \text{and} \quad \frac{\partial^2 H}{\partial y \partial x} = -x.$$ 

Hence $H(x,y)$ does not exist.

(iv) Page 81, equation (v): This should be

$$H(x,y) = -12 + \frac{13}{2} \left( x^2 + y^2 \right) - \frac{1}{4} (x^4 + y^4) - \frac{1}{2} x^2 y^2 = C.$$ 

(v) Page 81, line 12: The second derivative $H_{xx}$ should be

$$\frac{\partial^2 H}{\partial x^2} = 13 - 3x^2 - y^2.$$ 

(vi) Page 81, Fig 2.13: All the directional arrows in this phase diagram are pointing in the opposite direction to the flow.

**Exercise N2.3**

Determine which of the following systems are Hamiltonian.

(i) $\dot{x} + \partial V/\partial x = 0$,

(ii) $\dot{x} + \nu(1 - x^2)\dot{x} + x = 0$, $\nu \neq 0$,

(iii) $\dot{x} = y(1 + x^2)$, $\dot{y} = -xy^2 - \partial V/\partial x$,

where $V(x,t)$ is a function of $x$ and $t$ only.

**Exercise N2.4**

(i) Show that with $y = \dot{x}$ the damped linear oscillator

$$\frac{d^2 x}{dt^2} + \alpha \frac{dx}{dt} + \omega^2 x = 0, \quad \alpha \geq 0$$

is a Hamiltonian system only if $\alpha = 0$.

(ii) Show that the equations of motion for the Hamiltonian

$$H = \frac{1}{2} e^{-\alpha t} y^2 + \frac{1}{2} \omega^2 e^{\alpha t} x^2$$

can be written in the form $\ddot{x} + \alpha \dot{x} + \omega^2 x = 0$. 

Exercise N2.5

Derive the equations of motion for the Hamiltonian

\[ H(x, y) = \frac{y^2}{2f(x)} + V(x) \]

and hence show that, with a suitable choice of \( y, f(x) \) and \( V(x) \), the equation of motion

\[ \ddot{x} + G(x)\dot{x}^2 + F(x) = 0 \]

may be written in Hamiltonian form.

Exercise N2.6

The motion of a system is defined by the Hamiltonian \( H(x, y) \).

(i) Show that if a function \( f(x, y) \) can be expressed as a differentiable function of \( H \), then the value of \( f(x, y) \) is a constant along all phase curves.

(ii) If \( g(x, y) \) is constant along all phase curves, show that it may be expressed as a function of \( H \).

Exercise N2.7

Locate and classify, where possible, the fixed points and find the equations of the phase paths for the following systems,

(i) \( \dot{x} = x + y, \; \dot{y} = x - y + 1; \)
(ii) \( \dot{x} = x + y, \; \dot{y} = x + y + 2; \)
(iii) \( \dot{x} = y^3, \; \dot{y} = x^3; \)
(iv) \( \dot{x} = \sin y, \; \dot{y} = \cos x; \)
(v) \( \dot{x} = y^3, \; \dot{y} = y(x + y^3). \)

Note, the systems of parts (i), (iii) and (iv) are Hamiltonian.

Exercise N2.8

Locate and classify the fixed points of the following systems,

(i) \( \dot{x} = -6y + 2xy - 8, \; \dot{y} = y^2 - x^2; \)
(ii) \( \dot{x} = -2x - y + 2, \; \dot{y} = xy; \)
(iii) \( \dot{x} = 4 - 4x^2 - y^2, \; \dot{y} = 3xy; \)
(iv) \( \dot{x} = \sin y, \; \dot{y} = -(x + 2x^3). \)

Do exercises: 2.2\{i,ii\}, 2.4\{i,v\}, 2.7, 2.8, 2.12, 2.23, 2.29, 2.33.
Revision exercises: 2.14, 2.27, 2.43, 2.45, 2.46.
Solutions for exercises from chapter 2

Solution for exercises set in Course Notes

Solution to Exercise N2.1

The linearised system is

\[
\frac{dx}{dt} = Ax \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The matrix \( A \) has eigenvalues \( \lambda = \pm i \), so the linearised system has a centre.

Since \( r^2 = x^2 + y^2 \) and \( \tan \theta = y/x \) we have

\[
r\dot{r} = x \left\{ y - xr^2 \right\} + y \left\{ -x - yr^2 \right\} = -r^4
\]

and

\[
\dot{\theta} = \frac{1}{r^2} (xy - y\dot{x}) = -1.
\]

Since \( \dot{r} < 0 \) for \( r > 0 \) and \( \dot{\theta} = \text{constant} \) the phase curves spiral into the origin. Further, since, for small \( r \), \( r^3 \ll r \) the rate at which the phase curves approach the origin is slower than in the case of a conventional stable spiral.

Solution to Exercise N2.2

A limit cycle is an isolated periodic orbit, so we may use the linearity of the system to prove that it cannot exist.

Suppose that \( u(t) = (x(t), y(t)) \) is a limit cycle. Then for any real number \( \alpha \) the function \( \alpha u(t) \) is also a solution of the linear differential equations. Hence a periodic solution, if it exists, is not isolated.

Solution to Exercise N2.3

(i) Put \( y = \dot{x} \) give

\[
\dot{x} = y, \quad \dot{y} = -\frac{\partial V}{\partial x}
\]

so this system is Hamiltonian with the Hamiltonian \( H = \frac{1}{2}y^2 + V(x, t) \).

(ii) Putting \( y = \dot{x} \) as in part (i) gives

\[
\dot{x} = y = X(x, y), \quad \dot{y} = -\nu(1 - x^2)y - x = Y(x, y)
\]

This system is not Hamiltonian because

\[
\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \nu(1 - x^2)
\]

is zero for all \( x \) only if \( \nu = 0 \).

This analysis shows that the system is not Hamiltonian in these variables. The original system has a fixed point at the origin and this is either a stable spiral (\( 0 < \nu < 2 \)), or a stable node (\( \nu > 2 \)). Since Hamiltonian systems have only centres or saddles this suggests that no variable \( y = f(x, \dot{x}) \) can be found that makes this system Hamiltonian. However, the
example in exercise N2.4 shows that care is needed and that a time-dependent transformation that makes this system Hamiltonian may exist. I know of no proof that ensures either the existence or the nonexistence of such a transformation.

(iii) Assume that a Hamiltonian exists, then

\[ \frac{\partial H}{\partial y} = y(1 + x^2) \quad \Rightarrow \quad \frac{\partial^2 H}{\partial y \partial x} = 2xy \]

and

\[ \frac{\partial H}{\partial x} = xy^2 + \frac{\partial V}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 H}{\partial x \partial y} = 2xy. \]

Integration of the first equation gives

\[ H = \frac{1}{2} y^2 (1 + x^2) + f(x,t) \]

where \( f(x,t) \) is a function of \( x \) and \( t \) only. By differentiating this with respect to \( x \) we see that \( f(x,t) \) may be chosen to be \( V(x,t) \) and hence the Hamiltonian is

\[ H = \frac{1}{2} y^2 (1 + x^2) + V(x,t). \]

**Solution to Exercise N2.4**

(i) Suppose that a Hamiltonian \( H(x,y) \) exists, then

\[ X(x,y) = y = \frac{\partial H}{\partial y}, \quad Y(x,y) = -\alpha y - \omega^2 x = -\frac{\partial H}{\partial x}. \]

The first relation gives

\[ \frac{\partial^2 H}{\partial x \partial y} = 0 \quad \text{and the second gives} \quad \frac{\partial^2 H}{\partial x \partial y} = \alpha. \]

Hence the Hamiltonian exists only if \( \alpha = 0 \) and then it is \( H = \frac{1}{2} (y^2 + \omega^2 x^2) \).

(ii) We have

\[ \dot{x} = \frac{\partial H}{\partial y} = e^{-\alpha t} y, \quad \dot{y} = -\frac{\partial H}{\partial x} = -\omega^2 e^{\alpha t} x. \]

Hence

\[ \dot{x} = e^{-\alpha t} \dot{y} - \alpha e^{-\alpha t} y = e^{-\alpha t} (-\omega^2 e^{\alpha t} x) - \alpha e^{-\alpha t} (\dot{x} e^{\alpha t}), \]

or \( \dot{x} + \alpha \dot{x} + \omega^2 x = 0 \).

**Solution to Exercise N2.5**

The equations of motion are

\[ \dot{x} = \frac{\partial H}{\partial y} = \frac{y}{f(x)}, \quad \dot{y} = -\frac{\partial H}{\partial x} = \frac{y^2 f'(x)}{2f(x)^2} - V'(x) \]

so that

\[ \frac{d}{dt} (xf(x)) - \frac{1}{2} \dot{x}^2 f'(x) + V'(x) = 0, \]
or
\[ \ddot{x} + \frac{1}{2} x^2 \frac{f'(x)}{f(x)} + \frac{V'(x)}{f(x)} = 0. \]

By comparing this with the original equation we find
\[ \frac{1}{2} \frac{f'(x)}{f(x)} = G(x) \implies f(x) = \exp \left( 2 \int dx \, G(x) \right) \quad \text{and} \quad V(x) = \int dx \, f(x) F(x). \]

**Solution to Exercise N2.6**

(i) If \( F(z) \) is a differentiable function and \( g(x, y) = F(H(x, y)) \) then
\[
\frac{dg}{dt} = F'(H) \frac{\partial H}{\partial x} + F'(H) \frac{\partial H}{\partial y} = \left( \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \right) F'(H) = 0.
\]

(ii) If \( g(x, y) \) is a constant of the motion then
\[
\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial g}{\partial y} \frac{\partial H}{\partial x} = \frac{\partial (g, H)}{\partial (x, y)} = 0.
\]

Since the Jacobian of \( g \) and \( H \) is identically zero it follows that \( g \) and \( H \) are functionally related.

**Solution to Exercise N2.7**

(i) The fixed point is at \((-1/2, 1/2)\): expansion about the fixed point gives \( \dot{\xi} = \xi + \eta \), \( \dot{\eta} = \xi - \eta \), with \( A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) and eigenvalues \( \lambda = \pm \sqrt{2} \), so the fixed point is a saddle. In this case the system is Hamiltonian so \( \xi + \eta = \partial H / \partial \eta \) and \( \xi - \eta = -\partial H / \partial \xi \), for some Hamiltonian \( H(\xi, \eta) \). The phase curves are the contours of the Hamiltonian, which direct integration gives as,
\[
2H(\xi, \eta) = \eta^2 - \xi^2 + 2\xi \eta = \left( y - \frac{1}{2} \right)^2 - \left( x + \frac{1}{2} \right)^2 + 2 \left( y - \frac{1}{2} \right) \left( x + \frac{1}{2} \right),
\]
the contours of which are hyperbolas with asymptotes \( \xi = (1 \pm \sqrt{2})\eta \).

(ii) This system has no fixed point because \( \dot{y} = \dot{x} + 2 \) so both \( \dot{x} \) and \( \dot{y} \) cannot simultaneously be zero; integration gives \( y = x + 2t + a \) and,
\[
x = -\frac{1}{2} (a + 1) - t + b \exp(2t), \quad y = \frac{1}{2} (a - 1) + t + b \exp(2t)
\]
a and \( b \) being constants.

(iii) The only fixed point is at \((0,0)\), but as this is not a simple fixed point the usual classifications are not appropriate. The system is Hamiltonian with the Hamiltonian function
\[
H(x, y) = \frac{1}{4} (y^4 - x^4),
\]
and the phase curves are the contours of \( H(x, y) \).
CHAPTER 2. PLANE AUTONOMOUS SYSTEMS AND LINEARISATION

For \( c = 0 \) this gives the straight lines \( y = \pm x \), with the arrows as shown in figure N2.4, but note the motion along these lines is different than for a conventional saddle: a point initially at \((a, -a)\), \(a < 0\) subsequently moves to \((x(t), -x(t))\) where \(x(t) = a/\sqrt{1 + 2a^2t}\), so approaches the origin as \(t^{-1/2}\) not as \(e^{-t}\) as for a saddle; the approach is much slower because \(x^3 \ll x\) for small \(x\).

![Figure N2.4 Contour plot of the Hamiltonian \(H = (y^4 - x^4)/4\).](image)

The contours of this Hamiltonian should be compared with the saddle shown in JS figure 2.6, (page 69).

(iv) The fixed points are at \(((m + 1/2)\pi, n\pi)\), \(n\) and \(m\) being integers. Then \(\text{Tr}(A) = 0\) and \(\det(A) = (-1)^{n+m}\), so that for odd \(n + m\) the fixed points are saddles and for even \(n + m\) they are centres.

This system is Hamiltonian so the phase curves are given by the contours of the Hamiltonian,

\[H(x, y) = \cos y + \sin x\]

which are shown in figure N2.5.

![Figure N2.5 Contour plot of the Hamiltonian, \(H = \cos y + \sin x\)](image)

(v) All points on the \(x\)-axis are fixed points and these are the only fixed points. The fixed points are not isolated so the system is not typical. The equation for the phase curves is,

\[\frac{dy}{dx} = \frac{y(x + y^3)}{y^3}\]
or, on putting \( z = y^3 \),
\[
\frac{dz}{dx} - 3z = 3x,
\]
which has the solutions,
\[
y(x)^3 = Ae^{3x} - x - 1/3,
\]
for some constant \( A \). Some typical curves for various values of \( A \) are shown in figure N2.6.

![Figure N2.6](image-url)  
**Figure N2.6** Graph of the function \( y^3 = Ae^{3x} - x - 1/3 \), for various values of \( A \).

**Solution to Exercise N2.8**

(i) The fixed points are at the roots of,
\[
y = x, \quad x^2 - 3x - 4 = 0, \quad (2.12)
y = -x, \quad x^2 - 3x + 4 = 0. \quad (2.13)
\]
Equation N2.12 gives \( x = 4, 4 \) and \( -1, -1 \) and equation N2.13 has no real roots. The linearisation matrix is
\[
A(x, y) = 2 \begin{pmatrix} y & x - 3 \\ -x & y \end{pmatrix}
\]
so we have,
\[
\begin{align*}
x &= (4, 4), & \det(A) &= 80, & \text{Tr}(A) &= 16, & \lambda &= 8 \pm 4i, & \text{Unstable spiral}, \\
x &= (-1, -1), & \det(A) &= 20, & \text{Tr}(A) &= -4, & \lambda &= -2 \pm 4i, & \text{Stable spiral}.
\end{align*}
\]

(ii) The fixed points are at \( x = (0, 2) \) and \( (1, 0) \) and \( A = \begin{pmatrix} -2 & -1 \\ y & x \end{pmatrix} \) so we have,
\[
\begin{align*}
x &= (0, 2), & \det(A) &= 2, & \text{Tr}(A) &= -2, & \lambda &= -1 \pm i, & \text{Stable spiral}, \\
x &= (1, 0), & \det(A) &= -2, & \text{Tr}(A) &= -1, & \lambda &= 1, -2, & \text{Saddle}.
\end{align*}
\]

(iii) The fixed points are at \( (0, \pm 2) \) and \( (\pm 1, 0) \) and \( A = \begin{pmatrix} -8x & -2y \\ 3y & 3x \end{pmatrix} \) so we have,
\[
\begin{align*}
x &= (0, 2), & \det(A) &= 24, & \text{Tr}(A) &= 0, & \lambda &= \pm i\sqrt{24}, & \text{Centre}, \\
x &= (0, -2), & \det(A) &= 24, & \text{Tr}(A) &= 0, & \lambda &= \pm i\sqrt{24}, & \text{Centre}, \\
x &= (1, 0), & \det(A) &= -24, & \text{Tr}(A) &= -5, & \lambda &= -8, 3, & \text{Saddle}, \\
x &= (-1, 0), & \det(A) &= -24, & \text{Tr}(A) &= 5, & \lambda &= -3, 8, & \text{Saddle}.
\end{align*}
\]
(iv) This system is Hamiltonian with the Hamiltonian function,

\[ H(x, y) = \cos y - x^2 (1 + x^2)/2, \]

so the fixed points are either saddles or centres. The fixed points are at \( x = 0, y = n\pi, n = 0, \pm 1, \pm 2, \ldots \) and \( A = \begin{pmatrix} 0 & \cos y \\ -(1 + 6x^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^n \\ -1 & 0 \end{pmatrix} \) so \( \text{Tr}(A) = 0, \) \( \det(A) = (-1)^n. \) Hence for \( n = 0, \pm 2, \ldots \) the fixed points are centres and for odd \( n \) they are saddles. The contours of \( H(x, y), \) that is the phase curves of the system, are shown in figure N2.7.

![Figure N2.7 Contour plot of the Hamiltonian](image)

Figure N2.7 Contour plot of the Hamiltonian \( H = \cos y - x^2 (1 + x^2)/2. \)

Solutions of exercises from set book

**Solution to Exercise JS 2.1**

For each set of equations we give the linearisation matrix, \( A, \) its eigenvalues and the classification of the fixed point

(i) \( A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}, \lambda = \pm 2i: \) centre.

(ii) \( A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \lambda = -\frac{1}{2} \left(1 \pm \sqrt{13}\right): \) real, opposite sign, saddle.

(iii) \( A = \begin{pmatrix} -4 & 2 \\ 3 & -2 \end{pmatrix}, \lambda = -3 \pm \sqrt{7}: \) real, both negative, stable node.

(iv) \( A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \lambda = \frac{1}{2} \left(3 \pm \sqrt{17}\right): \) real, opposite sign, saddle.

(v) \( A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}, \lambda = 1, 2: \) real, both positive, unstable node.

(vi) \( A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \lambda = \frac{1}{2} (3 \pm i\sqrt{3}): \) unstable spiral.
2.7. SOLUTIONS FOR EXERCISES FROM CHAPTER 2

Solution to Exercise JS 2.2

(i) In this case $\text{Tr}(A) = 4$ and $\det(A) = 4$ and there is a double root at $\lambda = 2$; the eigenvalues are real and equal. The eigenvector of $A$ is $(1, 1)$ which suggests putting $\xi = x + y$ and $\eta = x - y$ to obtain the equations,

\[
\dot{\xi} = 2(\xi + \eta), \quad \dot{\eta} = 2\eta,
\]

which have the solutions,

\[
\eta(t) = \eta_0 e^{2t}, \quad \xi(t) = (\xi_0 + 2\eta_0 t)e^{2t}.
\]

If $\eta_0 = 0$, i.e. $x_0 = y_0$, then the motion is along the straight line $x = y = \xi_0 e^{2t}/2$ and is away from the origin. The phase curves in the $(\xi, \eta)$-plane are shown in figure N2.8, and you should be sure that you understand how the above equations produce this figure. These curves are typical of those produced when the eigenvalues are real and equal.

(ii) In this case the whole line $y = x$ comprises fixed points. Also $\dot{y} = 2\dot{x}$, so $y = 2x + c$ are the phase curves which intersect the line of fixed points at $(-c, -c)$; the time development is given by substituting this into the equation for $\dot{x}$ to give $\dot{x} = -(x + c)$. Two representative phase curves are shown in the figure N2.9.
CHAPTER 2. PLANE AUTONOMOUS SYSTEMS AND LINEARISATION

Solution to Exercise JS 2.3

(i) The fixed points are given by

\[ x - y = 0 \implies x = y \]
\[ x + y - 2xy = 0 \implies 2x - 2x^2 = 0, \quad x = 0, 1. \]

Hence the fixed points are at (0, 0) and (1, 1). The linearisation matrix is

\[ A(x, y) = \begin{pmatrix} 1 & -1 \\ 1 - 2y & 1 - 2x \end{pmatrix}. \]

The origin (0, 0): the eigenvalues of \( A(0, 0) \) are \( \lambda = 1 \pm i \), giving an unstable spiral.

The point (1, 1): the eigenvalues of \( A(1, 1) \) are \( \lambda = \pm \sqrt{2} \), giving a saddle.

(ii) The fixed points are at \( (\pm 1, 0) \) and the linearisation matrix is

\[ A(x, y) = \begin{pmatrix} 0 & (1 + y)e^y \\ -2x & 0 \end{pmatrix}. \]

The point (1, 0): the eigenvalues of \( A(1, 0) \) are \( \lambda = \pm i\sqrt{2} \), giving a centre.

The point (-1, 0): the eigenvalues of \( A(-1, 0) \) are \( \lambda = \pm \sqrt{2} \), giving a saddle.

Note that this system is a Hamiltonian system, with the Hamiltonian

\[ H(x, y) = \frac{1}{3}x^3 - x - (1 + y)e^{-y}, \]

so the fixed point at (1, 0) is also a centre of the nonlinear system.

Solution to Exercise JS 2.4

(i) The equations of motion can be written in the form,

\[ \dot{y} = x^3 - x = -\partial H/\partial x, \]
\[ \dot{x} = y = \partial H/\partial y, \]

with the Hamiltonian \( H(x, y) = y^2/2 + x^2/2 - x^4/4 \). The fixed points are at \( y = 0, x = 0, \pm 1 \); at \( x = 0, H \) has a local minimum, so this is a centre; at \( x = \pm 1, H \) has a saddle so these fixed points are saddles. The contours and potential function are similar to those given in the solution to JS exercise 1.2, with \( \alpha = -1 \).

(v) This is a Hamiltonian system with,

\[ H(x, y) = y^2/2 - (1 - \cos x) \cos x. \]

The potential \( V(x) = -(1 - \cos x) \cos x \) is stationary at \( \sin x = 0 \) and \( \cos x = 1/2 \); at \( x = 0 \), \( V(x) \simeq -x^2/2 \) so \( H \) has a saddle: at \( x = x_0, \cos x_0 = 1/2 \), \( V(x) \simeq -1/4 + 3(x - x_0)^2/4 \) so this fixed point is a centre. At \( x = \pm \pi, V(x) \simeq 2 - 3(x \pm \pi)^2/2 \) giving a saddle. Some contours of the Hamiltonian are shown in figure N2.10.
2.7. SOLUTIONS FOR EXERCISES FROM CHAPTER 2

Solution to Exercise JS 2.7

The terms $x\sqrt{x^2 + y^2}$ and $y\sqrt{x^2 + y^2}$ are ignored in a linear analysis so that the fixed point at the origin has the linear representation

$$\dot{x} = -y, \quad \dot{y} = x,$$

which are the equations of a centre.

The nonlinear problem is easiest to solve in the polar coordinates $(r, \theta)$, where $x = r \cos \theta$ and $y = r \sin \theta$. Thus on differentiating the relation $r^2 = x^2 + y^2$ with respect to $t$ we obtain

$$r \dot{r} = x \dot{x} + y \dot{y}$$

and on using the two equations of motion,

$$r \dot{r} = x(-y - xr) + y(x - yr) = -r^3, \quad \text{or} \quad \dot{r} = -r^2,$$

so that $\dot{r} < 0$ and the orbits spiral in towards the origin. Notice that a phase point reaches the origin at a finite time, in distinction to the linear stable spiral discussed in JS , page 73.

Solution to Exercise JS 2.8

If $\ddot{x} + x^2 + x = 0$ and if we define $y = \dot{x}$ the first-order equations are

$$\dot{x} = y, \quad \dot{y} = -x - y^2.$$

On the other hand, putting $z = \dot{x} - x$ gives $\dot{z} + \dot{x} + \dot{x}^2 + x = 0$, or

$$\dot{x} = z + x, \quad \dot{z} = -2x - (z + x)^2.$$

The original second-order equation is a special case of the system treated in exercise N2.5, with $F(x) = x$ and $G(x) = 1$, so $f(x) = e^{2x}$, $V(x) = \frac{1}{4}(2x - 1)e^{2x}$ and the Hamiltonian is

$$H = \frac{1}{2}y^2 e^{-2x} + \frac{1}{4}(2x - 1)e^{2x}, \quad y = e^{2x} \dot{x}.$$

Near the origin this Hamiltonian behaves as

$$H = -\frac{1}{4} + \frac{1}{2}(x^2 + y^2) + \cdots.$$
showing that there is a centre here. The phase curves are the contours of this function so are symmetric about reflections in the \(x\)-axis; some representative curves are shown in the following figure.

Alternatively, from the original equation with \(y = \dot{x}\)

\[
\frac{dy}{dx} = -y - \frac{x}{y} \quad \text{or} \quad y \frac{dy}{dx} + y^2 = -x.
\]

This equation may be written in the form

\[
\frac{1}{2} e^{-2x} \frac{d}{dx} \left( y^2 e^{2x} \right) = -x \implies y^2 e^{2x} + \frac{1}{2}(2x - 1)e^{2x} = \text{constant}. 
\] (2.14)

Note this is the same as the above Hamiltonian because the variable \(y\) has a different meaning in each case.

Similarly for the second representation

\[
\frac{dz}{dx} = -1 = (x + z) - \frac{x}{x + z}
\]

and putting \(z = -x + v(x)\) gives the same equation as above but with \(y \rightarrow v\) and hence the phase curves are the contours of

\[
(x + z)^2 e^{2x} + \frac{1}{2}(2x - 1)e^{2x} = \text{constant}. 
\] (2.15)

Some representative contours of these two functions are shown in the following figures.
2.7. SOLUTIONS FOR EXERCISES FROM CHAPTER 2

Solution to Exercise JS 2.11
If \( \dot{x} = \dot{y} = 0 \) at some point other than the origin we have \( Ax = 0 \) for \( x \neq 0 \) and so \( A \) is singular, that is \( \det(A) = ad - bc = 0 \). In this case both \( ax + by = 0 \) and \( cx + dy = 0 \) define the same line in the \((x, y)\)-plane on which \( Ax = 0 \) and this is a line of fixed points.

Solution to Exercise JS 2.12
This problem has five positive constants making a general analysis messy. Since there are three variables, \( H, P, \) and \( t \) we can remove three constants by rescaling: put \( x = c_2H/a_2, \ y = c_1c_2P/b_1a_2, \) and \( \tau = a_2t \) and writing \( \dot{x} = dx/d\tau \) etc we obtain

\[
\begin{align*}
\dot{x} &= a(b - x - y)x, \\
\dot{y} &= y(x - 1), \quad (x, y \geq 0),
\end{align*}
\]

with \( a = b_1/c_2, \ b = a_1c_2/b_1a_2. \) The fixed points are \((0, 0), \ (b, 0)\) and if \( b > 1, \ (1, b - 1). \) The linearisation matrix, \( A(x, y), \) is,

\[
A(x, y) = \begin{pmatrix} a(b - 2x - y) & -ax \\ y & x - 1 \end{pmatrix}.
\]

- At \((0, 0)\) the eigenvalues of the matrix \( A \) are \( ab \) and \(-1, \) giving a saddle.
- At \((b, 0)\) the eigenvalues are \(-ab \) and \( b - 1 \) giving a stable node for \( b < 1 \) and a saddle for \( b > 1. \)
- At \((1, b - 1)\) the eigenvalues are \( \{-a \pm \sqrt{a^2 - 4a(b - 1)}\}/2 \) giving a stable spiral if \( a < 4(b - 1) \) and a stable node if \( a > 4(b - 1). \)

Note that the solution given in JS (page 519), contains errors.

Solution to Exercise JS 2.14
The only fixed point is at \( P = a_1/c_1 \) and \( H = a_1c_2/(a_2c_1). \) Here the linearisation matrix is

\[
A = \begin{pmatrix} 0 & -a_1c_2 \\ a_2^2 & a_2 \\ c_2 & -a_2 \end{pmatrix}
\]

and the eigenvalues are

\[
\lambda = \frac{1}{2} \left( -a_2 \pm \sqrt{a_2(a_2 - 4a_1)} \right).
\]

- If \( a_2 < 4a_1 \) the fixed point is a stable spiral.
- If \( a_2 > 4a_1 \) the fixed point is a stable node.

Solution to Exercise JS 2.16
The number, \( x, \) of people ignorant of the rumour decreases at a rate proportional to the number of times an ‘x’ meets a ‘y’:

\[
\dot{x} = -\mu xy, \quad (\mu > 0).
\]

The increase in \( y \) is the sum of
(i) those who have just heard the rumour, $\mu xy$, and

(ii) those who have stopped spreading due to encounters with $z$, $-\mu yz$, and encounters with $y$, $-\mu y(y - 1)$, not $-\mu y^2$ which would include self-encounters.

Thus

$$\dot{y} = \mu \{xy - yz - y(y - 1)\} = \mu y(2x - N), \quad x + y + z = N + 1.$$ 

One set of fixed points is $y = 0$, any $x$, $z = N + 1 - x$ which is clearly of no interest. Another is $x = N/2$, $y = 0$, $z = 1 + N/2$, but this is not a simple fixed point (det$(A) = 0$) and is a special case of the first set.

The phase curves are given by

$$\frac{dy}{dx} = -(2x - N)/x \quad \text{or} \quad y = -2x + N \ln x + C.$$ 

With the initial conditions given, $y = 1$ and $x = N$ at $t = 0$,

$$y(x) = 1 + 2N - 2x + N \ln(x/N),$$

$$= 1 + N (2 - 2z + \ln z), \quad z = \frac{x}{N}.$$ 

A sketch of the function $\Psi(z) = 2 - 2z + \ln z$ is given in the following figure.

![Figure N2.14 Graph of $\Psi(z) = 2 - 2z + \ln z$.](image)

From this we see that $\Psi(z) = 0$ at $z = 1$ and $z \simeq 0.203$, so for large $N$, $y(x) = 0$ for $x/N \simeq 0.203$. An expansion in inverse powers of $N$ shows that $y(x) = 0$ when

$$\frac{x}{N} = 0.2032 \left(1 - \frac{1.685}{N} + \frac{2.390}{N^2} - \frac{4.098}{N^3} + O(N^{-4})\right).$$

**Solution to Exercise JS 2.23**

Let $(u_x, u_y)$ be the velocity of the boat relative to the water, with the directions as shown in the diagram.
If \( u \) is the speed of the boat relative to the water we have \( u_x = u \cos \theta, \ u_y = u \sin \theta \) and \( \tan \theta = y/x \) where \((x, y)\) are the coordinates of the boat. Relative to the fixed origin,

\[
\begin{align*}
\dot{x} &= -u_x = -\frac{ux}{\sqrt{x^2 + y^2}} \\
\dot{y} &= v - u_y = v - \frac{uy}{\sqrt{x^2 + y^2}}
\end{align*}
\]

Write these equation in the form

\[
x \frac{dy}{dx} = y + \frac{v \sqrt{x^2 + y^2}}{u} \quad \text{or} \quad x \frac{dy}{dx} - y = -\alpha \sqrt{x^2 + y^2}, \quad \alpha = \frac{v}{u}.
\]

Since \( x \frac{dy}{dx} - y = x^2 \frac{d}{dx} \left( \frac{y}{x} \right) \) this second equation becomes

\[
\int \frac{df}{\sqrt{1 + f^2}} = -\alpha \ln x, \quad y = xf.
\]

On putting \( f = \sinh \phi \) this integrates to

\[
\sinh^{-1} f = \ln \left( x^{-\alpha} \right) + C
\]

and since \( \sinh^{-1} w = \ln(w + \sqrt{w^2 + 1}) \) we obtain

\[
y + \sqrt{x^2 + y^2} = Ax^{1-\alpha}, \quad (2.16)
\]

for some positive constant \( A \).

If \( \alpha = v/u < 1 \) the speed of the boat relative to the water is larger that the speed of the river, so we expect that given any starting point \((x, y)\) there is a constant \( A \) such that the curve defined in equation N2.16 passes through \((x, y)\) and the origin. Since \( 1 - \alpha > 0 \) the right hand side of this equation is zero at the origin, so as \( x \to 0, \ y \to 0 \).

If \( v > u \) and the boat starts at \( y > 0 \) it cannot reach the origin. If initially \( y < 0 \) we set \(-y = w > 0\) so the equation becomes

\[
\sqrt{x^2 + w^2} - w = Ax^{-(\alpha-1)}.
\]
The constant $A$ depends only on the initial position and is positive. At the line $y = w = 0$, we have

$$1 = Ax^{-\alpha}$$

so the phase curve crosses the $x$-axis at some positive value of $x$, so cannot return to the origin. Hence if $v > u$ there is no solution that passes through the origin.

**Solution to Exercise JS 2.27**
The solution for this exercise is given in the course notes, page 48.

**Solution to Exercise JS 2.29**
Only $x$ and $y$ are coupled and since both are non-negative $\dot{x} < 0$ until a time at which either $x$ or $y$ is zero. Hence $x(t)$ is a decreasing function of $t$. If $x > \gamma/\beta$, $\dot{y} > 0$ and if $x < \gamma/\beta$, $\dot{y} < 0$.

Hence if $x(0) > \gamma/\beta$, $y(t)$ increases until $x(t) = \gamma/\beta$ were $y(t)$ reaches its maximum and then both $x(t)$ and $y(t)$ decrease.

The equation for $y(x)$ can be written in the form

$$\frac{dy}{dx} = \frac{\gamma}{\beta x} - 1 \implies y(x) = -x + \frac{\gamma}{\beta} \ln x + c$$

where $c$ is a constant. Graphs of the function $\overline{y}(x) = -x + \alpha \ln x$, for $\alpha = 1$, 2 and 4 are shown in the following figure.

![Figure N2.16](image)

**Solution to Exercise JS 2.33**
Typical shapes of $f(u)$ and $f'(u)$ are as shown in figure N2.17 of the course notes; $f(u)$ is odd and $f'(u)$ is even.
(i) One obvious fixed point is \( V_1 = V_2 = w \) where \( w\sigma = f(E - w) \) and from the sketch, figure N2.18, it is obvious that this equation has one and only one root. Putting \( V_1 = w + u, \ V = w + v \) gives, for small \( u \) and \( v, \)

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
-f'(-w) & -f'(E - w) \\
-f'(E - w) & -\sigma
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix}.
\]

The eigenvalues of the matrix are \( -\sigma \pm f'(E - w) \) so if \( f'(E - w) > \sigma \) they are real and of opposite sign so the fixed point is a saddle. Otherwise they are both real and negative and it is a stable node.

(ii) Now consider the possibility of other roots, \( V_1 = x, V_2 = y \) with \( x \neq y \) which are solutions of

\[
\begin{align*}
y &= f(E - x)/\sigma & \text{curve 1,} \\
x &= f(E - y)/\sigma & \text{curve 2.}
\end{align*}
\]

(2.17)

A sketch of these equations is shown in figure N2.19; in the example shown there are three roots the centre root being at \( x = y = w \) as discussed in part (i). For the two outer roots to be real it is clear that at the centre root the gradient, \( \tan \theta_1 \), of curve 1 must be less than the gradient, \( \tan \theta_2 \), of curve 2, that is,

\[-\frac{dy_1}{dx} > -\frac{dy_2}{dx} \quad \text{at} \quad x = y = w, \quad \sigma w = f(E - w).\]

But from equations N2.17, at this root

\[
\begin{align*}
\frac{dy_1}{dx} &= -f'(E - w)/\sigma & \text{and} & \frac{dy_2}{dx} &= -\sigma f'(E - w)
\end{align*}
\]

so that \( \theta_1 + \theta_2 = 3\pi/2 \). Thus the two extra roots are real if,

\[\sigma < f'(E - w), \quad \text{where} \quad \sigma w = f(E - w)\]

but if

\[\sigma > f'(E - w)\]

there is only one fixed point.
In the former case the two outer fixed points are at \((w_1, w_2)\) and \((w_2, w_1)\) where we chose \(w_1 > w\) and \(w_1\) and \(w_2\) are related by the equations,

\[
w_2 = f(E - w_1)/\sigma, \quad w_1 = f(E - w_2)/\sigma
\]

as shown in figure N2.19.

![Figure N2.19](image)

On putting \(V_1 = w_1 + u\) and \(V_2 = w_2 + v\) the equations of motion become,

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = 
\begin{pmatrix}
-\sigma & -f'_1 \\
-f'_2 & -\sigma
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

where \(f'_k = f'(E - w_k)\). The eigenvalues of this matrix are

\[
\lambda = -\sigma \pm \sqrt{f'_1 f'_2}
\]

which are real since \(f'_k > 0\). Further, by comparing the gradients of the curves at \((w_1, w_2)\) we see that

\[
-\frac{dy_1}{dx} < -\frac{dy_2}{dx}, \quad \text{or} \quad \frac{f'_1}{\sigma} < \frac{f'_2}{\sigma}
\]

so that both eigenvalues are real and negative, giving a stable node.

**Solution to Exercise JS 2.43**

From the equations \(\dot{N}_1 = 0\) on the line \(bN_1 + cN_2 = a_1/b_1\) and \(\dot{N}_2 = 0\) on the parallel line \(bN_1 + cN_2 = a_2/b_2\). These lines intersect the \(N_1\) axis at \(\frac{a_1}{b_1}\) and \(\frac{a_2}{b_2}\) respectively, so if \(\frac{a_1}{b_1} > \frac{a_2}{b_2}\) the arrangements of lines is as shown in the figure.
Thus there are fixed points at \((a_1/bh_1, 0)\) and \((0, a_2/cb_2)\), shown by the dots. The signs of the velocity functions show that the phase curves cross the line \(bN_1 + cN_2 = a_k/b_k\), \(k = 1, 2\) as shown by the arrows in the diagram. It follows that \(N_1 \to \frac{a_1}{a_1'}\) and \(N_2 \to 0\) as \(t \to \infty\).

**Solution to Exercise JS 2.45**

Near the fixed point the linearised equations are \(\dot{x} = Ax\) and \(\dot{y} = Bx\) where

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \quad \det(A) = -\det(B).
\]

The eigenvalues of \(A\) are given by

\[
2\lambda_A = \text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}.
\]

If the fixed point of \(A\) is a spiral then \(\det(A) > 0\), since \(\text{Tr}(A)^2 < 4\det(A)\), so \(\det(B) < 0\). The eigenvalues of \(B\) are thus given by

\[
2\lambda_B = \text{Tr}(B) \pm \sqrt{\text{Tr}(B)^2 + 4|\det(B)|}.
\]

These are real and have opposite sign, so this fixed point is a saddle.

If the fixed point of \(A\) is an unstable node, both eigenvalues are real and positive, so \(\text{Tr}(A)^2 > 4\det(A)\), but no conclusions about the nature of the eigenvalues of \(B\) can be made.

If the fixed point of \(A\) is a saddle, both eigenvalues are real and of opposite sign, so \(\text{Tr}(A)^2 > 4\det(A)\), (real eigenvalues), and \(\det(A) < 0\) (opposite signs). Hence \(\det(B) > 0\) but no conclusions about the nature of the eigenvalues of \(B\) can be made.

**Solution to Exercise JS 2.46**

The equations of motion are

\[
\frac{dx}{dt} = a + x^2y - (1 + b)x, \quad \frac{dy}{dt} = bx - yx^2.
\]
The fixed point is at $xy = b$ and $x = a$, that is $(a, b/a)$. The linearisation matrix is

$$A = \begin{pmatrix} 2xy - 1 - b & x^2 \\ b - 2xy & -x^2 \end{pmatrix} = \begin{pmatrix} b - 1 & a^2 \\ -b - a^2 & \end{pmatrix}$$

from which the eigenvalues may be computed: for the values of $(a, b)$ quoted we have:

a) $a = 1, b = 2$, $\lambda = \pm i$: centre of the linearised system:

b) $a = 1, b = 1$, $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$: stable spiral:

c) $a = 1, b = 5$, $\lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$, real and positive, unstable node.