Chapter 6  Exponentials and logarithms
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6.1 Solving exponential equations by taking logs

You were introduced to exponential equations in Topic 9 of S111. An exponential equation is an equation in which the unknown is in an exponent, such as

\[ 170 \times 1.23^t = 7 \, 000 \, 000 \, 000. \]

In the absence of a better method, equations of this sort can be solved by using trial and improvement or graphs. You are now ready to learn how to use logarithms to solve exponential equations.

The method involves using the third logarithm law which was introduced in Topic 9 of S111 (and also Chapter 10 of Maths for Science). For this purpose it is best to think of the law with its left- and right-hand sides swapped:

\[ \log(x^n) = n \log x. \]

The next example shows how you can use this logarithm law to help you to solve exponential equations.

**Example 1** Solving an exponential equation by taking logs

Solve the equation \( 0.7^x = 0.2 \), giving the solution to three significant figures.

**Solution**

The equation is

\[ 0.7^x = 0.2. \]

Take the logarithm to base 10 of both sides.

\[ \log_{10}(0.7^x) = \log_{10} 0.2 \]

Use the fact that \( \log(x^n) = n \log x \).

\[ x \log_{10} 0.7 = \log_{10} 0.2 \]

Divide both sides by the coefficient of the unknown.

\[ x = \frac{\log_{10} 0.2}{\log_{10} 0.7} \]

Use your calculator to evaluate the answer.

\[ x = 4.51 \text{ (to 3 s.f.)}. \]

The solution is approximately \( x = 4.51 \).

(Check: When \( x = 4.51 \), LHS \( = 0.7^{4.51} = 0.20016 \ldots \approx 0.2 = \text{RHS} \).)
So the basic procedure for solving an exponential equation is to take the logarithm of both sides of the equation – this is the ‘taking logs’ mentioned in the title of this subsection – and then apply the logarithm law mentioned on the opposite page to turn the awkward exponent into a straightforward coefficient. You can then solve the equation in the usual way.

You can use logarithms to any base in this procedure, as long as you are consistent. However, you will usually need to evaluate these logarithms, so it is best to use either logarithms to base 10 or natural logarithms, as these are easily available from your calculator.

### Activity 1  **Solving an exponential equation by taking logs**

Solve the equation \(1.4^x = 550\), giving the solution to three significant figures.

The method of taking logs is easiest to apply when one side of the equation consists only of a number raised to an exponent, where this exponent contains the unknown. So if you have to solve an exponential equation that is not in this form, then it is usually best to rearrange it into this form before you take logs.

This is demonstrated in the next example, which also illustrates the kind of problem that you can answer by solving exponential equations.

### Example 2  **Solving another exponential equation**

A population of insects currently numbers 200. If the size of the population increases by 10% each week, how long will it take to reach 400?

**Solution**

The starting number is 200 and the scale factor is 1.1, so the size of the population is modelled by the equation

\[ P = 200 \times 1.1^t, \]

where \(t\) is the number of weeks and \(P\) is the size of the population.

So the time \(t\) weeks for the population to reach 400 is given by the equation

\[ 200 \times 1.1^t = 400. \]

This equation can be solved as follows.

1. First divide both sides by 200 to obtain \(1.1^t\) by itself on one side. \(\Rightarrow\)
   \[ 1.1^t = 2 \]

2. Now take logs, and solve the equation using the method that you have seen. \(\Rightarrow\)
   \[ \log_{10}(1.1^t) = \log_{10}2 \]
   \[ t \log_{10}1.1 = \log_{10}2 \]
   \[ t = \frac{\log_{10}2}{\log_{10}1.1} \]
   \[ t = 7.27 \text{ (to 2 d.p.)} \]

So the insect population will take about \(7\frac{1}{4}\) weeks to reach 400.

(Check: When \(t = 7.27\), \(P = 200 \times 1.1^{7.27} = 399.90\ldots \approx 400.\))
Before you go on to solve similar equations yourself, there is something worth observing about Example 2. If the starting population of insects had been 300, say, and the question had asked you to find the time taken for it to reach double this number, which is 600, then the answer would have been the same. This is because the answer would be found by solving the equation

\[300 \times 1.1^t = 600,\]

and this simplifies to

\[1.1^t = 2,\]

which is the same equation as in the example. In fact, you can see that whatever the starting population is, the time taken for it to double would be the same. So the time taken for the population to double does not depend on the starting population, but depends only on the scale factor.

You can practise solving exponential equations by taking logs in the next activity.

### Activity 2

Solve the equation

\[170 \times 1.23^t = 700000000\]

by taking logs, giving your answer to three significant figures.

The method of taking logs can also be used for problems involving \textit{discrete} exponential change. As an example, an athlete who plans a new training schedule in which the distance in kilometres that she will run in week \(n\) is

\[20 \times 1.1^n.\]

The problem is to find the week of the schedule in which the athlete is first due to run more than 65 km. In other words, you have to find the smallest integer value of \(n\) for which

\[20 \times 1.1^n > 65.\]

To solve this, use the method of taking logs to find the solution of the equation

\[20 \times 1.1^n = 65;\]

then the required value of \(n\) is the smallest whole number greater than this solution.

If you have to solve an exponential equation in which the base is \(e\), then it is usually helpful to take \textit{natural logarithms} of both sides, rather than logarithms to base 10. This often allows you to obtain an exact answer in terms of natural logarithms, which you can then evaluate if necessary. This is illustrated in the next example.

In both physics and higher-level mathematics courses many exponential equations involve the base \(e\).
In this example the exponent is not simply the unknown, \(x\), but an expression involving the unknown, namely \(2x\). Taking logs is useful in cases like this too, whatever the base is.

**Example 3  Solving an exponential equation in which the base is \(e\)**

Find the exact solution of the equation \(e^{2x} = 3\), then evaluate it to three significant figures.

**Solution**

The equation is

\[ e^{2x} = 3. \]

- The base is \(e\), so take natural logarithms.

\[ \ln(e^{2x}) = \ln 3 \]

- Use the fact that taking the natural logarithm of a number is the inverse operation to raising \(e\) to a number.

\[ 2x = \ln 3 \]

- Now rearrange the equation in the usual way.

\[ x = \frac{1}{2} \ln 3 \]

- Evaluate the answer if required.

\[ x = 0.549 \text{ (to 3 s.f.)} \]

(Check: When \(x = 0.549\), LHS = \(e^{2 \times 0.549} = 2.998 \ldots \approx 3 = \text{RHS}\).)

**Activity 3  Solving exponential equations in which the base is \(e\)**

Find the exact solutions of the following equations, then evaluate them to three significant figures.

(a) \(e^t = 8\)  
(b) \(e^{x-1} = 5\)

The useful fact used in Example 3 and Activity 3 was as follows. Suppose that you take a number \(x\) and first raise the base \(e\) to that number, then take the natural logarithm of the result. Since the two operations are inverses of each other, the final result will just be \(x\), which gives the following identity:

\[ \ln(e^x) = x. \]

You can also apply the operations in the other order. If you take a number \(x\) and first take the natural logarithm of that number, then raise the base \(e\) to the result, then again the final result will just be \(x\), which gives the following second useful identity:

\[ e^{\ln x} = x. \]
Solutions and comments on Activities

**Activity 1**
The equation is
\[ 1.4^x = 550. \]
Taking logs gives
\[ \log_{10}(1.4^x) = \log_{10} 550 \]
\[ x \log_{10} 1.4 = \log_{10} 550 \]
\[ x = \frac{\log_{10} 550}{\log_{10} 1.4} \]
\[ x = 18.753 \ldots . \]
So the solution is \( x = 18.8 \) (to 3 s.f.).
(Check: When \( x = 18.8 \),
LHS = 1.4^{18.8} = 558.73 \ldots \approx 550 = RHS.)

**Activity 2**
The equation is
\[ 170 \times 1.23^t = 700000000. \]
Dividing by 170 gives
\[ 1.23^t = \frac{700000000}{170}. \]
Taking logs gives
\[ \log_{10}(1.23^t) = \log_{10} \left( \frac{700000000}{170} \right) \]
\[ t \log_{10} 1.23 = \log_{10} \left( \frac{700000000}{170} \right) \]
\[ t = \frac{\log_{10} \left( \frac{700000000}{170} \right)}{\log_{10} 1.23} \]
\[ t = 84.696 \ldots . \]
So the solution is \( t = 84.7 \) (to 3 s.f.).
(Notice that this is consistent with the solution to Activity 11. That is, it will take 84 to 85 years for Elvis impersonators to account for the entire population of the world, under the assumptions stated in the question.)

**Activity 3**
(a) \( e^t = 8 \)
\[ \ln (e^t) = \ln 8 \]
\[ t = \ln 8 \]
The solution is \( t = 2.08 \) (to 3 s.f.).
(Check: When \( t = 2.08 \),
LHS = \( e^{2.08} = 8.004 \ldots \approx 8 = RHS.)
(b) \( e^{x-1} = 5 \)
\[ \ln (e^{x-1}) = \ln 5 \]
\[ x - 1 = \ln 5 \]
\[ x = \ln 5 + 1 \]
The solution is \( x = 2.61 \) (to 3 s.f.).