S111 to MST124 - additional mathematics support

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5 Trigonometry

5.1 The Sine Rule

Angles and trigonometry were introduced in Topic 7 of S111. In this subsection and the next you will meet rules that relate the angles and side lengths of any triangle. These rules are most easily stated using the notation shown in Figure 1. Here the vertices of a triangle are labelled $A$, $B$ and $C$. The side lengths are labelled $a$, $b$ and $c$ in such a way that vertex $A$ is opposite side length $a$, vertex $B$ is opposite side length $b$, and vertex $C$ is opposite side length $c$. The angles at vertices $A$, $B$ and $C$ can be denoted by $\angle A$, $\angle B$ and $\angle C$, or just by $A$, $B$ and $C$.

The notation in Figure 1 is often used for a general triangle, as it helps you to remember which angle is related to which side, and it also makes the resulting formulas easier to remember. It will be used throughout this chapter.

In Topic 7 of S111 you saw how to solve right-angled triangles by using the trigonometric ratios sine, cosine and tangent. One way of making geometric problems easier to solve is to draw construction lines. So if you are trying to find a relationship between the sides and angles of the general triangle in Figure 1, then one approach is to draw a construction line to introduce right-angled triangles and use some of the facts that you already know about these.

Figure 2 shows a general triangle with three acute angles. A construction line has been drawn from $C$ at right angles to the opposite side. A line like this, drawn at right angles to another line, is called a perpendicular and the process of drawing such a line is called dropping a perpendicular.

Let $h$ be the length of the perpendicular. In the left-hand triangle you can use the fact that sine is opposite over hypotenuse to give

$$\sin A = \frac{h}{b},$$

so

$$h = b \sin A.$$  

In this formula, $A$ is short for $\angle A$.

In the right-hand triangle you can get a similar expression for the sine of the angle $B$:

$$\sin B = \frac{h}{a},$$

so

$$h = a \sin B.$$  

The two expressions above for $h$ are equal, so

$$b \sin A = a \sin B.$$
This equation can be rearranged to give the equation

\[ \frac{a}{\sin A} = \frac{b}{\sin B} \]

In a similar way, if you drop a perpendicular from the vertex \( A \) to the opposite side of the triangle, then you can show that

\[ \frac{b}{\sin B} = \frac{c}{\sin C} \]

Combining this equation with the one above gives the following rule.

**Sine Rule**

\[
\begin{align*}
\frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} \\
\text{or, equivalently,} & \\
\frac{\sin A}{a} &= \frac{\sin B}{b} = \frac{\sin C}{c}
\end{align*}
\]

The Sine Rule tells you that the ratio of a side length of a triangle to the sine of the angle opposite that side is the same no matter which side and its opposite angle you consider. So it can be used in the following way.

**Using the Sine Rule**

The Sine Rule can be used if you know one side length of a triangle and the opposite angle, and one further angle or side length.

For example, if you know the length \( a \) and the opposite angle \( A \), and also the angle \( B \), then you can find the length \( b \) by using the first equation in the Sine Rule,

\[ \frac{a}{\sin A} = \frac{b}{\sin B} \]

and rearranging this equation to make \( b \) the subject.

The next two examples illustrate how this can be done.

**Example 1 Using the Sine Rule to find a side length**

Find the length of the side \( BC \) in the triangle below, to two significant figures.
Solution

A side length and the opposite angle are known, together with one further angle, so the Sine Rule can be used.

The side $AB$ of length 8 cm is opposite the $50^\circ$ angle at $C$, so $c = 8$ and $C = 50^\circ$. Also the side $BC$ of unknown length is opposite the $70^\circ$ angle at $A$, so the unknown length is $a$ and $A = 70^\circ$.

By the Sine Rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

which gives

$$\frac{a}{\sin 70^\circ} = \frac{8}{\sin 50^\circ},$$

so

$$a = \frac{8 \sin 70^\circ}{\sin 50^\circ} = 9.813 \ldots.$$  

Hence the length $BC$ is 9.8 cm (to 2 s.f.).

In Example 1, the vertices and the sides were labelled as they are in the statement of the Sine Rule, and you may find it helpful to label your triangles in a similar way while you become familiar with the Sine Rule. When you feel confident using the Sine Rule, you can apply it directly without this labelling, as shown in the next example.

Example 2 \textit{Using the Sine Rule without vertex labels}

Find the length $x$ in the triangle below, to three significant figures.

![Triangle with angles 80°, 35°, and 10°]

Solution

A side length and the opposite angle are known, so the Sine Rule can be used. The side length 10 is opposite the angle of $35^\circ$, and the unknown side length $x$ is opposite the angle of $80^\circ$.

By the Sine Rule,

$$\frac{10}{\sin 35^\circ} = \frac{x}{\sin 80^\circ},$$

so

$$x = \frac{10 \sin 80^\circ}{\sin 35^\circ} = 17.169 \ldots.$$  

Hence $x = 17.2$ (to 3 s.f.).

Here is a similar question for you to try.
Activity 1 Using the Sine Rule to find a side length

Find the length $x$ in the triangle below, to three significant figures.

![Triangle with sides 7 m, 55°, and 60°](image)

In this subsection you have seen that the Sine Rule is useful for solving a triangle when you know the length of a side and the angle opposite this side. As you have seen, if you also know one other angle, then you can use the Sine Rule to calculate the side length opposite this angle. If instead you know one further side length, then you can often use the Sine Rule to calculate the angle opposite this side.

5.2 The Cosine Rule

Although the Sine Rule is useful in many cases, it can be used only if you know the length of a side and the opposite angle. If you know the lengths of two sides and the angle between them, as in Figure 3, then a different rule, called the Cosine Rule, can be used.

Figure 4 shows a general triangle with three acute angles. Suppose that you want to find a formula for the side length $a$ in terms of the side lengths $b$ and $c$, and the angle $A$. To do this, you can start by dropping a perpendicular from $C$, as shown. The side of length $c$ is then divided into two parts: if you call the length of one part $y$, then the other part has length $c - y$.

![General triangle with perpendicular](image)

Applying Pythagoras’ Theorem to each of the right-angled triangles gives

$$b^2 = y^2 + h^2, \quad (1)$$

$$a^2 = (c - y)^2 + h^2.$$

Expanding the brackets in the second of these equations gives

$$a^2 = c^2 - 2cy + y^2 + h^2.$$
The right-hand side of this equation contains the expression $y^2 + h^2$, which is equal to $b^2$ by equation (1). So the equation above can be rearranged as

$$a^2 = b^2 + c^2 - 2cy.$$  \hspace{1cm} (2)

The length $y$ in this equation can be replaced by using the fact that the right-angled triangle on the left in Figure 4 gives

$$\cos A = \frac{y}{b}, \text{ so } y = b \cos A.$$

Substituting this expression for $y$ in equation (2) gives the equation

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This equation is one form of the Cosine Rule. There are three forms of this rule, one for each of the three angles, as shown overleaf.

**Cosine Rule**

- $a^2 = b^2 + c^2 - 2bc \cos A$,
- $b^2 = c^2 + a^2 - 2ca \cos B$,
- $c^2 = a^2 + b^2 - 2ab \cos C$.

These three equations look quite complicated, but there are some patterns in the way that the letters appear that will help you to remember them. For example, only one length appears on the left-hand side of each equation, and this length is related to the angle whose cosine appears on the right-hand side. The other two lengths appear only on the right-hand side in each case.

You will see in the next section that if $\angle C$ is a right angle, then $\cos C = 0$. In this case, the third form of the Cosine Rule simply states that $c^2 = a^2 + b^2$. This is Pythagoras’ Theorem, as you would expect in a right-angled triangle. In fact, the Cosine Rule is similar to Pythagoras’ Theorem, with an extra correction term because the opposite angle is not a right angle in general.

The next two examples illustrate how to use the Cosine Rule.

**Example 3 Using the Cosine Rule to find a side length**

Use the Cosine Rule to find the length $AC$ in the triangle below, to two significant figures.
Solution

Label the sides with letters and identify which form of the Cosine Rule to use. The known angle is angle $B$, so use the second form.

By the Cosine Rule,

$$b^2 = c^2 + a^2 - 2ca \cos B.$$ 

Substitute in the values and do the calculation.

Substituting $a = 5.6$, $c = 7.7$ and $B = 78^\circ$ gives

$$b^2 = 7.7^2 + 5.6^2 - 2 \times 7.7 \times 5.6 \cos 78^\circ = 72.719 \ldots.$$ 

So $b = \sqrt{72.719} \ldots = 8.52 \ldots$. Hence the length $AC$ is 8.5 m (to 2 s.f.).

As with the Sine Rule, once you are familiar with the Cosine Rule you may prefer to apply it without labelling the vertices and sides of the triangle.

Example 4 Using the Cosine Rule without vertex labels

Find the length $x$ in the triangle below, to two significant figures.

Solution

The lengths of two sides and the included angle are known, so use the Cosine Rule.

By the Cosine Rule,

$$x^2 = 10^2 + 12^2 - 2 \times 10 \times 12 \cos 35^\circ = 47.403 \ldots.$$ 

So $x = \sqrt{47.403} \ldots = 6.88 \ldots = 6.9$ (to 2 s.f.).

Here is a similar question for you to try.
Activity 2 *Using the Cosine Rule to find a side length*

Find the length $x$ in the triangle below, to two significant figures.

![Triangle diagram]

The Cosine Rule can also be used if you know the lengths of all three sides of a triangle and you want to find an angle, as in the following example.

**Example 5 *Using the Cosine Rule to find an angle***

Find the angle $\theta$ in the triangle below, to the nearest degree.

![Triangle diagram]

**Solution**

The lengths of three sides are known, so use the Cosine Rule.

By the Cosine Rule,

$$4^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \cos \theta.$$  

This equation simplifies to

$$16 = 149 - 140 \cos \theta.$$  

Making $\cos \theta$ the subject of the equation gives:

$$140 \cos \theta = 133$$  

$$\cos \theta = \frac{133}{140} = 0.95.$$  

So $\theta = \cos^{-1}(0.95) = 18.194\ldots^\circ = 18^\circ$ (to the nearest degree).

You can use a similar method in the following activity.
Activity 3  Finding all the angles in a triangle

Use the Cosine Rule to calculate $\angle B$ of the isosceles triangle shown below, and deduce the other angles. Give your answers to the nearest degree.

![Isosceles triangle with sides 4, 4, and 3, and angles A, B, and C.]

5.3 A formula for the area of a triangle

Using trigonometry you can derive a formula for the area of a triangle that can be used if you know the lengths of two sides and the included angle. As with the Sine Rule and Cosine Rule, we will derive this formula for a triangle with three acute angles, but it can also be used for triangles with an obtuse angle.

For the triangle in Figure 5, the base is $b$ and the height is $h$. The area of the triangle is $\frac{1}{2}bh$. From the right-angled triangle on the left,

$$\sin \theta = \frac{h}{a}, \quad \text{so} \quad h = a \sin \theta.$$  

Substituting this formula for $h$ into $\frac{1}{2}bh$ gives the area of the triangle as $\frac{1}{2}bh = \frac{1}{2}b \times a \sin \theta = \frac{1}{2}ab \sin \theta$. This gives the following formula for the area of a triangle.

**Area of a triangle**

The area of a triangle with two sides of lengths $a$ and $b$, and included angle $\theta$, is

$$\text{area} = \frac{1}{2}ab \sin \theta.$$  

This area formula is more useful in practical situations than the formula $\frac{1}{2}bh$. For example, in surveying it gives the area of a triangle of land in terms of quantities that a surveyor can readily measure. The areas of most pieces of land can be found by breaking them down into convenient triangular shapes and using this formula in each triangle.

You can use the formula in the following activity.
Activity 4  Finding the areas of triangles

Find the area of each of the following triangles, correct to three significant figures.

(a)  
\[
\begin{align*}
\text{8 km} & \quad \text{10 km} \\
40^\circ & \\
\end{align*}
\]

(b)  
\[
\begin{align*}
60^\circ & \quad \text{6 cm} \\
7 \text{ cm} & \\
\end{align*}
\]

In many cases calculating the area of a triangle would be easier if there were a simple formula for the area of a triangle in terms of just the lengths of the three sides. Such a formula was derived by the Greek mathematician Heron around AD 62 and later found independently by Chinese mathematicians.

**Heron’s Formula**

The area of a triangle with sides of length \(a\), \(b\) and \(c\) is

\[
\text{area} = \sqrt{s(s-a)(s-b)(s-c)},
\]

where \(s = \frac{1}{2}(a + b + c)\).

The quantity \(s\) in Heron’s Formula is called the semi-perimeter of the triangle because it is half the perimeter.

Heron’s Formula can be proved by expressing one angle of the triangle in terms of the lengths of the sides and then using the formula \(\frac{1}{2}ab \sin \theta\) for the area of the triangle. However, it takes considerable algebraic skill to write the final result in the neat form given in Heron’s Formula.

In the next activity you are asked to use the two area formulas from this subsection to calculate the area of an equilateral triangle.

Activity 5  Using area formulas

For an equilateral triangle of side length 2:

(a) Calculate the area of the triangle using the formula: \(\text{area} = \frac{1}{2}ab \sin \theta\).

(b) Calculate the semi-perimeter \(s\) of the triangle and then use Heron’s Formula to calculate the area of the triangle.

Give your answers in surd form.
5.4 Trigonometric functions

So far in this unit the trigonometric ratios sine, cosine and tangent have always been applied to angles between 0° and 90°. However, if you use your calculator to find sin 150°, cos 1000° or even tan(−45°), then it gives the answers

\[
\sin 150° = \frac{1}{2}, \quad \cos 1000° = 0.173\ldots, \quad \tan(-45°) = -1.
\]

What do these values mean, and why are they needed?

In this section you will learn how to define the sine and cosine of every possible angle, and the tangents of most angles, and you will also see why defining these values is useful.

One reason why it is useful to define the sine and cosine of an angle greater than 90° is that some triangles have an obtuse angle, that is, an angle between 90° and 180°. In order to solve such triangles, you may want to use the Sine Rule or the Cosine Rule: these apply to triangles with an obtuse angle in the same way as they apply to other triangles. But to do that you need to know what the sine and cosine of an obtuse angle are!

Since the definitions of sine and cosine given in Topic 7 of S111 do not make sense for an obtuse angle, new definitions are needed. But once you know these new definitions, it is natural to define the sine and cosine of any angle, not just acute and obtuse ones.

The sines and cosines of general angles give rise to functions, called trigonometric functions, that turn out to be useful in situations that are not explicitly related to triangles. For example, these functions can be used to model many types of real-world behaviour with a repeating nature, such as the occurrence of high tides – as you will see.

Sine, cosine and tangent of a general angle

Topic 7 of S111, introduced the idea that an angle is a measure of rotation, or amount of turning, that can be measured in degrees. There, angles were discussed that take values up to and including 360°, that is, up to a rotation through one full turn. But it is possible to have a rotation through more than one full turn, so it makes sense to use general angles that measure more than 360°. You can also have rotations that are clockwise or anticlockwise, and it is useful to distinguish between these.

**Sign of an angle**

A general angle is a measure of rotation around a point, measured in degrees. Positive angles give anticlockwise rotations, and negative angles give clockwise rotations.
Some examples of angles corresponding to rotations around the origin, from the positive $x$-axis, are shown in Figure 7. The arrow indicates whether the rotation is anticlockwise or clockwise.

![Figure 7](image)

**Figure 7** Some general angles

The sine and cosine of a general angle $\theta$ are defined by using a point $P$ on the circle with radius 1 centred on the origin, which is called the unit circle. The position of the point $P$ is determined by the angle $\theta$; it is obtained by a rotation around the origin through the angle $\theta$ starting from the point on the $x$-axis with $x$-coordinate 1. If the angle $\theta$ is positive, then the rotation is anticlockwise; if the angle is negative, then the rotation is clockwise.

Some examples of how general angles give rise to points on the unit circle are shown in Figure 8.

![Figure 8](image)

**Figure 8** General angles and points on the unit circle

In Figure 8, the $x$- and $y$-axes divide the graph into four regions, each of which is known as a quadrant. These quadrants are numbered in order anticlockwise as shown in Figure 9, and they are useful for describing where points, such as those on the unit circle, lie.

![Figure 9](image)

**Figure 9** The four quadrants
### Activity 6  Plotting general angles on the unit circle

For each of the following angles, draw a sketch to illustrate how the point $P$ on the unit circle is rotated through that angle from its starting position on the $x$-axis. Your sketches should be similar to those in Figure 8. In each case state the quadrant in which $P$ lies.

(a) $60^\circ$  
(b) $225^\circ$  
(c) $390^\circ$  
(d) $-70^\circ$

It is possible that two different angles of rotation lead to the point $P$ being in the same position. For example, as you can see in Figure 10, the point $P$ is in the same position on the unit circle for the angle $510^\circ$ as for the angle $150^\circ$. This is because

$$510^\circ = 150^\circ + 360^\circ,$$

so $510^\circ$ gives exactly one full turn around the origin more than $150^\circ$.

Figure 11 shows the point $P$ after it has rotated through an acute angle $\theta$, so $P$ is in the first quadrant.

If you drop a perpendicular from $P$ to the $x$-axis, then you obtain a right-angled triangle with hypotenuse of length 1 in which one angle is $\theta$, as shown in Figure 12.

You can see that if $P$ has coordinates $(x, y)$, then

$$
sin \theta = \frac{y}{1} \quad \text{and} \quad cos \theta = \frac{x}{1},$$

which give

$$x = cos \theta \quad \text{and} \quad y = sin \theta.$$

These two equations are the key to defining the sine and cosine of a general angle. This is done as follows.
Sine and cosine of a general angle

For a general angle \( \theta \), let \( P \) be the point on the unit circle obtained by a rotation of \( \theta \) around the origin from the positive \( x \)-axis, and suppose that \( P \) has coordinates \((x, y)\). Then

\[
\cos \theta = x \quad \text{and} \quad \sin \theta = y.
\]

So the cosine and sine of a general angle \( \theta \) are just the \( x \)- and \( y \)-coordinates of the point \( P \) whose position on the unit circle is determined by the angle \( \theta \), as shown in Figure 13.

Figure 13  Defining \( \cos \theta \) and \( \sin \theta \)

To illustrate this definition, consider the angle \( \theta = 150^\circ \). For this angle, the point \( P \) lies in the second quadrant, as shown in Figure 14.

Figure 14  The point \( P \) obtained from the angle 150°

To find \( \sin \theta \) and \( \cos \theta \) in this case, you need to find the coordinates of this point \( P \). You can do this by using the right-angled triangle shown in Figure 15, which has one angle equal to \( 180^\circ - 150^\circ = 30^\circ \).

Figure 15  A right-angled triangle related to the angle 150°
In the triangle in Figure 15, the hypotenuse is of length 1, so
\[ \sin 30^\circ = \frac{\text{opp}}{1} \quad \text{and} \quad \cos 30^\circ = \frac{\text{adj}}{1}. \]

Hence
- the side opposite the angle of 30° has length \( \sin 30^\circ = \frac{1}{2} \),
- the side adjacent to the angle of 30° has length \( \cos 30^\circ = \frac{\sqrt{3}}{2} \).

Therefore the coordinates of the point \( P \) are \( (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \).

You can check that your calculator gives these values for \( \cos 150^\circ \) and \( \sin 150^\circ \).

It is straightforward to write down the sine or cosine of an angle that is a multiple of \( 90^\circ \), because for such an angle the point \( P \) lies on one of the axes. For example, Figure 16 shows that \( \cos 0^\circ = 1 \) and \( \sin 0^\circ = 0 \).

Similarly, Figure 17 shows that \( \cos 90^\circ = 0 \) and \( \sin 90^\circ = 1 \).

Now that you know the definition of the sine and cosine of any angle, the next question is: How can you define the tangent of a general angle? In any right-angled triangle with an acute angle \( \theta \),
\[ \tan \theta = \frac{\sin \theta}{\cos \theta}. \]

This equation can be used to define the tangent of a general angle.

**Activity 7 Finding sines and cosines from the definition**

(a) Find \( \cos 225^\circ \) and \( \sin 225^\circ \) (to four decimal places) by plotting the appropriate point \( P \) on the unit circle and using a suitable right-angled triangle. Check your answers with a calculator.

(b) Find \( \cos(-180^\circ) \) and \( \sin(-180^\circ) \) by plotting the appropriate point \( P \) on the unit circle. Check your answers with a calculator.

Remember that division by 0 is not allowed.
For example,
\[
\tan 150^\circ = \frac{\sin 150^\circ}{\cos 150^\circ} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -0.577\ldots
\]
You can check this value on a calculator. However, \tan 90^\circ cannot be defined because \cos 90^\circ = 0.

**Activity 8  Finding tangents from the definition**

Use your answers to Activity 7 to find \tan 225^\circ and \tan (-180^\circ).

Earlier in the subsection, the sine and cosine of a general angle \(\theta\) were defined geometrically in terms of the position of a point \(P\) on the unit circle. There is also a geometric interpretation of the tangent of \(\theta\). This interpretation will be useful in the next subsection when we draw graphs of the sine, cosine and tangent functions.

First consider the situation when the angle \(\theta\) is acute, so the point \(P\) is in the first quadrant, as shown in Figure 18. A vertical line has been drawn through the point on the \(x\)-axis with coordinate 1, and the line from the origin through \(P\) has been extended to meet this vertical line, at the point \(Q\). A new right-angled triangle \(\triangle OQN\) has thus been formed in which the side adjacent to the angle \(\theta\) has length 1. In this triangle,
\[
\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{1} = \text{opp},
\]
so the \(y\)-coordinate of \(Q\) is \(\tan \theta\).

The name tangent arises from the fact that in this diagram the line that passes through the point \((1, 0)\) and the point \(Q\) is a tangent to the unit circle; that is, it meets this circle at exactly one point.

**Figure 18  The geometric interpretation of \(\tan \theta\) when \(\theta\) is acute**

As the angle \(\theta\) increases, the point \(Q\) moves up the vertical line, so \(\tan \theta\) increases. As \(\theta\) reaches the value 90\(^\circ\), the line from the origin through \(P\) becomes vertical. At this point \(\tan 90^\circ\) is not defined.

For values of \(\theta\) between 90\(^\circ\) and 180\(^\circ\), the line through the origin and \(P\) once again meets the vertical line on the right, though now the point of intersection is below the \(x\)-axis, as shown in Figure 19. It can be shown, by considering the triangles in the diagram, that the \(y\)-coordinate of \(Q\) is again \(\tan \theta\), though the details are not given here.

It can also be shown that this interpretation works for any angle \(\theta\) where \(\tan \theta\) is defined; that is, the value of \(\tan \theta\) is always the \(y\)-coordinate of the point \(Q\).
5.5 Graphs of sine, cosine and tangent

In Subsection 5.4 you saw how the sine, cosine and tangent of any angle are defined – except when division by zero is involved. So you can think of each of the expressions

\[
\sin \theta, \quad \cos \theta \quad \text{and} \quad \tan \theta
\]

as a ‘rule’ that takes an input value \( \theta \) and produces an output value. A rule of this type, which transforms an input value into an output value, is often called a ‘function’. For this reason, sine, cosine and tangent are often called trigonometric functions.

To gain a better understanding of a function, it is often helpful to plot its graph, as you saw with quadratics. Let’s begin by plotting the graphs of sine, cosine and tangent in the range \( 0^\circ \) to \( 90^\circ \). This can be done by using the values in Table 1, which were obtained from a calculator and are stated to two decimal places.

<table>
<thead>
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<th>( \theta )</th>
<th>( 0^\circ )</th>
<th>( 10^\circ )</th>
<th>( 20^\circ )</th>
<th>( 30^\circ )</th>
<th>( 45^\circ )</th>
<th>( 60^\circ )</th>
<th>( 70^\circ )</th>
<th>( 80^\circ )</th>
<th>( 90^\circ )</th>
</tr>
</thead>
<tbody>
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<td>( \sin \theta )</td>
<td>0.00</td>
<td>0.17</td>
<td>0.34</td>
<td>0.50</td>
<td>0.71</td>
<td>0.87</td>
<td>0.94</td>
<td>0.98</td>
<td>1.00</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1.00</td>
<td>0.98</td>
<td>0.94</td>
<td>0.87</td>
<td>0.71</td>
<td>0.50</td>
<td>0.34</td>
<td>0.17</td>
<td>0.00</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>0.00</td>
<td>0.18</td>
<td>0.36</td>
<td>0.58</td>
<td>1.00</td>
<td>1.73</td>
<td>2.75</td>
<td>5.67</td>
<td>–</td>
</tr>
</tbody>
</table>

Using these values, the following graphs can be drawn. Only a few significant points are shown on the graphs.

Figure 19  The geometric interpretation of \( \tan \theta \) when \( \theta \) is obtuse

Figure 20  Graphs of sine, cosine and tangent functions for \( 0^\circ < \theta < 90^\circ \)
In Figure 20, the input variable $\theta$ is on the horizontal axis. The vertical axes are labelled with the names of the functions, in order to emphasise which function is involved and also to avoid using the variable $y$, which had a different meaning earlier in this section as one of the coordinates of the point $P$.

You can see that, as the angle $\theta$ increases from 0 to 90°:

- the value of $\sin \theta$ increases from 0 to 1;
- the value of $\cos \theta$ decreases from 1 to 0;
- the value of $\tan \theta$ increases from 0 and takes values that are arbitrarily large as the angle $\theta$ approaches 90°.

These changes take place because, as the point $P$ moves anticlockwise round the part of the unit circle in the first quadrant (see Figure 11 on page 14), the vertical coordinate of $P$, which is equal to $\sin \theta$, increases from 0 to 1, and the horizontal coordinate of $P$, which is equal to $\cos \theta$, decreases from 1 to 0. Also the point $Q$ (see Figure 18 on page 17) moves upwards from the point (1, 0), arbitrarily far as $\theta$ approaches 90°.

But what do these graphs look like for other values of $\theta$? In fact, the other parts of these graphs have similar shapes to the parts that are plotted in Figure 20, but these other parts are positioned differently in relation to the axes. The following activity allows you to see how these graphs are generated as the point $P$ moves on the unit circle in the way described in the previous subsection.

The graphs repeat endlessly along the horizontal axis. The parts of the graphs corresponding to values of $\theta$ between $-360°$ and $720°$ are shown in Figures 21, 22 and 23.

**Figure 21** The graph of the sine function
The graphs of these three trigonometric functions have various properties, such as symmetry characteristics, which can be useful when you are working with sines, cosines and tangents. Let’s have a look at some of these properties.

**Periodicity**

A key feature of these graphs is that their shapes repeat in a regular way. The sine and cosine graphs repeat every $360^\circ$ and we say that these functions are *periodic*, with period $360^\circ$. This is what you would expect because as the angle $\theta$ increases the position of the point $P$ on the unit circle repeats every $360^\circ$. The tangent function is also periodic but with a smaller period, of $180^\circ$. So the values of the tangent function repeat twice as often as the values of the sine and cosine functions.

Because of this property of periodicity, an equation such as $\sin \theta = 0.5$ has infinitely many solutions, and not just the solution $\sin^{-1}(0.5) = 30^\circ$. For example, $\sin 390^\circ = \sin 30^\circ = 0.5$ because $390^\circ = 360^\circ + 30^\circ$. Another solution is $150^\circ$, since $\sin 150^\circ = 0.5$ as you saw earlier.
**Mirror symmetry**

The sine and cosine graphs have *mirror symmetry* in any vertical line through a peak or trough on the graph. For example, the graph of the cosine function (part of which is shown in Figure 22) has mirror symmetry in the vertical axis. This property means that any angle and its negative have the same cosine value. This is what you would expect, because if the point $P$ is rotated around the origin from the positive $x$-axis by an angle of either $\theta$ or $-\theta$, as in Figure 24, then the resulting $x$-coordinate is the same. Facts like this can be written down as trigonometric identities; this fact gives

$$\cos(-\theta) = \cos(\theta)$$

for any angle $\theta$. For example, $\cos(-30^\circ) = \cos 30^\circ$.

**Rotational symmetry**

Each of the three graphs has *rotational symmetry* about any point where the graph crosses the $\theta$-axis – if you rotate the graph through a half-turn about such a point, then it lies exactly on top of where it was before. For example, the sine graph (part of which is shown in Figure 21) has rotational symmetry about the origin. This means that any angle $\theta$ and its negative $-\theta$ have sine values that have the same magnitude, but one of the sine values is negative while the other is positive. Exactly the same is true of the tangent graph; these two facts give the identities

$$\sin(-\theta) = -\sin(\theta) \text{ and } \tan(-\theta) = -\tan(\theta)$$

for any angle $\theta$. For example, $\sin(-30^\circ) = -\sin 30^\circ$.

**Asymptotes**

The graph of the tangent function differs from the other two graphs in that it is broken up into separate pieces and it takes values that are arbitrarily large. The breaks in the graph correspond to the values of $\theta$ where $\cos \theta$ is zero, such as $\theta = 90^\circ$.

For angles just below $90^\circ$, you can see that $\tan \theta$ is very large and positive. This is because $\tan \theta = \sin \theta / \cos \theta$, and $\sin \theta$ is close to 1 whereas $\cos \theta$ is very small and positive (if you divide a number close to 1 by a very small number, then the answer is a very large number). Similarly, for angles just above $90^\circ$, you can see that $\tan \theta$ is very large and negative.

This behaviour is described by saying that the tangent graph has an **asymptote** at $\theta = 90^\circ$. Informally, an asymptote is a line that a graph approaches but never reaches. Asymptotes are often indicated by dashed lines, as in Figure 23.

**Activity 9  The asymptotes of the tangent graph**

Write down the values of $\theta$ between $360^\circ$ and $720^\circ$ at which the asymptotes of the tangent graph occur.
**Relationships between the sine and cosine graphs**

Sine and cosine graphs have the same basic shape. Various relationships between sines and cosines can be obtained from this fact.

For example, try the following ‘thought experiment’. Imagine walking from left to right along the sine graph, starting from the origin. As the angle increases from $0^\circ$, the height you are above the horizontal axis increases from 0 to 1, then decreases from 1 through 0 to $-1$, and then increases from $-1$ to 0 again, and so on.

If you performed the same thought experiment with the cosine graph, again walking from left to right but this time starting at $-90^\circ$, then the heights would follow exactly the same pattern. This is because the graph of the cosine function is obtained by shifting the graph of the sine function to the left by a distance of $90^\circ$ on the $\theta$-axis. The corresponding trigonometric identity is

$$\sin \theta = \cos(-90^\circ + \theta)$$

for any angle $\theta$. For example, $\sin 60^\circ = \cos(-90^\circ + 60^\circ) = \cos(-30^\circ)$.

Similarly, if you walked along the cosine graph starting from $90^\circ$, but this time from right to left, then the same pattern of heights would occur. In this case, the resulting trigonometric identity is

$$\sin \theta = \cos(90^\circ - \theta)$$

for any angle $\theta$. For example, $\sin 60^\circ = \cos(90^\circ - 60^\circ) = \cos 30^\circ$.

You have already seen that the identity $\sin \theta = \cos(90^\circ - \theta)$ holds for acute angles. So now you know that it holds for all angles.

Similarly, the identity

$$\cos \theta = \sin(90^\circ - \theta)$$

for any angle $\theta$, can be seen from the sine and cosine graphs by using a thought experiment similar to those above; you might like to try this.

### Two other trigonometric identities

The identity

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

holds for all acute angles. In fact it holds for all angles $\theta$, except where $\cos \theta = 0$, since it was used to define $\tan \theta$ for non-acute angles!

Finally, the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

also holds for every angle $\theta$. For example, suppose that $P$ is the point on the unit circle corresponding to an obtuse angle $\theta$. As shown in Figure 25, the line from the origin to $P$ has length 1 and it is the hypotenuse of a right-angled triangle. The length of the vertical side of the triangle is $\sin \theta$ (because $\sin \theta$ is positive). Since the $x$-coordinate of $P$ is negative, the length of the horizontal side of the triangle can be found by multiplying the $x$-coordinate of $P$ by $-1$. So the length of the horizontal side is $-\cos \theta$. Then, by Pythagoras’ Theorem,

$$(\sin \theta)^2 + (-\cos \theta)^2 = 1^2,$$

so

$$\sin^2 \theta + \cos^2 \theta = 1.$$
Modelling real-world phenomena

The sine, cosine and tangent functions can be used in modelling many real-world phenomena; you will meet many examples in your studies in physics. As the unit circle is used in the definition of sine and cosine, it may not surprise you that circular motion can be analysed mathematically using sines and cosines. If you go on to study higher-level courses, then you will see that this is exploited to analyse the circular motion of many types of objects, from satellites to children’s roundabouts and spinning tops.

What is perhaps a little more surprising is that sines and cosines can be used to describe any periodic phenomenon such as the heights of tides, sea temperatures during the year, the motion of waves or the time when the Sun rises during the year.

So far in this section you have seen how the sine, cosine and tangent of any angle are defined, and you have also explored the graphs of these functions. The next section introduces another way of measuring angles.

Radians

This section is concerned with another way of measuring angles, different from degrees. The number of degrees in a full turn, 360, is a matter of convention that has its roots in ancient Babylonian mathematics. This convention has persisted for so many centuries because it has worked well and there was no good reason for changing to another measurement system when applying trigonometry in practical situations such as surveying.

However, when mathematicians studied circular motion and other periodic phenomena, it became apparent that the equations involved are often simpler if you use a different unit for measuring angles, called a radian. In higher-level mathematics courses, angles are almost always measured in radians.

5.6 Defining a radian

You can understand what a radian is by thinking about arcs on the circumference of a circle. Figure 26 shows such an arc, and the corresponding angle at $O$, the centre of the circle. The angle at the centre is said to be subtended by the arc. The definition of a radian is as follows.

**Radian**

One radian is the angle subtended at the centre of a circle by an arc that is the same length as the radius.

This definition is illustrated in Figure 27.
From this definition, you can find the number of radians in a full turn. The circumference of the circle in Figure 27 has length $2\pi r$, and each arc of length $r$ subtends an angle of 1 radian. So the number of radians in a full turn is

$$\frac{2\pi r}{r} = 2\pi.$$

In other words, $360^\circ$ is the same angle as $2\pi$ radians.

$$2\pi \text{ radians} = 360^\circ.$$  

This gives

$$1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}.$$  

Since

$$\frac{180^\circ}{\pi} = 57.295\ldots,$$

one radian is approximately $57^\circ$.

Because a full turn is $2\pi$ radians, the number of radians in a simple fraction of a full turn can be conveniently expressed in terms of $\pi$. It is usual to leave these numbers in this form, rather than finding decimal approximations. For example, a half-turn is an angle of $\pi$ radians, and a quarter-turn is an angle of $\frac{\pi}{2}$ radians, also written as $\frac{\pi}{2}$ radians or $\pi/2$ radians. Reasoning in this way, we can build up Table 2.

### Table 2  A conversion table for common angles

<table>
<thead>
<tr>
<th>Angle in degrees</th>
<th>Angle in radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>$\pi/6$</td>
</tr>
<tr>
<td>45°</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>60°</td>
<td>$\pi/3$</td>
</tr>
<tr>
<td>90°</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>180°</td>
<td>$\pi$</td>
</tr>
<tr>
<td>360°</td>
<td>$2\pi$</td>
</tr>
</tbody>
</table>

For example, $30^\circ$ is one twelfth of a full turn, so $30^\circ$ is $2\pi/12 = \pi/6$ radians.
Since
\[ 1 \text{ radian} = \frac{180^\circ}{\pi}, \]
the factor \( \frac{180}{\pi} \) can be used to convert an angle measured in radians into degrees, and vice versa.

**Converting between degrees and radians**

- Angle in radians = \( \frac{\pi}{180} \times \) angle in degrees,
- Angle in degrees = \( \frac{180}{\pi} \times \) angle in radians.

Here are some examples of these types of conversions.

**Example 6 Converting between degrees and radians**

(a) Convert 270° to radians.
(b) Convert \( \frac{5\pi}{6} \) radians to degrees.

**Solution**

(a) Applying the degrees-to-radians conversion formula gives

\[ \text{angle in radians} = \frac{\pi}{180} \times 270 = \frac{3\pi}{2}. \]

So 270° = \( \frac{3\pi}{2} \) radians.

(b) Applying the radians-to-degrees conversion formula gives

\[ \text{angle in degrees} = \frac{180}{\pi} \times \frac{5\pi}{6} = 150. \]

So \( \frac{5\pi}{6} \) = 150°.

Here are some similar questions for you to try.

**Activity 10 Converting between degrees and radians**

(a) Convert the following angles from degrees to radians.
   (i) 75°   (ii) 225°

(b) Convert the following angles from radians to degrees.
   (i) \( \frac{3\pi}{4} \) radians   (ii) \( \frac{4\pi}{5} \) radians
### 5.7 Formulas involving radians

Some formulas related to circles have a simpler form if angles are measured in radians rather than degrees.

An example is the formula for the length of an arc on the circumference of a circle of radius \( r \), in terms of the angle \( \theta \) subtended. You know that

- an arc that subtends an angle of 1 radian has length \( r \).

Hence

- an arc that subtends an angle of \( \theta \) radians has length \( r \theta \).

**Length of an arc of a circle**

\[
\text{arc length} = r \theta,
\]

where \( r \) is the radius of the circle and \( \theta \) is the angle subtended by the arc, measured in radians.

An arc of a circle is sometimes called a *circular arc*, and it can occur as part of another shape. The *centre* and *radius* of a circular arc are the centre and radius of the circle whose circumference it is part of. The next example involves a curved section of road whose edges are circular arcs.

**Example 7  Planning a road barrier**

The diagram below shows a plan for a bend \( ABCD \) in a new road. \( AB \) and \( CD \) are circular arcs with centre \( O \). A barrier is to be placed along \( CD \). What is the length of this barrier, to the nearest metre?

![Diagram of a bend with a road barrier](image)

**Solution**

- Find the radius of the arc and the angle that it subtends at the centre of the circle.

The radius of the arc \( CD \) is the length \( OA + AD \), which is 40 metres.

The arc \( CD \) subtends an angle of \( \pi/6 \) radians at the centre of the circle.

- Use the formula for arc length.
So the length of the arc $CD$ is $r\theta$, where $r = 40$ and $\theta = \pi/6$.

Hence the length of the arc $CD$ in metres is

$$40 \times \frac{\pi}{6} = \frac{20\pi}{3} = 20.94\ldots$$

So the length of the barrier is 21 m (to the nearest metre).

Another formula that’s simpler when radians are used is the formula for the area of a sector. A sector of a circle is the part of the circle lying between two radii, as shown in Figure 29.

Here’s how to find this formula. First, the area of a sector is proportional to the angle $\theta$ of the sector. Next, if the angle of the sector is $\pi$ radians, a half-turn, then the sector is a semicircle, which has area $\frac{1}{2}\pi r^2$. That is,

an angle of $\pi$ radians gives a sector of area $\frac{1}{2}\pi r^2$.

Hence

an angle of 1 radian gives a sector of area $\frac{1}{2}r^2$,

so

an angle of $\theta$ radians gives a sector of area $\frac{1}{2}r^2\theta$.

This result is summarised overleaf.

**Area of a sector of a circle**

area of sector $= \frac{1}{2}r^2\theta$,

where $r$ is the radius of the circle and $\theta$ is the angle of the sector, measured in radians.

The following example uses the formula for the area of a sector to find the area of the road bend in Example 7.

**Example 8** Finding the area of the bend in the road

Calculate the area of the road bend $ABCD$ in Example 7.

**Solution**

Identify the area that is required and work out how you can find it from the areas of shapes that you know how to calculate.

The area of $ABCD$ can be found by subtracting the area of the sector $OAB$ from the area of the sector $OCD$.

Use the formula for the area of a sector.

For the sector $OCD$, the radius $r$ is 40 metres and the angle is $\pi/6$. So the area in square metres of the sector $OCD$ is

$$\frac{1}{2} \times 40^2 \times \frac{\pi}{6} = \frac{400\pi}{3} = 418.879\ldots$$

For the sector $OAB$, the radius $r$ is 30 metres and the angle is $\pi/6$. So the area in square metres of the sector $OAB$ is

$$\frac{1}{2} \times 30^2 \times \frac{\pi}{6} = 75\pi = 235.619\ldots$$
Hence the area in square metres of the bend in the road is

\[
\text{area of sector } OCD \ - \ \text{area of sector } OAB
\]

\[
= 418.879\ldots - 235.619\ldots
\]

\[
= 183.259\ldots
\]

\[
= 183 \text{ (to the nearest integer)}.
\]

So the area of the bend is approximately 183 square metres.

The next activity involves using the formulas for arc length and the area of a sector to calculate lengths and areas in a gothic window. An example of a gothic window is shown in Figure 30.

The angles given in the activity are measured in degrees, so before you can use the formulas, the first step is to convert from degrees to radians.

**Activity 11  Finding lengths and areas in a gothic window**

The following diagram shows a gothic window. Each of the two curves at the top is a circular arc whose centre is the lowest point of the other circular arc.

*Figure 30  A gothic window*
(a) Convert 60° to radians.

(b) Calculate the length of metal edging that is required to fit around the entire perimeter of the window, in metres, to two decimal places.

(c) Calculate the area of the window, in square metres, to two decimal places, using the following steps.

(i) Calculate the area of the triangle formed by the three dashed lines.

(ii) Calculate the area of the sector formed by the horizontal dashed line, one slant dashed line and one of the circular arcs.

(iii) Hence calculate the area of the window above the horizontal dashed line.

(iv) Calculate the area of the whole window.

When the size of an angle is given in radians, the word ‘radians’ is often omitted. For example, you might say that the size of an angle is $\pi/3$, rather than $\pi/3$ radians. So if you see the size of an angle given with no degrees mentioned, then you can assume that it is measured in radians.

This convention is particularly useful when you are using trigonometry and the angles are measured in radians. For example, $\sin(\pi/3)$ means the sine of the angle that measures $\pi/3$ radians.

You can use your calculator to find the trigonometric ratios of angles measured in radians, but first it is important to check that your calculator is set to measure angles in radians rather than degrees. The next activity shows you how to do that.

**Activity 12 Using radians on your calculator**

In this activity, the angles are measured in radians. Find the values of the following expressions, giving your answers correct to 3 s.f.

(a) $\sin 1$

(b) $\cos \frac{\pi}{3}$

(c) $\tan^{-1}(1)$
5.8 Useful trigonometric ratios and identities

In this subsection, you will see how the trigonometric ratios for the angles $30^\circ$, $45^\circ$ and $60^\circ$ can be worked out directly, without using your calculator. These values are quite memorable and useful to know. They can be calculated from triangles in which these angles occur.

For example, in an equilateral triangle, the interior angles are each $60^\circ$. Figure 31 shows an equilateral triangle with sides of length 2 units, in which a vertical line divides the base of the triangle into two equal parts, each of length 1. (Choosing the equilateral triangle to have sides of length 2 makes the calculations easier.)

From the right-angled triangle on the left-hand side of the equilateral triangle you can see that

$$\cos 60^\circ = \frac{1}{2}.$$  

The length, $x$, of the third side of this right-angled triangle can be calculated by using Pythagoras’ Theorem:

$$1^2 + x^2 = 2^2, \quad \text{so} \quad x^2 = 4 - 1 = 3.$$  

Hence $x = \sqrt{3}$ units, as shown in Figure 32. Since the side opposite the angle $60^\circ$ is of length $\sqrt{3}$,

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}.$$  

The right-angled triangle in Figure 32 can also be used to find the trigonometric ratios for $30^\circ$, the third angle in the triangle. You can see that the ratios are as follows:

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan 30^\circ = \frac{1}{\sqrt{3}}.$$  

You can find the trigonometric ratios for the angle $45^\circ$ by using a right-angled isosceles triangle in a similar way, in the next activity.

Activity 13 Finding the sine, cosine and tangent of $45^\circ$

For the triangle below, find the length $x$ by using Pythagoras’ Theorem, and then calculate $\sin 45^\circ$, $\cos 45^\circ$ and $\tan 45^\circ$.

The trigonometric ratios for the angles $30^\circ$, $45^\circ$ and $60^\circ$ are used frequently, so they are listed in Table 3. If you study mathematics further, then you will find it useful to remember these values, or remember how to find them from the particular triangles discussed in this subsection.
There are various relationships between sines, cosines and tangents of angles. These relationships can help you to remember the values in Table 3, and they are often helpful in other ways.

In the next activity you will find two such relationships.

**Activity 14  Finding relationships between sines and cosines**

The diagram below shows a general right-angled triangle. Since its two acute angles add up to $90^\circ$, one is marked $\theta$ and the other is marked $90^\circ - \theta$.

\[ \text{A} \quad \text{B} \quad \text{C} \quad \begin{array}{c} \theta \\
\theta \quad 90^\circ - \theta \\
\theta \quad \theta \\
\theta \quad \theta \\
\theta \quad \theta \end{array} \]

(a) Write down expressions for $\sin \theta$, $\cos \theta$, $\sin(90^\circ - \theta)$ and $\cos(90^\circ - \theta)$, in terms of the side lengths $a$, $b$ and $c$.

(b) Use the results of part (a) to show that if $\theta$ is an acute angle, then

\[ \cos \theta = \sin(90^\circ - \theta) \quad \text{and} \quad \sin \theta = \cos(90^\circ - \theta). \]

In Activity 14 you were asked to prove the following two results.

\[ \cos \theta = \sin(90^\circ - \theta) \]
\[ \sin \theta = \cos(90^\circ - \theta) \]

These equations are examples of **identities**, as they are true for every acute angle $\theta$. They tell you that if two angles add up to $90^\circ$, then the sine of one angle is the cosine of the other, and vice-versa. For example,

\[ \cos 30^\circ = \sin 60^\circ, \quad \cos 45^\circ = \sin 45^\circ \quad \text{and} \quad \cos 60^\circ = \sin 30^\circ. \]

This explains the repeated values that you can see in the sine and cosine columns of Table 3.

You can obtain another useful identity from the definitions of sine, cosine and tangent. The definitions are

\[ \sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}. \]

From these equations you can see that

\[ \frac{\sin \theta}{\cos \theta} = \frac{\frac{\text{opp}}{\text{hyp}}}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{hyp}} \times \frac{\text{hyp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}} = \tan \theta, \]

which gives the identity below.

---

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30^\circ$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$45^\circ$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>
You might like to check this identity for some of the values in Table 3.

Finally, a very neat identity can be obtained by considering a right-angled triangle whose hypotenuse has length 1, as shown in Figure 33, and using Pythagoras’ Theorem. In this triangle,

\[ \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\ominus}{1} = \text{opp} \quad \text{and} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\ominus}{1} = \text{adj}. \]

Thus, by Pythagoras’ Theorem,

\[(\sin \theta)^2 + (\cos \theta)^2 = 1^2 = 1.\]

It is conventional to write \((\sin \theta)^2\) and \((\cos \theta)^2\) as \(\sin^2 \theta\) and \(\cos^2 \theta\), so the identity above is usually written as follows.

\[\sin^2 \theta + \cos^2 \theta = 1\]

Again, you might like to check this identity for some of the values in Table 3.

In Section 5.4 you have seen that \(\sin \theta\), \(\cos \theta\) and \(\tan \theta\) can also be defined for angles other than acute angles, and the identities above also hold for such angles.

### Table 4  Sine, cosine and tangent of special angles

<table>
<thead>
<tr>
<th>(\theta) in degrees</th>
<th>(\theta) in radians</th>
<th>(\sin \theta)</th>
<th>(\cos \theta)</th>
<th>(\tan \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{\sqrt{3}})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{1}{\sqrt{2}})</td>
<td>(\frac{1}{\sqrt{2}})</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{2})</td>
<td>(\sqrt{3})</td>
</tr>
<tr>
<td>90°</td>
<td>(\frac{\pi}{2})</td>
<td>1</td>
<td>0</td>
<td>–</td>
</tr>
</tbody>
</table>

### Table 5  A pattern in the table

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(\theta)</th>
<th>(\sin \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{1}{\sqrt{2}})</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\sqrt{3}}{2})</td>
</tr>
<tr>
<td>90°</td>
<td>(\frac{\pi}{2})</td>
<td>1</td>
</tr>
</tbody>
</table>

In equations of this type involving \(\sin \theta\) and \(\cos \theta\), there is a solution only if the number on the right-hand side is between –1 and 1, inclusive.

### 5.9 Solving trigonometric equations

When you use trigonometry to find an unknown angle \(\theta\) in a triangle, the final step involves solving a trigonometric equation such as

\[ \sin \theta = \frac{5}{8}, \quad \cos \theta = -0.87 \quad \text{or} \quad \tan \theta = 4. \]

Equations like these crop up frequently in trigonometry.
As you know, you can obtain a solution of an equation like those above by using the \( \sin^{-1} \), \( \cos^{-1} \) or \( \tan^{-1} \) function on your calculator. If you know that the angle \( \theta \) that you are trying to find is acute, then the solution given by your calculator will be the angle that you want. In general, however, the solution given by your calculator is only one of many possible solutions.

For example, consider the equation

\[
\sin \theta = \frac{5}{6}.
\]

One solution of this equation is

\[
\theta = \sin^{-1} \left( \frac{5}{6} \right) \approx 56^\circ.
\]

However, the graph in Figure 34 shows that there are other solutions. Each of the points marked with a black dot corresponds to a value of \( \theta \) for which \( \sin \theta = \frac{5}{6} \). The dot between \( 0^\circ \) and \( 90^\circ \) corresponds to the solution on the opposite page, \( \theta \approx 56^\circ \), but you can see that there is another solution between \( 90^\circ \) and \( 180^\circ \), and since the graph of \( y = \sin \theta \) repeats every \( 360^\circ \), there are infinitely many solutions altogether.

\[
\text{Figure 34} \quad \text{Some points on the graph of } y = \sin \theta \text{ that have } y\text{-coordinate } \frac{5}{6}
\]

When you need to solve a trigonometric equation like those on the opposite page, you often need solutions other than the one provided by your calculator. When you use the Sine Rule to find an angle \( \theta \) of a triangle, you obtain an equation of the form

\[
\sin \theta = \text{a number},
\]

where the number on the right-hand side is positive, and less than or equal to 1. You can use the \( \sin^{-1} \) function on your calculator to find one solution of the equation, but, except when \( \theta = 90^\circ \), there is always a second solution, which is \( 180^\circ \) minus the first solution (as illustrated in Figure 34 for the equation \( \sin \theta = \frac{5}{6} \)). Both of the two solutions can occur as the angle \( \theta \) of the triangle, and you need more information about the triangle in order to decide which is the correct angle.

So being able to find all the solutions of trigonometric equations is a useful skill. Once you know how to find all the solutions of an equation of this type, you can choose any particular solution that you might need for the situation that you’re working with. For example, you might know that the angle that you are trying to find is between \( 90^\circ \) and \( 180^\circ \).

You can use drawing a graph to find approximate solutions of trigonometric equations, just as you can for any other type of equation.
However, as you have seen for other kinds of equations, it is also useful to have a non-graphical method for finding solutions. This often allows you to obtain solutions more quickly, it can enable you to obtain exact solutions rather than approximate ones, and it gives you a greater understanding of the mathematics.

So in this subsection you will learn a useful method for finding all the solutions of trigonometric equations like those at the beginning of this subsection.

The key to finding all the solutions of trigonometric equations is to understand how the sine, cosine and tangent of any angle are related to the sine, cosine and tangent of an acute angle. So let’s look at that next.

**Sines, cosines and tangents of related angles**

Think back to the way that the sine, cosine and tangent of a general angle were defined. Suppose that the angle is \( \theta \): remember that you think of it drawn on a pair of coordinate axes, as shown in Figure 35. It is measured from the positive direction of the \( x \)-axis, anticlockwise if \( \theta \) is positive, and clockwise if \( \theta \) is negative.

![Figure 35](image)

An angle \( \theta \) drawn on a pair of coordinate axes

The angle \( \theta \) corresponds to a point on the unit circle (the circle of radius 1 centred at the origin), and \( \sin \theta \) and \( \cos \theta \) are defined to be the \( y \)- and \( x \)-coordinates of this point, respectively. The value of \( \tan \theta \) is defined by

\[
\tan \theta = \frac{\sin \theta}{\cos \theta}.
\]

For example, consider the angle 25°. Figure 36 shows the point \( P \) on the unit circle that corresponds to 25°.

![Figure 36](image)

The point \( P \) corresponding to the angle 25°

The \( y \)-coordinate of \( P \) is 0.423, to three decimal places, so \( \sin 25^\circ \approx 0.423 \).
The $x$-coordinate of $P$ is 0.906, to three decimal places, so
\[ \cos 25° \approx 0.906. \]

Using more precise values for $\cos 25°$ and $\sin 25°$ gives
\[ \tan 25° = \frac{\sin 25°}{\cos 25°} = \frac{0.42261\ldots}{0.90630\ldots} \approx 0.466. \]

The angle 25°, shown in Figure 36, is referred to as a *first-quadrant* angle, because the corresponding point $P$ lies in the first quadrant. Similarly, any angle that corresponds to a point on the unit circle that lies in the second quadrant is called a *second-quadrant* angle, and so on. Figure 37 reminds you of how the quadrants are labelled. So, for example, any acute angle is a first-quadrant angle, and any obtuse angle is a second-quadrant angle.

Now consider what happens if you reflect the point $P$ corresponding to 25° in the $y$-axis. The resulting point $Q$ is shown in Figure 38. The angle corresponding to $Q$ is a second-quadrant angle, namely $180° - 25° = 155°$.

The $x$-coordinate of $Q$ is the negative of the $x$-coordinate of $P$, and the $y$-coordinate of $Q$ is the same as the $y$-coordinate of $P$.

**Figure 38** The point $Q$ corresponding to the angle 155°

The sine and cosine of 155° are given by the coordinates of $Q$. These coordinates give:
\[ \sin 155° \approx 0.423, \quad \cos 155° \approx -0.906 \]
and
\[ \tan 155° = \frac{0.42261\ldots}{-0.90630\ldots} \approx -0.466. \]

So the sine, cosine and tangent of 155° are exactly the same as the sine, cosine and tangent of 25°, except for some of the signs. The cosine and tangent of 155° are negative, whereas the cosine and tangent of 25° are positive. Another way to describe the relationship between the sines, cosines and tangents of these two angles is to say that they have the same *magnitudes*.

There are two other angles in the range 0° to 360° that also have the same sine, cosine and tangent as 25°, except for some of the signs, as shown in Figure 39.

The first of these other angles is obtained by rotating the point $P$ through a half-turn about the origin (or alternatively by reflecting the point $Q$ in the $x$-axis). This gives the point $R$ shown in Figure 39(a).

The second of the other angles is obtained by reflecting the point $P$ in the $x$-axis (or alternatively by reflecting the point $R$ in the $y$-axis). This gives the point $S$ shown in Figure 39(b).

Remember that the *magnitude* of a number is the number without its negative sign, if it has one.
In the next activity you are asked to work out the angles corresponding to $R$ and $S$, and find their sines, cosines and tangents.

**Activity 15** *Finding the cosines, sines and tangents of angles related to $25^\circ$*

(a) Work out the angles, measured anticlockwise from the positive $x$-axis as shown in Figure 39, corresponding to the points $R$ and $S$.

(b) By considering how the coordinates of $R$ and $S$ are related to the coordinates of $P$ (or $Q$), write down the coordinates of $R$ and $S$, to three significant figures.

(c) Using your answers to part (b), find the sine, cosine and tangent of each of the two angles in part (a), to three significant figures. (For the tangents, you will need to use more precise values of the cosines and sines, to avoid rounding errors. You can find the numbers that you need on page 35.)

You have now seen that the four angles

\[
25^\circ, \\
180^\circ - 25^\circ = 155^\circ, \\
180^\circ + 25^\circ = 205^\circ, \\
360^\circ - 25^\circ = 335^\circ
\]

all have the same sine, cosine and tangent, except for the signs.

In general, you can see that, for any acute angle $\phi$, the four angles

\[
\phi, \\
180^\circ - \phi, \\
180^\circ + \phi \quad \text{and} \quad 360^\circ - \phi
\]

all have the same sine, cosine and tangent, except for the signs. These four related angles are shown in Figure 40.
Figure 40  Four angles with the same sine, cosine and tangent, except for the signs

One of the four related angles lies in each of the four quadrants. The summary diagram in Figure 41 should help you to remember them.

Figure 41  The related angles diagram

The signs of the sines, cosines and tangents are determined by the signs that the $x$- and $y$-coordinates take in the different quadrants, as follows.

- In the first quadrant, $x$ and $y$ are both positive, so sine, cosine and tangent are all positive.
- In the second quadrant, $x$ is negative and $y$ is positive, so sine is positive and cosine is negative, and hence tangent is negative.
- In the third quadrant, $x$ and $y$ are both negative, so sine and cosine are both negative, and hence tangent is positive.
- In the fourth quadrant, $x$ is positive and $y$ is negative, so sine is negative and cosine is positive, and hence tangent is negative.

The sign of the tangent is worked out from the signs of the sine and cosine by using the fact that

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

for any angle $\theta$. 
There is a useful way to remember these signs, which is shown in Figure 42. The letters tell you which of sine, cosine and tangent are positive in which quadrants:

- A stands for all,
- S stands for sine,
- T stands for tangent,
- C stands for cosine.

To remember this diagram, you might like to think of the word CAST or the mnemonic phrase ‘All Silly Tom Cats’.

**Using the CAST diagram and the related angles diagram to solve trigonometric equations**

You can use the information in the CAST diagram and the related angles diagram to help you to solve simple trigonometric equations. The method is demonstrated in the example on the next page.

You usually also need to use your calculator. Whenever you use your calculator for trigonometry, remember to check that it is set to use the units for angles that you are working with – degrees or radians.

**Example 9 Solving trigonometric equations**

Find all the solutions between 0° and 360° of the following equations. Give your answers to the nearest degree.

(a) \( \cos \theta = 0.8 \)
(b) \( \tan \theta = -4 \)

**Solution**

(a) ○ Use the CAST diagram to find the quadrants of the solutions. ○

The cosine of \( \theta \) is positive, so \( \theta \) is a first- or fourth-quadrant angle.

○ Use your calculator to find the first-quadrant angle. ○

One solution of the equation is
\[
\theta = \cos^{-1}(0.8) = 37^\circ \quad \text{(to the nearest degree)}.
\]

○ Use the related angles diagram to find the related fourth-quadrant angle. ○

The other solution is
\[
\theta = 360^\circ - 37^\circ = 323^\circ \quad \text{(to the nearest degree)}.
\]

(Check: A calculator gives
\[
\cos 37^\circ \approx 0.798, \quad \cos 323^\circ \approx 0.798.
\]

(b) ○ Use the CAST diagram to find the quadrants of the solutions. ○

The tangent of \( \theta \) is negative, so \( \theta \) is a second- or fourth-quadrant angle.

○ Use your calculator to find the related first-quadrant angle whose tangent has the same magnitude but is positive. ○

The related first-quadrant angle is
\[
\tan^{-1}(4) = 76^\circ \quad \text{(to the nearest degree)}.
\]

○ Use the related angles diagram to find the related second and fourth-quadrant angles. ○

---

**Figure 42** The CAST diagram
The solutions are
\[ \theta = 180^\circ - 76^\circ = 104^\circ \text{ (to the nearest degree)}, \]
\[ \theta = 360^\circ - 76^\circ = 284^\circ \text{ (to the nearest degree)}. \]

(Check: A calculator gives
\[ \tan 104^\circ = -4.010\ldots \approx -4, \]
\[ \tan 284^\circ = -4.010\ldots \approx -4. \])

Before you do the next activity, check that your calculator is set to use degrees rather than radians, if you haven’t done so already.

**Activity 16  Solving trigonometric equations**

Find all the solutions between 0° and 360° of the following equations. Give your answers to the nearest degree.

(a) \( \sin \theta = 0.2 \)  
(b) \( \cos \theta = -0.6 \)

Once you have found all the solutions of a trigonometric equation in the interval 0° to 360°, it is straightforward to find any other solutions that you want. The trigonometric functions repeat every 360°, so adding or subtracting a multiple of 360° to a solution gives another solution. For example, in Example 9(a) it was found that the solutions of the equation \( \cos \theta = 0.8 \) in the interval 0° to 360° are approximately 37° and 323°. So some other approximate solutions are, for example,
\[ 37^\circ + 360^\circ = 397^\circ, \]
\[ 37^\circ - 2 \times 360^\circ = -683^\circ, \]
\[ 323^\circ - 360^\circ = -37^\circ. \]

**5.10 Solving trigonometric equations in radians**

If you go on to higher-level mathematics courses, then you will often work with angles in radians rather than degrees. So in this subsection you will have a chance to practise solving trigonometric equations in radians. You can use exactly the same method as in Subsection 5.9, just with the angles converted to radians.

Remember that 2\( \pi \) radians is the same as 360°, so the boundary values of the quadrants in radians are
\[ 0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2} \quad \text{and} \quad 2\pi, \]
as shown in Figure 43.
Figure 44(a) repeats the CAST diagram from earlier, and Figure 44(b) shows the related angles diagram in radians.

\[ \begin{align*}
\text{S} & \quad \text{A} \\
\text{T} & \quad \text{C} \\
\phi & \quad \phi \\
\pi - \phi & \quad \phi \\
\pi + \phi & \quad 2\pi - \phi \\
\end{align*} \]

Figure 43 The quadrants

Figure 44 (a) The CAST diagram. (b) The related angles diagram in radians.

In the example below a trigonometric equation is solved in radians.

**Example 10  Solving a trigonometric equation in radians**

Find all the solutions between 0 and \( 2\pi \) of the equation

\[ \sin \theta = -\frac{\sqrt{3}}{2}, \]

giving exact answers in radians.

**Solution**

The equation is

\[ \sin \theta = -\frac{\sqrt{3}}{2}. \]

Use the CAST diagram to find the quadrants of the solutions. The sine of \( \theta \) is negative, so \( \theta \) is a third- or fourth-quadrant angle.

Find the related first-quadrant angle, either by using your calculator or by recognising that \( \sqrt{3}/2 \) appears as a sine value in the special angles table (Table 2 on page 202).

The related first-quadrant angle is

\[ \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3}. \]

Use the related angles diagram to find the solutions.

The solutions are

\[ \begin{align*}
\theta &= \pi + \frac{\pi}{3} = \frac{4\pi}{3}, \\
\theta &= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.
\end{align*} \]

(Check: A calculator gives

\[ \begin{align*}
\sin \left( \frac{4\pi}{3} \right) &= -\frac{\sqrt{3}}{2}, \\
\sin \left( \frac{5\pi}{3} \right) &= -\frac{\sqrt{3}}{2}.
\end{align*} \] )
Here is a similar activity for you to try. When you do it, don’t solve the equations in degrees and then convert the solutions to radians. Instead, work with radians throughout, as in Example 10; this will be useful practice for doing mathematics at higher levels. Make sure that your calculator is set to use radians.

### Activity 17  Solving trigonometric equations in radians

Find all the solutions between 0 and $2\pi$ radians of the following equations, giving your answers in radians. In parts (a) and (b) give exact answers, and in part (c) give answers to three significant figures.

(a) $\cos \theta = \frac{\sqrt{3}}{2}$  
(b) $\tan \theta = -\sqrt{3}$  
(c) $\cos \theta = 0.4$

Finally in this subsection, notice that you can use the CAST diagram and the related angles diagram, together with the table of special angles (Table 2 on page 24), to find the exact values of the sines, cosines and tangents of some more angles.

For example, the special angles table tells you that $\cos \left(\frac{\pi}{3}\right) = \frac{1}{2}$, so, by the related angles diagram, the cosine of each of the following angles is either $\frac{1}{2}$ or $-\frac{1}{2}$:

$$\pi - \frac{\pi}{3}, \quad \pi + \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}.$$  

These angles simplify to

$$\frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad \frac{5\pi}{3}.$$  

They lie in the second, third and fourth quadrants, respectively, so, by the CAST diagram,

$$\cos \left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad \cos \left(\frac{4\pi}{3}\right) = -\frac{1}{2}, \quad \cos \left(\frac{5\pi}{3}\right) = \frac{1}{2}.$$  

You can use this method to find the exact value of the sine, cosine or tangent of any angle that is linked by the related angles diagram to one of the special angles.
Activities and comments on Activities

Activity 1
The side length 7 m is opposite the angle of 55°, and the unknown side length $x$ is opposite the angle of 60°. By the Sine Rule,
\[
\frac{7}{\sin 55°} = \frac{x}{\sin 60°}
\]
so
\[
x = \frac{7 \sin 60°}{\sin 55°} = 7.400 \ldots
\]
Thus $x = 7.40$ m (to 3 s.f.).

Activity 2
The unknown side length $x$ is opposite the angle of 55°, and this angle is between two known side lengths. So we can apply the Cosine Rule to the triangle to give
\[
x^2 = 6^2 + 10^2 - 2 \times 6 \times 10 \cos 55° = 67.170 \ldots
\]
Thus $x = \sqrt{67.170} \ldots = 8.19 \ldots = 8.2$ (to 2 s.f.).

Activity 3
We first calculate $\angle B$ using the Cosine Rule. The side opposite $\angle B$ has length 4, and this appears on the left-hand side of the equation, with the other two lengths on the right-hand side:
\[
4^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \cos B,
\]
so
\[
16 = 25 - 24 \cos B.
\]
Hence
\[
\cos B = \frac{25 - 16}{24} = \frac{9}{24} = 0.375.
\]
A calculator gives $\cos^{-1}(0.375) = 67.975 \ldots$, so $\angle B = 68°$ to the nearest degree.
As the triangle is isosceles, $\angle A = \angle B$. So $\angle A = 68°$ to the nearest degree.
The third angle can be found by using the fact that the angles of a triangle add up to 180°. Using the unrounded value of $\angle B$, we obtain the following value for $\angle C$:
\[
180 - 2 \times 67.975 \ldots = 44.048 \ldots
\]
So $\angle C = 44°$ to the nearest degree.

Activity 4
(a) By the area formula, the area of the triangle in km$^2$ is
\[
\frac{1}{2} \times 8 \times 10 \sin 40° = 25.71 \ldots
\]
Thus the area is 25.7 km$^2$ (to 3 s.f.).
(b) By the area formula, the area of the triangle in cm$^2$ is
\[
\frac{1}{2} \times 6 \times 7 \sin 60° = 18.18 \ldots
\]
Thus the area is 18.2 cm$^2$ (to 3 s.f.).

Activity 5
(a) Since the side length of the equilateral triangle is 2 and each angle is 60°, the area is
\[
\frac{1}{2} \times 2 \times 2 \sin 60° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}.
\]
(b) Each side has length 2, so the semi-perimeter is $s = \frac{1}{2}(2 + 2 + 2) = 3$. By Heron’s Formula, the area is
\[
\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{3(3-2)(3-2)(3-2)} = \sqrt{3},
\]
as in part (a).

Activity 6
(a) $P$ lies in the first quadrant.
(b) $P$ lies in the third quadrant.
(c) $P$ lies in the first quadrant.
(d) \( P \) lies in the fourth quadrant.

\[ \begin{align*}
\cos 225^\circ &= -\frac{1}{\sqrt{2}} = -0.7071 \text{ (to 4 d.p.)} \\
\sin 225^\circ &= -\frac{1}{\sqrt{2}} = -0.7071 \text{ (to 4 d.p.)}
\end{align*} \]

These values agree with those given by a calculator.

\[ \begin{align*}
\tan 225^\circ &= \frac{\sin 225^\circ}{\cos 225^\circ} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1 \\
\tan(-180^\circ) &= \frac{\sin(-180^\circ)}{\cos(-180^\circ)} = \frac{0}{-1} = 0
\end{align*} \]

\[ \begin{align*}
\cos(-180^\circ) &= -1 \quad \text{and} \quad \sin(-180^\circ) = 0.
\end{align*} \]

These values agree with those given by a calculator.

\[ \begin{align*}
\theta &= 450^\circ \quad \text{and} \quad \theta = 630^\circ.
\end{align*} \]

\[ \begin{align*}
\text{angle in radians} &= \frac{\pi}{180} \times 75 = \frac{5\pi}{12}.
\end{align*} \]

So \( 75^\circ = 5\pi/12 \) radians.
(ii) Applying the degrees-to-radians conversion formula gives

\[
\text{angle in radians} = \frac{\pi}{180} \times 225 = \frac{5\pi}{4}.
\]

So \(225^\circ = \frac{5\pi}{4}\) radians.

(b) (i) Applying the radians-to-degrees conversion formula gives

\[
\text{angle in degrees} = 180\times\frac{3\pi}{4} = 135.
\]

So \(\frac{3\pi}{4}\) radians = \(135^\circ\).

(ii) Applying the radians-to-degrees conversion formula gives

\[
\text{angle in degrees} = 180\times\frac{4\pi}{5} = 144.
\]

So \(\frac{4\pi}{5}\) radians = \(144^\circ\).

Activity 11

(a) Applying the degrees-to-radians conversion formula gives

\[
\text{angle in radians} = \frac{\pi}{180} \times 60 = \frac{\pi}{3}.
\]

So \(60^\circ = \frac{\pi}{3}\) radians.

(Alternatively, you could have looked up the angle \(60^\circ\) in Table 2 on page 24.)

(b) The length of each of the two arcs in the diagram is given by \(r\theta\), where \(r = 1.5\) and \(\theta = \pi/3\). So the length of each arc in metres is

\[1.5 \times \frac{\pi}{3} = 0.5\pi.
\]

Hence the total length of edging required is

\[2 \times 0.5\pi + 2 \times 2.5 + 1.5 = \pi + 6.5 = 9.64\text{ m} \text{ (to 2 d.p.).}
\]

(c) (i) The area of the triangle is given by the formula

\[
\text{area} = \frac{1}{2}ab\sin \theta,
\]

with \(a = b = 1.5\) and \(\theta = 60^\circ\). So it is

\[
\frac{1}{2} \times 1.5^2 \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{16} = 0.974\ldots \text{ m}^2.
\]

(ii) The area of the sector is given by the formula

\[
\text{area} = \frac{1}{2}r^2\theta,
\]

with \(r = 1.5\) and \(\theta = \pi/3\). So it is

\[
\frac{1}{2} \times 1.5^2 \times \frac{\pi}{3} = \frac{3\pi}{8} = 1.178\ldots \text{ m}^2.
\]

(iii) There are two sectors visible in the diagram of the window, and their overlap is the triangle formed by the dashed lines. So the area of the window above the horizontal dashed line is

\[
2 \times \text{area of sector} - \text{area of triangle}
= 2 \times 1.178\ldots - 0.974\ldots
= 1.381\ldots \text{ m}^2.
\]

(Alternatively, you could subtract the area of the triangle from the area of one sector to find the area of one of the two thin segments at the top of the window. Then you could calculate the area of the window above the horizontal dashed line by adding twice the area of the segment to the area of the triangle.)

(iv) The area of the window below the horizontal dashed line is

\[1.5 \times 2.5 = 3.75 \text{ m}^2,
\]

so the area of the whole window is

\[1.381\ldots + 3.75 = 5.13 \text{ m}^2 \text{ (to 2 d.p.).}
\]

Activity 12

Remember to set your calculator to work in radians before entering these calculations.

(a) \(\sin 1 = 0.841\) (to 3 s.f.).

(b) \(\cos \frac{\pi}{3} = \frac{1}{2} = 0.5\).

(c) \(\tan^{-1}(1) = \frac{1}{4}\pi \text{ or } 0.785\) (to 3 s.f.).

Activity 13

The length of the hypotenuse, \(x\), can be calculated by applying Pythagoras’ Theorem:

\[x^2 = 1^2 + 1^2 = 2, \quad \text{so} \quad x = \sqrt{2}.
\]

So the hypotenuse has length \(\sqrt{2}\) and the adjacent and opposite sides have length 1. Thus

\[
\sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = \frac{1}{1} = 1.
\]
Activity 14

(a) In this triangle,
\[ \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}. \]

Also,
\[ \sin(90^\circ - \theta) = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}, \quad \cos(90^\circ - \theta) = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}. \]

(b) By the equations in part (a),
\[ \cos \theta = \frac{b}{c} = \sin(90^\circ - \theta) \]
and
\[ \sin \theta = \frac{a}{c} = \cos(90^\circ - \theta). \]

Activity 15

(a) The point \( R \) corresponds to the angle \( 180^\circ + 25^\circ = 205^\circ \):

The point \( S \) corresponds to the angle \( 360^\circ - 25^\circ = 335^\circ \):

(b) The coordinates of \( R \) are
\((-0.906, -0.423).\)
The coordinates of \( S \) are
\((0.906, -0.423).\)
Both pairs of coordinates are given to three significant figures.

(c) The sine, cosine and tangent of the angle corresponding to \( R \) are
\[ \sin(205^\circ) \approx -0.423, \quad \cos(205^\circ) \approx -0.906 \]
and
\[ \tan(205^\circ) = \frac{-0.42261\ldots}{-0.90630\ldots} \approx 0.466. \]

Similarly, the sine, cosine and tangent of the angle corresponding to \( S \) are
\[ \sin(335^\circ) \approx -0.423, \quad \cos(335^\circ) \approx 0.906 \]
and
\[ \tan(335^\circ) = \frac{-0.42261\ldots}{0.90630\ldots} \approx -0.466. \]

All these values are given to three significant figures.

Activity 16

(a) The equation is
\[ \sin \theta = 0.2. \]
The sine of \( \theta \) is positive, so \( \theta \) is a first- or second-quadrant angle.

One solution is
\[ \theta = \sin^{-1}(0.2) = 12^\circ \] (to the nearest degree).
The other solution is
\[ \theta = 180^\circ - 12^\circ = 168^\circ \] (to the nearest degree).

(Check: A calculator gives
\[ \sin 12^\circ = 0.207\ldots \approx 0.2, \]
\[ \sin 168^\circ = 0.207\ldots \approx 0.2. \])

(b) The equation is
\[ \cos \theta = -0.6. \]
The cosine of \( \theta \) is negative, so \( \theta \) is a second- or third-quadrant angle.

The related first-quadrant angle is
\[ \theta = \cos^{-1}(0.6) = 53^\circ \] (to the nearest degree).
The solutions are
\[ \theta = 180^\circ - 53^\circ = 127^\circ \] (to the nearest degree),
\[ \theta = 180^\circ + 53^\circ = 233^\circ \] (to the nearest degree).

(Check: A calculator gives
\[ \cos 127^\circ = -0.6018\ldots \approx -0.6, \]
\[ \cos 233^\circ = -0.6018\ldots \approx -0.6. \])

Activity 17

(a) The equation is
\[ \cos \theta = \frac{\sqrt{3}}{2}. \]
The cosine of \( \theta \) is positive, so \( \theta \) is a first- or fourth-quadrant angle.

One solution is
\[ \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}. \]
The other solution is
\[ \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \].

(b) The equation is
\[ \tan \theta = -\sqrt{3} \].
The tangent of \( \theta \) is negative, so \( \theta \) is a second- or fourth-quadrant angle.

The related first-quadrant angle is
\[ \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \].
The solutions are
\[ \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \],
\[ \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \].

(c) The equation is
\[ \cos \theta = 0.4 \].
The cosine of \( \theta \) is positive, so \( \theta \) is a first- or fourth-quadrant angle.

One solution is
\[ \theta = \cos^{-1}(0.4) = 1.1592\ldots = 1.16 \text{ (to 3 s.f.)} \].
The other solution is
\[ \theta = 2\pi - 1.1592\ldots = 5.1239\ldots = 5.12 \text{ (to 3 s.f.)} \].
(You can check these solutions on your calculator.)