UNIT
S111 to MST124 – additional mathematics support
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4 Quadratic expressions and equations

Multiplying out pairs of brackets

4.1 Pairs of brackets

In Chapter 2 you learned how to multiply out, or expand, brackets in expressions such as

\[ 2x(-3y + 2z). \]

There, you used the following strategy.

**Strategy** *To multiply out brackets*

Multiply each term inside the brackets by the multiplier of the brackets.

As a reminder, here are some examples of multiplying out brackets:

\[ a(b + c) = ab + ac, \]
\[ -2(x - y) = -2x + 2y, \]
\[ 3m(-2n + 3r - 6) = -6mn + 9mr - 18m. \]

You should also remember that if the multiplier follows the brackets, then you can apply the same strategy. The next activity gives you a chance to revise multiplying out brackets.

**Activity 1** *Multiplying out brackets*

Multiply out the brackets in each of the following expressions.

(a) \( a(2b + 3c) \)  
(b) \( -r(2s - 3t) \)  
(c) \( (n - 1)n \)

There is an important identity that involves a product of two brackets:

\[ (n - 1)(n + 1) = n^2 - 1. \]

This can also be proved by using algebra to multiply out the brackets in \((n - 1)(n + 1)\). Such products of brackets occur in many situations in mathematics, so it is important to be able to multiply them out correctly.

You can multiply out two brackets of the form

\[ (a + b)(c + d) \]

in two steps, as follows.

- First, keep \((a + b)\) as one expression and use it as the multiplier to expand the right bracket \((c + d)\), to obtain

\[ (a + b)(c + d) = (a + b)c + (a + b)d. \]
• Second, expand each of the \((a + b)\) brackets on the right-hand side:

\[
(a + b)(c + d) = (a + b)c + (a + b)d
\]

\[
= ac + bc + ad + bd.
\]

Alternatively, you could first expand the left bracket \((a + b)\), using \((c + d)\) as the multiplier. You can check that the answer is the same!

If you examine the effect of multiplying out the brackets in \((a + b)(c + d)\), then you can see that each term in the second bracket is multiplied by each term in the first bracket. This always happens when you multiply out two brackets, and it gives the following strategy.

**Strategy**  
To multiply out two brackets

Multiply each term inside the first bracket by each term inside the second bracket, and add the resulting terms.

When you use this strategy to multiply out

\[(a + b)(c + d),\]

you have to multiply each of the terms \(a\) and \(b\) in the first bracket by each of the terms \(c\) and \(d\) in the second bracket. It is a good idea to be systematic about the order in which you do these four multiplications.

Figure 1 shows the order that is usually used in this course.

The acronym FOIL may help you to remember the order in which these pairs are multiplied. It stands for:

1. First: \(ac\),
2. Outer: \(ad\),
3. Inner: \(bc\),
4. Last: \(bd\).

Note that the answer obtained before the strategy above was a different arrangement of these four terms.

If \(a, b, c\) and \(d\) represent positive numbers, then the rule for multiplying out brackets can be thought of in terms of areas of rectangles as follows. In Figure 2 the sides of the large rectangle are \(a + b\) units and \(c + d\) units, and this rectangle can be split into four smaller rectangles with areas \(ac\), \(ad\), \(bc\) and \(bd\) square units. Adding these four areas shows that the above expansion of the brackets is correct in this context.

**Figure 1**  
Order of multiplications

Here is the result:

\[(a + b)(c + d) = ac + ad + bc + bd.\]

Of course, it is not essential to follow the order given in the above diagram. If you already have some experience of multiplying out pairs of brackets and have developed your own method of doing it, then that’s fine as long as you obtain the correct answer!

If \(a, b, c\) and \(d\) represent positive numbers, then the rule for multiplying out brackets can be thought of in terms of areas of rectangles as follows. In Figure 2 the sides of the large rectangle are \(a + b\) units and \(c + d\) units, and this rectangle can be split into four smaller rectangles with areas \(ac\), \(ad\), \(bc\) and \(bd\) square units. Adding these four areas shows that the above expansion of the brackets is correct in this context.
As usual, you have to be careful when multiplying out brackets if minus signs are present. For example, you can use the strategy above to multiply out the product

\[(2s - t)(u - 3v),\]

but you must remember that the terms in the first bracket are \(2s\) and \(-t\), and the terms in the second bracket are \(u\) and \(-3v\). It may help to mark these terms, as you did in Section 3 of Chapter 2, before multiplying each term in the first bracket by each term in the second bracket:

\[
2s \times u = 2su, \\
2s \times (-3v) = -6sv, \\
(-t) \times u = -tu, \\
(-t) \times (-3v) = 3tv.
\]

The resulting four terms on the right-hand side have each been simplified in one step, as you learned to do in Chapter 2.

Here are some more examples of multiplying out brackets.

**Example 1  Multiplying out two brackets**

Multiply out the brackets in each of the following expressions.

(a) \((x + 1)(x + 2)\) \hspace{1cm} (b) \((x - 1)(x - 2)\) \hspace{1cm} (c) \((x + 1)^2\)

(d) \((a + 2b)(3c - d)\) \hspace{1cm} (e) \((n - 1)(n + 1)\)

**Solution**

(a) \((x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2\)
(b) \((x - 1)(x - 2) = x^2 - 2x - x + 2 = x^2 - 3x + 2\)
(c) \((x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = x^2 + 2x + 1\)
(d) \((a + 2b)(3c - d) = 3ac - ad + 6bc - 2bd\)
(e) \((n - 1)(n + 1) = n^2 + n - n - 1 = n^2 - 1\)

The next activity gives you lots of practice in multiplying out brackets.

**Activity 2  Multiplying out two brackets**

Multiply out the brackets in each of the following expressions, simplifying the answer where appropriate.

(a) \((x + 2)(x + 4)\) \hspace{1cm} (b) \((a + 2b)(3x + 4y)\) \hspace{1cm} (c) \((x - 3)^2\)

(d) \((a - 2b)(3c - d)\) \hspace{1cm} (e) \((p - 1)(-2 + 3q)\) \hspace{1cm} (f) \((n - 2)(n + 2)\)

The strategy on page 4 can also be used to multiply out two brackets that contain more than two terms. Here a little more care is needed to ensure that all possible pairs of terms are included. For example, in the product

\[(a + b)(c + d + e),\]

each of the two terms in the first bracket must multiply each of the three terms in the second bracket, so there are \(2 \times 3 = 6\) product terms in the answer. We obtain these terms in order, beginning with the products involving \(a\) and then the products involving \(b\):

\[
(a + b)(c + d + e) = a(c + d + e) + b(c + d + e) \\
= ac + ad + ae + bc + bd + be.
\]
Activity 3  Multiplying out longer brackets

Multiply out the brackets in the expression \((2a - b)(c - 3d + 2e)\).

4.2 Squaring brackets

You have now multiplied out many pairs of brackets, including ones involving squaring, such as

\[(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = x^2 + 2x + 1.\]

You often need to square brackets in this way, so it is a good idea to become familiar with this operation. Once again, there is a useful pattern to spot, which you can see in these examples:

\[
\begin{align*}
(x + 1)^2 &= x^2 + 2x + 1, \\
(x + 2)^2 &= x^2 + 4x + 4, \\
(x + 3)^2 &= x^2 + 6x + 9, \\
(x + 4)^2 &= x^2 + 8x + 16.
\end{align*}
\]

In each case, in the expansion on the right,

- the coefficient of \(x\) is twice the number in the bracket on the left
- the constant term is the square of the number in the bracket on the left.

In general,

\[
(x + p)^2 = (x + p)(x + p) = x^2 + xp + px + p^2 = x^2 + 2px + p^2.
\]

In other words, to square a bracket with two terms, you add together the square of the first term, the square of the last term, and twice the product of the two terms.

A similar pattern occurs when there is a negative sign in the brackets; for example:

\[
\begin{align*}
(x - 1)^2 &= x^2 - 2x + 1, \\
(x - 2)^2 &= x^2 - 4x + 4, \\
(x - 3)^2 &= x^2 - 6x + 9, \\
(x - 4)^2 &= x^2 - 8x + 16.
\end{align*}
\]

Here the only difference is that a minus sign appears on each side.

Squaring brackets

\[
(x + p)^2 = x^2 + 2px + p^2
\]

and

\[
(x - p)^2 = x^2 - 2px + p^2.
\]

So, if you are asked to square brackets, then you may be able to write down the answer directly using one of these general identities (if the expression in brackets is \(x + p\) or \(x - p\), or of this form). Alternatively, you can multiply
out the brackets in the usual way. Here are some questions for you to try.

**Activity 4  Squaring brackets**

Multiply out the following brackets.

(a) \((x + 7)^2\)  
(b) \((u - 10)^2\)  
(c) \((t + \frac{1}{2})^2\)

(d) \((3x + 2)^2\)  
(e) \((2s - 5t)^2\)

Squaring brackets can also be used to find tricks for squaring numbers in your head. These tricks can even form the basis of a career!

Dr Arthur T. Benjamin is an American professor of mathematics and a professional magician. In a performance called ‘Mathemagics’, he does rapid mental calculations including squaring 2-, 3- and 4-digit numbers faster than his audience can square them with a calculator. He then explains how he uses simple algebraic techniques to do these mental feats.

To illustrate how algebra can explain such tricks, imagine that you want to square a two-digit number ending in 5. Such a number is of the form \(10m + 5\), where \(m\) is a natural number; for example, \(65 = 10 \times 6 + 5\), so \(m = 6\) in this case, and \(45 = 10 \times 4 + 5\), so \(m = 4\) in this case.

Here is a quick way to calculate the square of a number of this form.

To square \(10m + 5\), where \(m\) is a natural number:

- calculate \(m(m + 1)\)
- the required square is \(100m(m + 1) + 25\).

For example, to find \(65^2\), where \(m = 6\):

- \(6 \times 7 = 42\)
- \(4200 + 25 = 4225\).

In the second step you can obtain the answer quickly by putting the digits 25 immediately after the number you found in the first step.

The secret of the trick is just that the square of \(10m + 5\) is

\((10m + 5)^2 = 100m^2 + 100m + 25\).

Now \(100m^2 + 100m\) can be factorised as \(100m(m + 1)\), so the right-hand side can be rearranged to give

\((10m + 5)^2 = 100m(m + 1) + 25\).

The key idea here is to replace one apparently complicated calculation by several much easier ones. The difficulty is spotting what those easier calculations are. However, by using algebra you can manipulate one expression into another relatively easily and so not only find a simpler way of doing the calculation but also show that the new calculation works for all numbers of this form. This illustrates some of the power of using algebra and you’ll see another example of this in the next section.
4.3 Differences of two squares

In Example 1 and Activity 2 on page 5 you were asked to multiply out pairs of brackets in which one bracket contains a sum of two numbers and the other bracket contains a difference of the same two numbers, such as

\[(n - 1)(n + 1) = n^2 - 1.\]

This is another type of product that occurs so often in mathematics that it is worth learning the pattern in the answer. Here is the general case:

\[(x - p)(x + p) = x^2 + xp - px - p^2 = x^2 - p^2.\]

Because the form of the right-hand side in this identity is a difference of two squares, namely \(x^2\) minus \(p^2\), the identity is given this name.

**Difference of two squares**

\[(x - p)(x + p) = x^2 - p^2\]

So, if you are asked to find the product of two brackets of the form above, one a sum and the other the corresponding difference, then you may be able to write down the answer directly using this identity. For example,

\[(2n - 1)(2n + 1) = (2n)^2 - 1^2 = 4n^2 - 1.\]

Alternatively, you can just multiply out the brackets in the usual way.

Here are some similar questions for you to try.

**Activity 5 Using a difference of two squares**

Use the difference of two squares identity to multiply out the following brackets.

(a) \((u - 12)(u + 12)\)  
(b) \((x - 2y)(x + 2y)\)  
(c) \((10 - a)(10 + a)\)

Finally, the following story shows that the difference of two squares identity may help you to win a quiz!

A television quiz once asked a contestant to calculate \(51^2 - 49^2\). At first sight, this appears to be quite a tricky calculation:

\[51^2 - 49^2 = 2601 - 2401 = 200.\]

However, by using a difference of two squares, the answer can be found in your head:

\[51^2 - 49^2 = (51 - 49)(51 + 49) = 2 \times 100 = 200.\]
### 4.4 Quadratic expressions

If you expand the brackets in the product \((x + 1)(3x - 4)\), then you obtain

\[
(x + 1)(3x - 4) = 3x^2 - 4x + 3x - 4 = 3x^2 - x - 4.
\]

The resulting expression involves three terms:

- a term in \(x^2\), namely \(3x^2\)
- a term in \(x\), namely \(-x\)
- a constant term, namely \(-4\).

An expression of the form

\[
ax^2 + bx + c,
\]

where \(a, b, c\) are numbers, and \(a \neq 0\),

is called a **quadratic expression** in \(x\), or a quadratic in \(x\), or just a **quadratic**.

The numbers \(a\), \(b\) and \(c\) are called the **coefficients** of the quadratic.

The coefficients \(b\) and \(c\) in a quadratic expression \(ax^2 + bx + c\) can equal 0, but \(a\) must be non-zero, so that the expression includes a term in \(x^2\). Also, you can write the terms of a quadratic in any order, but in this course we usually put them in the order above, with the term in \(x^2\) first.

As usual, the variable in a quadratic expression can be any letter. For example:

- \(x^2 + 2x - 3\) is a quadratic in \(x\), with \(a = 1\), \(b = 2\) and \(c = -3\)
- \(3t^2 + 1\) is a quadratic in \(t\), with \(a = 3\), \(b = 0\) and \(c = 1\)
- \(2x + 1\) is not a quadratic, as there is no term in \(x^2\)
- \(3y^3 + y^2 + 1\) is not a quadratic, because it includes a power of \(y\) higher than a square.

**Activity 6 Identifying quadratic expressions**

Which of the following expressions are quadratics? For those that are quadratics, state the values of \(a\), \(b\) and \(c\).

(a) \(9x^2 - 12x + 4\)  
(b) \(3y - 5\)  
(c) \(-6 - 7s^2\)  
(d) \(2x^3 + x^2\)

### 4.5 Quadratic equations

Any equation that can be expressed in the form

\[
ax^2 + bx + c = 0
\]

(by rearranging if necessary) is called a **quadratic equation** in \(x\). In this equation, \(x\) is an unknown, and \(a\), \(b\) and \(c\) are numbers with \(a \neq 0\). One of the key techniques of algebra that you will learn in this course is how to solve a quadratic equation, that is, how to find the values of \(x\) that satisfy the equation.

Remember that the statement \(a \neq 0\) is read as ‘\(a\) is not equal to 0’ or as ‘\(a\) is non-zero’.

The Latin word ‘quadrare’ means ‘to square’.
Activity 7  Checking a solution

Show that $x = 2$ is a solution of the quadratic equation

$$2x^2 - x - 6 = 0.$$

There are many problems in mathematics that require you to solve a quadratic equation. You have already met some examples in Topic 7 of S111, where you used Pythagoras’ Theorem to find side lengths in right-angled triangles. You will meet several more examples in this chapter and the next.

The importance of quadratic equations has sometimes been called into question. For example, there was a debate in Parliament in 2003 about quadratic equations and whether they were ‘irrelevant’. The closing speech of the debate included the following stirring words in support of the teaching of quadratic equations, and of mathematics in general!

‘Quadratic equations allow us to analyse the relationships between variable quantities, and they are the tool for understanding variable rates of change. It is in variable rates of change that quadratic equations are seen in economics, science and engineering. Examples of the use of quadratic equations include acceleration, ballistics and financial comparisons . . .

In conclusion, the teaching of quadratic equations, and of the mathematics curriculum overall, is key to a future workforce that can develop and use mathematical models in daily life. As research in a book of quotations reveals, Napoleon said: “The advancement and perfection of mathematics are intimately connected with the prosperity of the state.”’

Alan Johnson, Minister for Lifelong Learning, Further and Higher Education, 26 June 2003.

So how do quadratic equations occur? In Chapter 3 you saw that equations arise from problems in which you have to find an unknown number, by using the following strategy.

Strategy  To find an unknown number

- Represent the number that you want to find by a letter.
- Express the information that you know about the number as an equation.
- Solve the equation.
Here is a problem (not a practical problem, however) in which finding an unknown number leads to a quadratic equation.

Find a number with the property that if you square the number and add 6, then the answer is 5 times the number.

If you try some simple numbers, then you soon find that one answer is 2, because

\[2^2 + 6 = 10 \quad \text{and} \quad 5 \times 2 = 10.\]

But if you continue to search, then you find another solution, namely 3, because

\[3^2 + 6 = 15 \quad \text{and} \quad 5 \times 3 = 15.\]

The fact that there is more than one solution to this problem may seem strange at first – could there be further solutions?

To investigate, we can follow the strategy above. First, let \(x\) represent a number that has the given property. Then \(x^2 + 6\) must be the same as \(5x\); that is,

\[x^2 + 6 = 5x.\]

This equation can be rearranged to give the quadratic equation

\[x^2 - 5x + 6 = 0.\]

Thus if you know how to solve a quadratic equation, then you can solve the problem. In this case you can find two solutions without using the quadratic equation, as you saw, but if the numbers had been different, then using the quadratic equation might have been essential.

The above problem is just a puzzle, but it is intriguing because there are two possible solutions. In fact, it illustrates a type of problem used by Babylonian teachers almost 4000 years ago as part of the mathematical training of their scribes! For example, one of their problems was:

If the sum of the area of a square and its side length is \(\frac{3}{4}\), then what is the side length of the square?

The ancient Babylonians developed many skills such as agriculture, irrigation, writing and arithmetic. Their systems of trade required reliable methods of calculating areas and volumes, to find the volumes of grain in storage containers, for example. So they invented many techniques for solving numerical problems, including ones leading to quadratic equations. The problems that they used for teaching were often puzzles. Other problems involved measurements of fields and construction projects, usually in somewhat unrealistic settings.

You will see more examples of how quadratic equations arise later in this chapter, after you have met a method of solving them.

When using mathematics to solve a practical problem, you should not add an area to a length, as the units are different.
4.6 Factorising quadratics of the form $x^2 + bx + c$

In this subsection you will learn a technique that can often be used to solve a quadratic equation. The technique is to factorise the quadratic expression that appears in the equation; that is, to express the quadratic expression as a product of simpler expressions. Initially, we consider only quadratic expressions of the form $x^2 + bx + c$ (so the coefficient of $x^2$ is 1), where $b$ and $c$ are integers.

You saw earlier that multiplying out brackets like $(x + 2)(x + 3)$ leads to a quadratic expression:

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6.$$

This shows that $x^2 + 5x + 6$ can be written as a product of the simpler expressions $x + 2$ and $x + 3$, each of which is called a factor of $x^2 + 5x + 6$.

But now suppose that you want to do the reverse process; that is, you want to find a factorisation of $x^2 + 5x + 6$:

$$x^2 + 5x + 6 = (\_)(\_),$$

where each of the two expressions in brackets on the right-hand side contains a term in $x$ and a constant term. Because the term in $x^2$ in the quadratic expression is just $x^2$, the factorisation must be of the form

$$x^2 + 5x + 6 = (x\_)(x\_),$$

where there is a positive or negative constant term in each of the gaps indicated. You can find these missing numbers by comparing the coefficients of the terms on both sides of the equation. When you multiply out the brackets on the right-hand side:

- the constant term that you get is the product of the two missing numbers, so this product is equal to the constant term on the left-hand side, which is 6
- the coefficient of the term in $x$ that you get is the sum of the two missing numbers, so this sum is equal to the coefficient of $x$ on the left-hand side, which is 5.

It is not hard to guess a pair of numbers whose product is 6 and whose sum is 5, namely 2, 3, so this gives the required factorisation:

$$x^2 + 5x + 6 = (x + 2)(x + 3),$$

as expected.

A more systematic approach is to consider all factor pairs of 6 and choose the pair whose sum is 5. The only factorisations of 6 as a product of two positive numbers are

$$1 \times 6 \quad \text{and} \quad 2 \times 3.$$

Among these factor pairs, only the pair 2, 3 has sum 5.

This approach is the basis of the following strategy for trying to factorise any quadratic expression of the form $x^2 + bx + c$, where $b$ and $c$ are integers.
**Strategy** To factorise $x^2 + bx + c$, where $b$ and $c$ are integers

Fill in the gaps in the brackets on the right-hand side of the equation

$$x^2 + bx + c = (\, \, \,)(\, \, \,)$$

with two numbers whose product is $c$ and whose sum is $b$:

$$x^2 + bx + c.$$  

\[ \uparrow \quad \uparrow \]

sum product

You can search systematically for integers with these properties as follows:

- write down the factor pairs of $c$, the constant term
- choose (if possible) a pair whose sum is $b$, the coefficient of $x$.

If there is no pair of integers with these two properties, then a factorisation of the quadratic of the form $x^2 + bx + c = (\, \, \,)(\, \, \,)$, with integers in the gaps, is not possible. Here we concentrate on those quadratics that can be factorised using integers, but you should be aware that this is not possible for all quadratics.

Here is an example of this strategy in action.

---

**Example 2**

**Factorising a quadratic expression**

Factorise the quadratic expression $x^2 + 6x + 8$.

**Solution**

Find a pair of numbers whose product is 8 and whose sum is 6.

The positive factor pairs of 8 are

1, 8, 2, 4.

The only pair whose sum is 6 is 2, 4.

If you spot that the pair 2, 4 has product 8 and sum 6 straight away, then there is no need to write down any other factor pairs.

Thus

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

(Check: Multiplying out the brackets gives 

$$(x + 2)(x + 4) = x^2 + 4x + 2x + 8$$

$$= x^2 + 6x + 8.$$)

When using the strategy on the opposite page, you will often need to consider factor pairs of the constant term $c$ that include negative factors; for example, $-1, -8$ and $-2, -4$ are factor pairs of 8. However, in
Example 2 you needed to consider only positive factor pairs, because:
- the product of the factors, 8, is positive, so the factors must have the same sign as each other
- the sum of the factors, 6, is positive, so both factors must be positive.
So you needed to consider only the factor pairs 1, 8 and 2, 4.
The quadratics for you to factorise in Activity 8 are of this type.
The strategy on the opposite page suggests that you write down the possible factor pairs systematically. However, if you can spot the required factor pair, then there is no need to write down the others.

### Activity 8  Factorising quadratic expressions

Factorise each of the following quadratic expressions.

(a) \( x^2 + 3x + 2 \)  
(b) \( x^2 + 11x + 24 \)

In the solution to Activity 8(b), all positive factor pairs of 24 were listed for completeness. But when you solve such a problem, you could omit the pair 1, 24, for example, since its sum is clearly not 11. Your aim is to find a factor pair that does work!

In the next example, the constant term \( c \) is again positive but \( b \), the coefficient of \( x \), is negative. In this case, you need to consider only negative factor pairs, because:
- the product of the factors is positive, so the factors must have the same sign as each other
- the sum of the factors is negative, so both factors must be negative.

### Example 3  Factorising a quadratic expression

Factorise the quadratic expression \( x^2 - 5x + 6 \).

**Solution**

\( \neq \) Find a pair of numbers whose product is 6 and whose sum is \(-5\). \( \neq \)
The negative factor pairs of 6 are
- \(-1, -6\), \(-2, -3\).
The only pair whose sum is \(-5\) is \(-2, -3\).
Thus
\[ x^2 - 5x + 6 = (x - 2)(x - 3). \]
(Check: Multiplying out the brackets gives
\( (x - 2)(x - 3) = x^2 - 3x - 2x + 6 \)
\[ = x^2 - 5x + 6. \])
Here are some quadratics of this type for you to factorise.

Activity 9  Factorising quadratic expressions

Factorise each of the following quadratic expressions.
(a) $x^2 - 10x + 24$  (b) $t^2 - 4t + 3$  (c) $x^2 - 6x + 9$

In the next example, the constant term $c$ is negative, so the numbers in the factor pairs of $c$ must have opposite signs. This leads to more factor pairs than if $c$ is positive.

Example 4  Factorising a quadratic expression

Factorise the quadratic expression $x^2 - 7x - 8$.

Solution  
Find a pair of numbers whose product is $-8$ and whose sum is $-7$.  
The factor pairs of $-8$ are 
$1, -8$,  $2, -4$,  $-1, 8$,  $-2, 4$.  
The only pair whose sum is $-7$ is $1, -8$.
Thus
$x^2 - 7x - 8 = (x + 1)(x - 8)$.

(Check: Multiplying out the brackets gives
$(x + 1)(x - 8) = x^2 - 8x + x - 8$
$= x^2 - 7x - 8$.)

Here are some quadratics of this type for you to factorise.

Activity 10  Factorising quadratic expressions

Factorise each of the following quadratic expressions.
(a) $x^2 - x - 2$  (b) $u^2 + 4u - 12$

Factorising special quadratic expressions

Some special quadratic expressions can be factorised more easily. For example, if there is no constant term (that is, if $c = 0$), then $x$ is always a common factor. For example:
$x^2 + 4x = x(x + 4)$,
$x^2 - 6x = x(x - 6)$.

Another special factorisation occurs when the quadratic expression is a difference of two squares. For example:
$x^2 - 1 = (x - 1)(x + 1)$,  because  $x^2 - 1 = x^2 - 1^2$,
$x^2 - 9 = (x - 3)(x + 3)$,  because  $x^2 - 9 = x^2 - 3^2$. 
Finally, you can sometimes recognise that a quadratic expression is a **perfect square**: that is, it is equal to the square of a simpler expression, because it is of the form

\[ x^2 + 2px + p^2 \quad \text{or} \quad x^2 - 2px + p^2. \]

In these cases

\[ x^2 + 2px + p^2 = (x + p)^2 \quad \text{or} \quad x^2 - 2px + p^2 = (x - p)^2, \]

as you saw in Subsection 4.2. For example, the quadratic expression

\[ x^2 - 6x + 9 \]

from Activity 9 is of the form

\[ x^2 - 2px + p^2, \quad \text{with} \quad p = 3. \]

This shows, without using the strategy, that

\[ x^2 - 6x + 9 = (x - 3)^2. \]

### Activity 11  **Factorising special quadratic expressions**

Factorise each of the following quadratic expressions.

(a) \( x^2 - x \)  
(b) \( u^2 - 16 \)  
(c) \( t^2 - 9t \)  
(d) \( x^2 + 10x + 25 \)

### 4.7 Solving quadratic equations by factorisation

Suppose now that you want to solve the quadratic equation

\[ x^2 - 5x + 6 = 0. \]  \hspace{1cm} (1)

In Example 3 you saw that you can factorise the quadratic expression

\[ x^2 - 5x + 6 \quad \text{as} \quad (x - 2)(x - 3). \]

So you can rewrite equation (1) as

\[ (x - 2)(x - 3) = 0. \]

How does this rewriting of the equation help? Well, this new equation states that the product of the two numbers \( x - 2 \) and \( x - 3 \) is 0. So you can use the following property of numbers.

**If the product of two or more numbers is 0, then at least one of the numbers must be 0.**

You can apply this property to the above factorisation to deduce that

\[ x - 2 = 0 \quad \text{or} \quad x - 3 = 0. \]

If \( x - 2 = 0 \), then \( x = 2 \), and if \( x - 3 = 0 \), then \( x = 3 \). So the quadratic equation (1) has two solutions:

\[ x = 2 \quad \text{and} \quad x = 3. \]

Here is a check that these values of \( x \) do indeed satisfy equation (1):

- when \( x = 2 \), \[ x^2 - 5x + 6 = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0, \]
- when \( x = 3 \), \[ x^2 - 5x + 6 = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0. \]
In general, you can use the following strategy.

**Strategy**  To solve $x^2 + bx + c = 0$ by factorisation

1. Find a factorisation:
   
   $x^2 + bx + c = (x + p)(x + q).

2. Then $(x + p)(x + q) = 0$, so
   
   $x + p = 0$ or $x + q = 0$,

   and hence the solutions are
   
   $x = -p$ and $x = -q$.

Note that the two solutions of a quadratic equation may be the same; in this case the equation is said to have a repeated solution.

Once you have found the solutions, it is a good idea to check that they both satisfy the equation $x^2 + bx + c = 0$.

Here is an example of this strategy in action.

**Example 5  Solving a quadratic equation**

Solve $x^2 - 7x - 8 = 0$ by factorisation.

**Solution**

The equation is: 

$x^2 - 7x - 8 = 0$

Factorise: 

$(x + 1)(x - 8) = 0$

So: 

$x + 1 = 0$ or $x - 8 = 0$

So: 

$x = -1$ or $x = 8$

(Check: When $x = -1$,

$x^2 - 7x - 8 = (-1)^2 - 7 \times (-1) - 8 = 1 + 7 - 8 = 0$.

When $x = 8$,

$x^2 - 7x - 8 = 8^2 - 7 \times 8 - 8 = 64 - 56 - 8 = 0$.)

Here are some quadratic equations for you to solve.

**Activity 12  Solving equations by factorisation**

Solve each of the following quadratic equations by factorisation.

(a) $x^2 + 3x + 2 = 0$  
(b) $x^2 - 10x + 24 = 0$  
(c) $t^2 - 16 = 0$

(d) $u^2 - u - 12 = 0$  
(e) $x^2 - 6x + 9 = 0$  
(f) $x^2 - 9x = 0$

Factorisation is an efficient method of solving quadratic equations when it can be applied. However, as stated earlier, not all quadratics of the form $x^2 + bx + c$ can be factorised in the form $(x + p)(y + q)$ using integers $p$ and $q$. Later in this chapter you will meet a formula for solving a quadratic equation which avoids the need for factorisation.
4.8 Factorising quadratics of the form \( ax^2 + bx + c \)

You have now seen how to factorise quadratic expressions of the form \( x^2 + bx + c \), whenever this is possible using integers. It’s also possible to factorise many expressions in which the coefficient of \( x^2 \) is not 1, as you will now see.

Sometimes, factorising a quadratic expression of the form \( ax^2 + bx + c \), where \( a \) is not 1, can be reduced to the case when the first term is \( x^2 \) because \( a \) is a common factor of the coefficients. For example, in the quadratic \( 2x^2 + 10x + 12 \), each of the coefficients is a multiple of 2, so

\[
2x^2 + 10x + 12 = 2(x^2 + 5x + 6).
\]

You saw earlier that \( x^2 + 5x + 6 = (x + 2)(x + 3) \), so

\[
2x^2 + 10x + 12 = 2(x + 2)(x + 3).
\]

**Activity 13** Factorising when the coefficients have a common factor

Factorise each of the following quadratic expressions.

(a) \( 3x^2 - 3x - 36 \)  
(b) \( -5x^2 + 15x - 10 \)

However, consider the quadratic expression

\[
2x^2 - x - 6.
\]

The presence of the coefficient 2 in the term \( 2x^2 \) means that a factorisation of the form \((x + p)(x + q)\) is not possible; also, 2 is not a common factor of the coefficients. A factorisation using integers may still be possible though, and below are two methods that you can use to try to find one. You can use either method, but if you have already met one of them and are confident in using it, then you may prefer to continue to use your method.

The first method is based on checking all possibilities.

**Example 6** Factorising a general quadratic expression – first method

Factorise \( 2x^2 - x - 6 \).

**Solution**

First note that the terms in \( x \) in the brackets must be \( 2x \) and \( x \). Then try to find a factorisation of the form

\[
2x^2 - x - 6 = (2x \underline{\text{ } }) (x \underline{\text{ } }),
\]

where the gaps each contain a positive or negative integer.

The two missing integers must have product \(-6\).

The possible factor pairs of \(-6\) are

\(-1, 6, \quad 1, -6, \quad -2, 3, \quad 2, -3.\)

These four factor pairs lead to eight possible cases:

\[
\begin{align*}
(2x - 1)(x + 6) & \quad \text{or} \quad (2x + 6)(x - 1), \\
(2x + 1)(x - 6) & \quad \text{or} \quad (2x - 6)(x + 1), \\
(2x - 2)(x + 3) & \quad \text{or} \quad (2x + 3)(x - 2), \\
(2x + 2)(x - 3) & \quad \text{or} \quad (2x - 3)(x + 2). \\
\end{align*}
\]
By multiplying out each of these pairs of brackets in turn, you find that one of these cases is the required factorisation, specifically,

\[(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6 = 2x^2 - x - 6.\]

Thus

\[2x^2 - x - 6 = (2x + 3)(x - 2).\]

You can often use this first method efficiently by deciding which are the most likely cases (for example, by considering which signs are possible) and checking these cases first. But the method can be time consuming because there may be many cases to consider. For example, if the first term of the quadratic is \(6x^2\), then you have to consider brackets starting with \(6x\) and \(x\), and also with \(3x\) and \(2x\).

In the second method, you need to consider fewer cases but it is not immediately clear why the method works. Indeed, explaining why it works requires a higher level of mathematics than is appropriate in this course. At this stage you should concentrate on learning the technique, which is described in the green text.

**Example 7** *Factorising a general quadratic expression – second method*

Factorise \(2x^2 - x - 6\).

**Solution**

The quadratic expression is of the form \(ax^2 + bx + c\), where \(a = 2\), \(b = -1\) and \(c = -6\).

\(\Rightarrow\) First, find two numbers whose product is \(ac\) and whose sum is \(b\). \(\Rightarrow\)

For this quadratic expression, \(ac = 2 \times (-6) = -12\) and \(b = -1\).

The possible factor pairs of \(-12\) are \(-1,12\), \(-1,12\), \(-2,6\), \(-2,6\), \(-3,4\), \(-3,4\).

The only pair whose sum is \(-1\) is \(-3,-4\).

\(\Rightarrow\) Next, rewrite the quadratic expression, splitting the term in \(x\) using the above factor pair. \(\Rightarrow\)

Since \(-1 = 3 - 4\),

\[2x^2 - x - 6 = 2x^2 + 3x - 4x - 6.\]

\(\Rightarrow\) Finally, group the four terms in pairs and take out common factors to give the required factorisation. \(\Rightarrow\)

Then

\[2x^2 - x - 6 = \underline{2x^2 + 3x} - 4x - 6 = x(2x + 3) - 2(2x + 3) = (x - 2)(2x + 3).\]

\(\Rightarrow\) Here, the first pair of terms on the RHS has a common factor of \(x\) and the second pair has a common factor of \(-2\), and then both expressions \(x(2x + 3)\) and \(-2(2x + 3)\) have a common factor of \((2x + 3)\). \(\Rightarrow\)

Thus

\[2x^2 - x - 6 = (x - 2)(2x + 3).\]
Note that the second method works in whichever order you split the middle term:

\[ 2x^2 - x - 6 = 2x^2 - 4x + 3x - 6 = 2x(x - 2) + 3(x - 2) = (2x + 3)(x - 2). \]

In the following activity you can use either the first method or the second method.

**Activity 14  Factorising a general quadratic expression**

Factorise each of the following quadratic expressions.

(a) \( 2x^2 - 5x + 3 \)  
(b) \( 6x^2 + 7x - 3 \)  
(c) \( 8x^2 - 10x - 3 \)

Once you have factorised a quadratic expression, you can solve the corresponding quadratic equation. In Examples 6 and 7 you saw that

\[ 2x^2 - x - 6 = (2x + 3)(x - 2). \]

Therefore the equation

\[ 2x^2 - x - 6 = 0 \]

can be written as

\[ (2x + 3)(x - 2) = 0. \]

Hence the solutions of this equation satisfy

\[ 2x + 3 = 0 \quad \text{or} \quad x - 2 = 0. \]

If \( 2x + 3 = 0 \), then \( 2x = -3 \) so \( x = -\frac{3}{2} \).

If \( x - 2 = 0 \), then \( x = 2 \).

Hence the solutions are

\[ x = -\frac{3}{2} \quad \text{and} \quad x = 2. \]

Here are some general quadratic equations for you to solve.

**Activity 15  Solving general quadratic equations**

Use your answers to Activity 14 to solve each of the following quadratic equations.

(a) \( 2x^2 - 5x + 3 = 0 \)  
(b) \( 6x^2 + 7x - 3 = 0 \)  
(c) \( 8x^2 - 10x - 3 = 0 \)

Before you start to solve a quadratic equation it is a good idea to check that it is in its simplest form as this can make it easier to work with. Here are some things that you can do to simplify a quadratic equation.

**Simplifying a quadratic equation**

- If the coefficient of \( x^2 \) is negative, then multiply the equation through by \(-1\) to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.
For example, to solve the equation
\[-5x^2 + 15x - 10 = 0,\]
you can multiply through by \(-1\) to obtain
\[5x^2 - 15x + 10 = 0.\]
Then divide the equation through by the common factor 5 to obtain
\[x^2 - 3x + 2 = 0,\]
which factorises to give
\[(x - 1)(x - 2) = 0.\]
So the solutions of this equation are \(x = 1\) and \(x = 2\).

Finally, the ancient Babylonian problem mentioned on page 11 leads to the equation
\[x^2 + x = \frac{3}{4},\]
which can be rearranged as
\[x^2 + x - \frac{3}{4} = 0.\]
You can obtain an equivalent quadratic equation with coefficients that are integers by clearing fractions in the usual way. Multiplying through by 4 gives the equation
\[4x^2 + 4x - 3 = 0.\]
This quadratic expression can be factorised using either the first or second method to give
\[4x^2 + 4x - 3 = (2x - 1)(2x + 3).\]
The solutions satisfy \(2x - 1 = 0\) or \(2x + 3 = 0\), so they are \(x = \frac{1}{2}\) and \(x = -\frac{3}{2}\). However, the problem was to find the side length of a square, so \(x\) is positive. Thus the answer to this problem is \(x = \frac{1}{2}\).

### 4.9 The quadratic formula

The quadratic formula, given below, provides a systematic way to find the exact solutions of any quadratic equation.

#### The quadratic formula

The solutions of the quadratic equation
\[ax^2 + bx + c = 0\]
are given by
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.

You will see why the formula works later in the chapter, but for now you should concentrate on how to use it. Here is an example.

**Example 8 Using the quadratic formula**

Use the quadratic formula to solve the equation \(3x^2 - 2x - 5 = 0\). 

The first person to give a formula for solving quadratic equations was the Indian mathematician Brahmagupta, in 628. He described the formula in words, but it was essentially the same as the modern quadratic formula.
Solution

Check that the equation is in the form $ax^2 + bx + c = 0$, and find the values of $a$, $b$ and $c$.

Here $a = 3$, $b = -2$ and $c = -5$. Substituting into the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-5)}}{2 \times 3}$$

$$= \frac{2 \pm \sqrt{4 + 60}}{6}$$

$$= \frac{2 \pm \sqrt{64}}{6}$$

$$= \frac{2 \pm 8}{6}$$

$$= \frac{2 + 8}{6} \text{ or } \frac{2 - 8}{6}$$

$$= \frac{10}{6} \text{ or } \frac{-6}{6}$$

$$= \frac{5}{3} \text{ or } -1.$$  

So the solutions are $x = \frac{5}{3}$ and $x = -1$.

The quadratic equation in Example 8 could alternatively have been solved by using factorisation, as shown below:

$$3x^2 - 2x - 5 = 0$$

$$(3x - 5)(x + 1) = 0$$

$$3x - 5 = 0 \text{ or } x + 1 = 0$$

$$x = \frac{5}{3} \text{ or } x = -1.$$  

This working is shorter and simpler than the working for the quadratic formula. So it is always worth checking whether a given quadratic equation can be factorised easily before you resort to using the quadratic formula. Factorising is often the quickest way to solve a quadratic equation, and the least likely to lead to mistakes.

The quadratic equation in Example 9 below cannot easily be factorised. In fact, its solutions turn out to be irrational, which confirms that it cannot be factorised using any rational numbers.

Example 9 Using the quadratic formula again

Use the quadratic formula to solve the equation $2x^2 + 4x - 7 = 0$. 

Solution

Here \( a = 2, b = 4 \) and \( c = -7 \). Substituting into the quadratic formula gives

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times (-7)}}{2 \times 2}
\]

\[
= \frac{-4 \pm \sqrt{16 + 56}}{4}
\]

\[
= \frac{-4 \pm \sqrt{72}}{4}
\]

\[
= \frac{-4 \pm 6\sqrt{2}}{4}
\]

\[
= -1 \pm \frac{3}{2}\sqrt{2}.
\]

So the solutions are

\[
x = -1 + \frac{3}{2}\sqrt{2} \quad \text{and} \quad x = -1 - \frac{3}{2}\sqrt{2}.
\]

The answers to Example 9 were left in surd form, so that they are exact. If they had been the answers to a problem in a real-life context, or were to be used to plot points on a graph, then it would be more sensible to give them as decimal approximations.

Notice also that the surds in Example 9 are expressed in their simplest form, by writing \( \sqrt{72} \) as \( 6\sqrt{2} \). If you give the solutions to a quadratic equation as surds, then you should write them in their simplest form, using the methods that you learned in Chapter 1. Sometimes you might find it helpful to use your calculator – the calculator recommended for the course can simplify surds.

Notice that in the last line of the working in Example 9, the expression

\[
\frac{-4 \pm 6\sqrt{2}}{4}
\]

was expanded to give

\[-1 \pm \frac{3}{2}\sqrt{2}.
\]

An alternative way to simplify expression (2) is to cancel the common factor 2 in the numerator and denominator, to give

\[
x = \frac{-2 \pm 3\sqrt{2}}{2},
\]

and then state the solutions as

\[
x = \frac{-2 + 3\sqrt{2}}{2} \quad \text{and} \quad x = \frac{-2 - 3\sqrt{2}}{2}.
\]

Either of these ways of writing the solutions is just as simple, and just as acceptable, as the other.

You can practise using the quadratic formula in the next activity.
Activity 16  Using the quadratic formula

Use the quadratic formula to solve the following quadratic equations.
(a) \( x^2 + 6x + 1 = 0 \)  
(b) \( 3x^2 - 8x - 2 = 0 \)

Remember to check that your quadratic equation is in the form \( ax^2 + bx + c = 0 \) before you apply the quadratic formula! If it is not in this form, then you must rearrange it before you identify the values of \( a \), \( b \) and \( c \). For example, if you want to solve the equation
\[ 3x^2 + 2x = 4, \]
then you should first rearrange it as
\[ 3x^2 + 2x - 4 = 0, \]
which gives \( a = 3 \), \( b = 2 \) and \( c = -4 \).

Another thing to think about before you start to solve a quadratic equation is whether it is in its simplest form. A helpful strategy is outlined below.

Simplifying a quadratic equation

- If the coefficient of \( x^2 \) is negative, then multiply the equation through by \(-1\) to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.

For example, you can simplify the quadratic equation
\[ -2x^2 - 2x + 6 = 0 \]
by multiplying through by \(-1\) (to make the coefficient of \( x^2 \) positive) and dividing through by 2 (to make all the coefficients smaller). This gives
\[ x^2 + x - 3 = 0, \]
which you can then proceed to solve.

Activity 17  Rearranging and solving a quadratic equation

Clear the fraction in the quadratic equation
\[ -x^2 = -x - \frac{3}{2} \]
and use the quadratic formula to solve it.

Now that you have a way of solving quadratic equations that you could not solve by factorising, you can solve many more problems involving quadratic functions. Here is one for you to try.
Activity 18  The picture framer’s problem

A picture framer always mounts photographs on a white rectangular backboard that has an area 50% larger than the area of the photograph, and whose dimensions are such that the white border around the photograph is the same width all the way round. The diagram below shows a mounted 12 inch by 18 inch photograph, with the width of the white border labelled as $x$. All the labelled lengths are in inches.

In this question you are asked to find the dimensions of the backboard, as follows.

(a) Explain in words why the width and height of the backboard, in inches, are $12 + 2x$ and $18 + 2x$, respectively.

(b) Use the fact that the area of the backboard is 50% larger than the area of the photograph to find the area of the backboard, in square inches.

(c) Find an algebraic expression for the area of the backboard in terms of $x$.

(d) Use your answers to parts (b) and (c) to show that $x^2 + 15x - 27 = 0$.

(e) Solve the quadratic equation in part (d), and hence find the width of the white border and the dimensions of the backboard to the nearest tenth of an inch.
There is a story that the ancient Greeks believed that the most aesthetically pleasing shape of rectangle was the shape such that if you cut it into a square and a smaller rectangle, as shown in Figure 4, then the smaller rectangle has the same shape – that is, the same aspect ratio – as the original rectangle. This shape is known as the golden rectangle.

You can work out the aspect ratio of the golden rectangle as follows. If the length and width of the larger rectangle are 1 and $x$, respectively (in any units), then the larger and smaller rectangles have aspect ratios $1 : x$ and $x : (1 - x)$, respectively, as you can see from Figure 4. Since these two aspect ratios are equivalent, and the first aspect ratio is equivalent to $x : x^2$, the number $x$ must satisfy the equation $x^2 = 1 - x$. You can use the quadratic formula to find the solutions of this equation. The solutions are \( \frac{1}{2}(-1 \pm \sqrt{5}) \), only one of which, $\frac{1}{2}(-1 + \sqrt{5})$, is positive. So the aspect ratio of the golden rectangle is $1 : \frac{1}{2}(-1 + \sqrt{5})$, and this ratio is known as the golden ratio.

Finally in this subsection, it is worth noticing how the quadratic formula relates to the equation of the axis of symmetry of a parabola. The quadratic formula can be rearranged slightly as

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.
\]

You can see from this form of the formula that the value of $x$ halfway between the two solutions of the quadratic equation $ax^2 + bx + c = 0$ is $x = -b/(2a)$.

**Completing the square**

In this section you will learn a useful way of rearranging a quadratic expression, which is called completing the square. This method allows you to understand why the graph of any quadratic function is the same shape as the graph of an equation of the form $y = ax^2$, but shifted horizontally or vertically or both.

The method also gives you an alternative way to find the vertex of a parabola, and an alternative way to solve a quadratic equation. Once you have learned the method, it can be a quick way of doing these things.

Completing the square is also the idea behind the quadratic formula – it is why the quadratic formula works, as you will see.
4 Quadratic expressions and equations

4.10 Shifting parabolas

We begin by looking at what happens when you start with a parabola of the form $y = ax^2$ and shift it relative to the axes.

Figure 5 shows the graph of the equation $y = 2x^2$. As you know, its vertex is the origin. A second parabola has been drawn on the same axes – this parabola is exactly the same shape as the first parabola, but it is shifted three units to the right and one unit up.

Let’s consider whether we can use the equation of the first parabola to work out an equation for the second. Let $(x, y)$ be any point on the second parabola. Then $(x, y)$ is the result of shifting a corresponding point on the first parabola, as shown in the diagram. This corresponding point is 3 units left and 1 unit down from $(x, y)$, so its coordinates are $(x - 3, y - 1)$.

Because this point lies on the first parabola, its coordinates satisfy the equation

\[
\text{y-coordinate} = 2 \times \text{x-coordinate}^2.
\]

So

\[
y - 1 = 2(x - 3)^2.
\]

Since $x$ and $y$ refer to the second parabola, this is an equation that is satisfied by every point $(x, y)$ on the second parabola, so it is the equation of the second parabola. It can be rearranged to get $y$ by itself on the left-hand side, as follows:

\[
y = 2(x - 3)^2 + 1.
\]

The equation can be left in this form, or it can be multiplied out to give the more usual form:

\[
y = 2(x^2 - 6x + 9) + 1
= 2x^2 - 12x + 18 + 1
= 2x^2 - 12x + 19.
\]
In general, suppose that the parabola with equation \( y = ax^2 \) is shifted right by \( h \) units and up by \( k \) units. The numbers \( h \) and \( k \) can be positive, negative or zero, so the actual shift could be to the right or left, or neither, and up or down, or neither. Then each point \((x, y)\) on the second parabola is a shift of the point \((x - h, y - k)\) on the first parabola, as illustrated in Figure 6.

By the same argument as before, we have

\[
y - k = a(x - h)^2,
\]

or equivalently,

\[
y = a(x - h)^2 + k, \tag{3}
\]

and this is the equation of the second parabola.

Any equation of form (3) can be multiplied out to give the usual form of the equation of a parabola, \( y = ax^2 + bx + c \).

However, the really crucial fact is that you can always go in the other direction, too. That is, any equation of the form \( y = ax^2 + bx + c \), where \( a, b \) and \( c \) are constants with \( a \neq 0 \), can be rearranged into form (3) (where the constants \( h \) and \( k \) can be positive, negative or zero). You’ll learn how to do this in this section.

The fact that the equation of every quadratic function is equivalent to an equation of form (3) explains why the graph of every quadratic function is a shift of a parabola of the form \( y = ax^2 \).

Since \( h \) and \( k \) can be positive, negative or zero, the fact stated above can be restated as below.

**Completed-square form**

Every expression of the form \( ax^2 + bx + c \), where \( a \neq 0 \), can be rearranged into the form

\[
a \left( x + \text{a number} \right)^2 + \text{a number},
\]

where each of the two numbers in this expression can be positive, negative or zero.

This is called the completed-square form.
The process of finding the completed-square form of a quadratic expression is called **completing the square**.

It was mentioned earlier that one reason for completing the square is that it gives you a quick way to find the vertex of a parabola. To see how to do this, suppose that you have rearranged the equation of a parabola into the form of equation (3); that is,

\[ y = a(x - h)^2 + k. \]

This tells you that the parabola has the same shape as the parabola with equation \( y = ax^2 \), but shifted \( h \) units to the right and \( k \) units up, as shown in Figure 6 on the previous page. Since the vertex of the parabola with equation \( y = ax^2 \) is \((0, 0)\), the vertex of the shifted parabola is \((h, k)\).

This useful result is summarised below.

The parabola with equation

\[ y = a(x - h)^2 + k \]

has vertex \((h, k)\).

For example, suppose that you have rearranged the equation of a parabola into the completed-square form

\[ y = 3(x + 5)^2 + 8. \]

Here \( h = -5 \) and \( k = 8 \), so the vertex is \((-5, 8)\).

It can be difficult to remember exactly how to obtain the coordinates of the vertex from the completed-square form. One way to remember it is to use the fact that the vertex always corresponds to the minimum or maximum value of \( y \) (depending on whether the parabola is u-shaped or n-shaped). For example, consider again the completed-square form

\[ y = 3(x + 5)^2 + 8. \]

The parabola that is the graph of this equation is u-shaped, because when you multiply out the brackets, the coefficient of \( x^2 \) will be 3, which is positive. So the vertex of this parabola corresponds to the **minimum** value of \( y \).

Notice that the equation contains the expression \((x + 5)^2\), and the minimum value of this expression is 0, because the square of a number is never negative. So the minimum value of \(3(x + 5)^2\) is also 0, and hence the minimum value of the whole expression \(3(x + 5)^2 + 8\) is 8. That is, the \(y\)-coordinate of the vertex is 8.

This minimum value occurs when the expression that is squared is zero, that is, when \( x + 5 = 0 \) or \( x = -5 \). So the \(x\)-coordinate of the vertex is \(-5\), and hence the vertex is \((-5, 8)\), as found above.

Here is another example of finding the vertex from a completed-square form in this way.

**Example 10**  **Finding the vertex of a parabola from its completed-square form**

State whether the parabola with equation

\[ y = -(x - 2)^2 - 3 \]
is u-shaped or n-shaped, and write down the coordinates of its vertex.

**Solution**

The parabola is n-shaped, because the coefficient of \( x^2 \) is \(-1\), which is negative.

The minimum value of \((x - 2)^2\) is 0, so the maximum value of \(- (x - 2)^2\) is 0, and hence the maximum value of \(-(x - 2)^2 - 3\) is \(-3\).

This occurs when \(x - 2 = 0\), that is, when \(x = 2\).

The vertex is \((2, -3)\).

Here are some similar examples for you to try.

### Activity 19  Finding the vertices of parabolas from completed-square forms

For each of the following equations, state whether the parabola is u-shaped or n-shaped, and write down the coordinates of its vertex.

(a) \(y = (x + 1)^2 + 5\)  
(b) \(y = -2(x + 3)^2 + 7\)  
(c) \(y = 7(x - 1)^2 - 4\)  
(d) \(y = -(x + \frac{1}{2})^2 - 1\)  
(e) \(y = x^2 + 3\)  
(f) \(y = (x - 2)^2\)

In the next subsection you will learn the basic method for completing the square in a quadratic expression.

### 4.11 Completing the square in quadratics of the form \(x^2 + bx + c\)

In this subsection you will learn how to complete the square in quadratic expressions in which the coefficient of \(x^2\) is 1. Other quadratics are covered in Subsection 4.13.

The completed-square form of a quadratic in which the coefficient of \(x^2\) is 1 is

\[
\left( x + \text{a number} \right)^2 + \text{a number}.
\]

This is the expression given in the pink box on page 28, with \(a = 1\).

We begin by looking at completing the square in quadratics in which not only does \(x^2\) have coefficient 1, but the constant term is zero – that is, expressions such as \(x^2 + 8x\), \(x^2 + 10x\) or \(x^2 - 6x\). In other words, we will look at quadratics of the form \(x^2 + bx\).

### Completing the square in quadratics of the form \(x^2 + bx\)

To see how to complete the square in a quadratic expression of this form, first consider the following examples of expanding squared brackets.

\[
\begin{align*}
(x + 1)^2 &= x^2 + 2x + 1 \\
(x - 2)^2 &= x^2 - 4x + 4 \\
(x + 3)^2 &= x^2 + 6x + 9
\end{align*}
\]

In general, for any number \(p\), positive, negative or zero,

\[
(x + p)^2 = x^2 + 2px + p^2.
\]
The expression on the right-hand side of each equation above is of the form $x^2 + bx$ plus an extra number. If you subtract this extra number from both sides of each equation, and swap the sides, then you obtain

\[
x^2 + 2x = (x + 1)^2 - 1,
\]

\[
x^2 - 4x = (x - 2)^2 - 4,
\]

\[
x^2 + 6x = (x + 3)^2 - 9,
\]

and in general,

\[
x^2 + 2px = (x + p)^2 - p^2.
\]

The expressions on the right-hand sides of the equations above are in completed-square form, so they are the completed-square forms of the expressions on the left. In each case the constant term in the brackets on the right-hand side is half of the coefficient of the term in $x$ on the left.

For example:

\[
x^2 + 6x = (x + 3)^2 - 9
\]

Half of the coefficient of $x$

Also, in each case the number that is subtracted on the right-hand side is the square of the constant term in the brackets.

For example:

\[
x^2 + 6x = (x + 3)^2 - 9
\]

The square of the number in brackets

You can see why this is by considering the general case:

\[
x^2 + 2px = (x + p)^2 - p^2
\]

Half of the coefficient of $x$ The square of the number in brackets

You can now see how to write down the completed-square form of any quadratic expression of the form $x^2 + bx$. First you write down $(x \underline{\phantom{p}})^2$, filling the gap with the number that is half of $b$, the coefficient of $x$. This ensures that you have a squared bracket that, when expanded, gives the terms $x^2 + bx$. However, it also gives an extra term, which is the square of the number in the gap. So you need to subtract this term to obtain a final completed-square form that is equivalent to $x^2 + bx$. 
Here is an example.

Example 11 Completing the square in a quadratic of the form $x^2 + bx$

Write the quadratic expression $x^2 - 10x$ in completed-square form.

Solution

$$x^2 - 10x = (x - 5)^2 - 25$$

Halve this coefficient and write it here. Square the constant term in brackets and subtract it.

Once you have found a completed-square form, you can check that it is correct by multiplying it out. For example, multiplying out the expression on the right-hand side of the equation in the solution to Example 11 gives

$$(x - 5)^2 - 25 = x^2 - 10x + 25 - 25$$

$$= x^2 - 10x,$$

which is the same as the left-hand side, as expected.

Here are some examples of completing the square for you to try.

Activity 20 Completing the square in quadratics of the form $x^2 + bx$

Write the following quadratic expressions in completed-square form, and check your answers by multiplying out.

(a) $x^2 + 16x$  (b) $x^2 - 12x$  (c) $t^2 - 2t$  (d) $x^2 + 3x$

Completing the square in quadratics of the form $x^2 + bx + c$

To complete the square in a quadratic of the form $x^2 + bx + c$ whose constant term $c$ is not zero, you just concentrate on the terms in $x^2$ and $x$, and complete the square for these terms in the same way as before. Then you have to collect the constant terms. This is illustrated in the example below.

Example 12 Completing the square in quadratics of the form $x^2 + bx + c$

Write the following quadratic expressions in completed-square form.

(a) $x^2 + 8x + 10$  (b) $x^2 - 3x + 5$

Solution

(a) First complete the square for the sub-expression $x^2 + 8x$, leaving the $+ 10$ unchanged.

$$x^2 + 8x + 10 = (x + 4)^2 - 16 + 10$$

Then collect the constant terms.

$$= (x + 4)^2 - 6$$
(Check: 
\[(x + 4)^2 - 6 = x^2 + 8x + 16 - 6 \]
\[= x^2 + 8x + 10. )\n
(b) First complete the square for the sub-expression \(x^2 - 3x\). 
\[x^2 - 3x + 5 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 5 \]
\[= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{20}{4} \]
\[= \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}. \]
\[\text{(Check: } \left(x - \frac{3}{2}\right)^2 + \frac{11}{4} = x^2 - 3x - \frac{9}{4} + \frac{11}{4} \]
\[= x^2 - 3x + 5. )\n
### Activity 21  Completing the square in quadratics of the form \(x^2 + bx + c\)

Write the following quadratic expressions in completed-square form, and check your answers by multiplying out.

(a) \(x^2 + 6x - 3\)  
(b) \(x^2 - 4x + 9\)  
(c) \(p^2 - 12p - 5\)  
(d) \(x^2 + x + 1\)

Here is a summary of the method that you have seen in this subsection.

**Strategy** To complete the square in a quadratic of the form \(x^2 + bx + c\)

1. Rewrite the expression with the \(x^2 + bx\) part changed to 
\[(x + p)^2 - p^2,\]
where the number \(p\) is half of \(b\).
2. Collect the constant terms.

Later in the section you will see how to complete the square in quadratics in which the coefficient of \(x^2\) is not 1. Before that, the next subsection tells you how to solve a quadratic equation by completing the square.

### 4.12 Solving quadratic equations by completing the square

As mentioned earlier, completing the square gives you another method of solving quadratic equations (and hence of finding the \(x\)-intercepts of parabolas).

To solve a quadratic equation in this way, you only ever need to complete the square in expressions of the form \(x^2 + bx + c\), that is, in quadratics whose term in \(x^2\) has coefficient 1. This is because you can always divide a quadratic equation through by the coefficient of \(x^2\) to give a quadratic equation whose term in \(x^2\) has coefficient 1. For example, if the quadratic
equation is
\[5x^2 - 3x + 10 = 0,\]
then you can divide through by 5 to give
\[x^2 - \frac{3}{5}x + 2 = 0.\]
Dividing through by the coefficient of \(x^2\) can turn some of the coefficients in the equation from whole numbers into fractions, but that doesn’t matter, as fractions are treated in the same way as any other number when completing the square.

Once you have completed the square in the equation, you can solve the equation as follows. First rearrange it so that the square term and the constant term are on different sides, then take the square root of both sides and finally rearrange the equation again to obtain \(x\) by itself on one side. This is illustrated in the example below.

**Example 13  Solving a quadratic equation by completing the square**

Solve the quadratic equation \(4x^2 + 8x - 1 = 0\).

**Solution**

\[4x^2 + 8x - 1 = 0\]

\(\textcircled{1}\) Divide through by the coefficient of \(x^2\).

\[x^2 + 2x - \frac{1}{4} = 0\]

\(\textcircled{2}\) Complete the square.

\[(x + 1)^2 - 1 - \frac{1}{4} = 0\]
\[(x + 1)^2 - \frac{5}{4} = 0\]

\(\textcircled{3}\) Get the constant term on the right.

\[(x + 1)^2 = \frac{5}{4}\]

\(\textcircled{4}\) Take the square root of both sides.

\[x + 1 = \pm \sqrt{\frac{5}{4}}\]
\[x + 1 = \pm \frac{1}{2}\sqrt{5}\]

\(\textcircled{5}\) Finally, get \(x\) by itself on the left.

\[x = -1 \pm \frac{1}{2}\sqrt{5}\]

The solutions are \(x = -1 + \frac{1}{2}\sqrt{5}\) and \(x = -1 - \frac{1}{2}\sqrt{5}\).

**Activity 22  Solving quadratic equations by completing the square**

Solve the following quadratic equations by completing the square.

(a) \(x^2 + 6x - 5 = 0\)      (b) \(2x^2 - 12x - 5 = 0\)
The derivation of the quadratic formula

You can use the technique of completing the square to rearrange the general quadratic equation

\[ ax^2 + bx + c = 0, \]

to obtain \( x \) by itself on the left-hand side. The method is the same as in Example 13, but \( a, b \) and \( c \) are not replaced by particular numbers – they just stay as they are throughout the manipulation. When you do this rearrangement, you end up with the quadratic formula.

The manipulation is given below – read it through if you would like to know why the quadratic formula works.

The equation is

\[ ax^2 + bx + c = 0, \]

where \( a \neq 0 \).

The first step is to divide through by the coefficient of \( x^2 \):

\[ x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \]

Then you complete the square:

\[ \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \]
\[ = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0. \]

Next you get the constant terms on the right, and combine them into a single fraction:

\[ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \]
\[ = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \]
\[ = \frac{b^2 - 4ac}{4a^2}. \]

Now you can take the square root of both sides:

\[ x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \]
\[ = \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \]

The last step is to get \( x \) by itself on the left-hand side:

\[ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
\[ = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \]

This is the quadratic formula!
4.13 Completing the square in quadratics of the form $ax^2 + bx + c$

So far you have seen how to complete the square in quadratic expressions in which the coefficient of $x^2$ is 1. In this subsection you will see how to complete the square in quadratic expressions in which the coefficient of $x^2$ is not 1. Although this is not needed to solve quadratic equations, it is useful for finding the vertices of parabolas.

When you are faced with a problem that is different from those that you have seen before, a useful strategy is to try to change it into a form that you recognise. If you have to complete the square in a quadratic expression in which the coefficient of $x^2$ is not 1, then you can turn this problem into a problem of completing the square when the coefficient is 1, by taking the coefficient of $x^2$ out as a factor. You don’t need to take the factor out of all the terms, but just the terms in $x^2$ and $x$. For example, to complete the square in the quadratic expression

$$2x^2 + 8x - 7,$$

you can take the common factor 2 out of the first two terms to obtain

$$2(x^2 + 4x) - 7.$$

Then you can complete the square in the quadratic *inside the brackets* in the way that you have seen. To obtain the final completed-square form for the whole expression you just need to simplify the results.

You can take the coefficient of $x^2$ out of the terms in $x^2$ and $x$ in a quadratic expression even if it isn’t a *common* factor of these terms. You’ll see this shortly, but first the example below illustrates the method when the coefficient is a common factor.

**Example 14 Completing the square in quadratics of the form $ax^2 + bx + c$**

Write the following quadratic expressions in completed-square form.

(a) $2x^2 + 8x - 7$  (b) $-x^2 + 8x - 7$

**Solution**

(a) ✿ Concentrate on the sub-expression $2x^2 + 8x$. First take the coefficient of $x^2$ out of the sub-expression as a common factor. ✿

$$2x^2 + 8x - 7 = 2(x^2 + 4x) - 7$$

✿ Now the brackets contain a quadratic in which the coefficient of $x^2$ is 1. Complete the square in it in the usual way, keeping it enclosed within its brackets. ✿

$$= 2((x + 2)^2 - 4) - 7$$

✿ Multiply out the outer brackets. Don’t multiply out the inner brackets, because you want the square $(x + 2)^2$ to appear in the final expression. ✿

$$= 2(x + 2)^2 - 8 - 7$$

✿ Collect the constant terms. ✿

$$= 2(x + 2)^2 - 15$$
(Check:  \[2(x + 2)^2 - 15 = 2(x^2 + 4x + 4) - 15 \]
\[= 2x^2 + 8x + 8 - 15 \]
\[= 2x^2 + 8x - 7.\])

(b) \(\triangleleft\) Concentrate on the sub-expression \(-x^2 + 8x\). First take the minus sign out of the sub-expression. \(\triangleleft\)

\[-x^2 + 8x - 7 = -(x^2 - 8x) - 7\]

\(\triangleleft\) Complete the square in the quadratic inside the brackets. \(\triangleleft\)

\[= -(x - 4)^2 - 16 - 7\]

\(\triangleleft\) Multiply out the outer brackets. \(\triangleleft\)

\[= -(x - 4)^2 + 16 - 7\]

\(\triangleleft\) Collect the constant terms. \(\triangleleft\)

\[= -(x - 4)^2 + 9\]

(Check: \[-(x - 4)^2 + 9 = -(x^2 - 8x + 16) + 9 \]
\[= -x^2 + 8x - 16 + 9 \]
\[= -x^2 + 8x - 7.\])

In Example 14(a) the coefficient of \(x^2\) was a common factor of the sub-expression consisting of the terms in \(x^2\) and \(x\), but of course this is not always the case. For example, consider the quadratic expression

\[2x^2 + 5x + 1.\]

To complete the square in a quadratic like this, you begin by taking out the coefficient of \(x^2\) from the sub-expression just as if it were a common factor – this will create fractions. For the quadratic here, you obtain

\[2 \left( x^2 + \frac{5}{2}x \right) + 1.\]

You can then go on to complete the square using the method demonstrated in Example 14. The final answer is

\[2 \left( x + \frac{5}{4} \right)^2 - \frac{17}{8}.\]

**Activity 23**  \(\textbf{Completing the square in quadratics of the form} \ ax^2 + bx + c\)

Write the quadratic expressions below in completed-square form, and check your answers by multiplying out.

Use the completed-square forms to write down the vertices of the corresponding parabolas.

(a) \(2x^2 - 4x - 1\) \hspace{1cm} (b) \(-x^2 - 8x - 18\)
Here is a summary of the method that you have seen in this subsection.

**Strategy**  To complete the square in a quadratic of the form $ax^2 + bx + c$

1. Rewrite the expression with the coefficient $a$ of $x^2$ taken out of the $ax^2 + bx$ part as a factor. This generates a pair of brackets.
2. Complete the square in the simple quadratic inside the brackets, remembering to keep it enclosed within its brackets. This generates a second pair of brackets, inside the first pair.
3. Multiply out the outer brackets.
4. Collect the constant terms.

Writing the equation of a parabola in completed-square form gives you another way to find some of the information that you need to sketch it. You can read off the coordinates of the vertex immediately, and you can use the completed-square form to solve the quadratic equation that gives the $x$-intercepts.

In this section you have seen how to complete the square in any quadratic expression. You have seen that this method explains why the graphs of all quadratic functions are shifts of the graphs of equations of the form $y = ax^2$, and that it also explains why the quadratic formula works. You have also seen how to use the method to solve quadratic equations and to find vertices of parabolas.

**Graphs of quadratic functions**

The graph of an equation of the form 

$$y = ax^2 + bx + c,$$  \hspace{1cm} (4) 

where $a$, $b$ and $c$ are constants with $a \neq 0$, has a shape called a *parabola*. In this section you will explore the different shapes of parabolas obtained by plotting graphs of this form, and the different positions of these parabolas relative to the axes.

One feature that you will see is that every parabola has a line of symmetry, as shown in Figure 7. Another name for a line of symmetry is an *axis of symmetry*, and this is the phrase that is usually used when discussing parabolas. The axis of symmetry of a parabola cuts the parabola at exactly one point. This point is called the *vertex* of the parabola, also shown in Figure 7.

This section also uses the following terminology: Whenever you have an equation that expresses one variable in terms of another variable, you can think of it as a rule that takes an input value and produces an output value. For example, if the equation is $y = x^2$, then inputting $x = 3$ gives the output $y = 9$, inputting $x = -1$ gives $y = 1$, and so on. A rule that takes input values and produces output values like this is called a *function*.

A function whose rule is of the form $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are constants with $a \neq 0$, is called a *quadratic function*. So this section is about the graphs of quadratic functions.
4.14 Graphs of equations of the form $y = ax^2$

We begin by looking at the graphs obtained when the constants $b$ and $c$ in equation (4) are both zero, that is, when the equation has the form

$$y = ax^2; \quad \text{where } a \neq 0.$$  

For example, the equations

$$y = 2x^2, \quad y = x^2 \quad \text{and} \quad y = -x^2$$  

are of this form, with $a$ equal to 2, 1 and $-1$, respectively.

We begin by looking at the simple equation

$$y = x^2.$$  

You can plot a graph of this equation by constructing a table of values. You choose some appropriate values for $x$, and calculate the corresponding values of $y$ by substituting into the equation. You can choose both positive and negative values of $x$, and Table 1 shows some results obtained in this way.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

Notice the symmetry in the $y$-values in Table 1: the values decrease until they reach 0, and then they increase again in the same steps. This happens because $(-x)^2 = x^2$, for any $x$.

If you plot the seven points given by Table 1 and join them with a smooth curve, then you obtain the graph in Figure 8.

![Figure 8](image_url)

Figure 8 The graph of $y = x^2$

Notice that the curve in Figure 8 has a line of symmetry (the y-axis), as expected. This is because of the symmetry of the $y$-values.

In the next activity you are asked to plot the graph of another simple quadratic function.
Activity 24  Plotting the graph of $y = -x^2$

(a) By constructing a table of values, plot the graph of the equation $y = -x^2$.

(b) Compare your graph to the graph of $y = x^2$ in Figure 8, and write a sentence or two to explain what you see.

In Activity 24 you saw that the graphs of $y = ax^2$ when $a = 1$ and when $a = -1$ are exactly the same shape, but mirror images of each other, reflected in the $x$-axis. This is because changing $a$ from 1 to $-1$ in the equation $y = ax^2$ changes all the $y$-values to their negatives, as you can see if you compare Table 1 to the table in the solution to Activity 24.

The same thing happens for any value of $a$: if you change the value of $a$ to its negative, then all the $y$-values obtained from the equation $y = ax^2$ change to their negatives, so the parabola changes to its mirror image, reflected in the $x$-axis.

Activity 25  Exploring the graph of $y = ax^2$

By constructing tables of values, plot the graphs of $y = ax^2$ for some different values of $a$. Try, for example, $a = 2$; $a = -2$; $a = 3$; $a = -3$.

You should have found in Activity 25 that the value of $a$ appears to affect how narrow the graph of $y = ax^2$ is. More specifically, it is the magnitude of $a$ that affects the width of the graph – the magnitude of a number is its value without its negative sign, if it has one. The larger the magnitude of $a$, the narrower the parabola becomes.

To see why this is, think of a point, other than $(0, 0)$, on the graph of $y = x^2$, say. Now imagine changing the graph to make it the graph of $y = 2x^2$. The point with the same $x$-value now has a $y$-value twice as large, so it has moved up. All the points on the parabola except $(0, 0)$ move up in this way. The further away they are from the vertex $(0, 0)$, the larger their $y$-value is to start with, so the more they move up. This has the effect of making the parabola more narrow. Similarly, if you change $y = -x^2$ to $y = -2x^2$, then the points move down, so again the parabola becomes narrower. The axis of symmetry of the graph is not affected.

Because the point $(0, 0)$ on the parabola does not move when you change the value of $a$, the vertex of the parabola does not change. So for all values of $a$, the vertex of the graph of $y = ax^2$ is at the origin.

Another property that you saw in Activity 25 is that, as expected, if $a$ is positive, then the graph of $y = ax^2$ is the same way up as the graph of $y = x^2$, while if $a$ is negative, then it is the other way up. A parabola that is the same way up as the graph of $y = x^2$ is called a u-shaped parabola, while one that is the other way up is called an n-shaped parabola. These two possibilities are illustrated in Figure 9.
Here is a summary of what you have learned about the graphs of equations of the form \( y = ax^2 \) in this subsection.

**The graph of the equation** \( y = ax^2 \)

The vertex is \((0,0)\).

If \(a\) is positive, then the graph is u-shaped.

If \(a\) is negative, then the graph is n-shaped.

The larger the magnitude of \(a\), the narrower the parabola.

### 4.15 Graphs of equations of the form \( y = ax^2 + bx + c \)

In this subsection you will explore the graphs of general quadratic functions. In the first activity you are asked to explore the graphs that are obtained when the coefficient \(c\) in the equation \( y = ax^2 + bx + c \) is zero. That is, you will explore the graphs of equations of the form \( y = ax^2 + bx \).

#### Activity 26  Graphs of equations of the form \( y = ax^2 + bx \)

(a) By constructing tables of values, explore the effect of changing \(b\) in the equation \( y = ax^2 + bx + c \).

Set \(a = 1\) and \(c = 0\), and try some different values of \(b\), for example, \(b = 4; b = 3\).

(b) Notice that although the position of the vertex changes, the following features of the graph remain the same:

- its shape
- the fact that its axis of symmetry is vertical
- which way up it is
- the fact that it goes through \((0,0)\).

In Activity 26 you should have found that for all values of \(a\) and \(b\) that you tried, the graph of \( y = ax^2 + bx \) is exactly the same as the graph of \( y = ax^2 \), but shifted to a different position relative to the axes. The word ‘shifted’ here means that the parabola is just slid to a new position – it is not rotated in any way, so its axis of symmetry remains vertical.

You should also have found that for all values of \(a\) and \(b\) that you tried, the graph of \( y = ax^2 + bx \) passes through the point \((0,0)\). This is because substituting \(x = 0\) in the equation \( y = ax^2 + bx \) gives \( y = 0 \).

In the next activity you are asked to look at the effect of changing the value of the constant term \(c\) in the equation \( y = ax^2 + bx + c \). You may find it helpful to download a graph-plotting app to help you with the activities in this section.
Activity 27  Graphs of equations of the form \( y = ax^2 + bx + c \)

For the graphs you drew in Activity 26, now explore the effect of changing \( c \) in the equation \( y = ax^2 + bx + c \).

In Activity 27 you should have found that for any values of \( a \), \( b \) and \( c \), the graph of \( y = ax^2 + bx + c \) is exactly the same as the graph of \( y = ax^2 + bx \), except that it is shifted vertically up or down, so that it crosses the \( y \)-axis at \((0, c)\) instead of \((0, 0)\).

This is because adding the constant \( c \) to the right-hand side of the equation \( y = ax^2 + bx \) just changes all the \( y \)-values by \( c \) units, which causes the graph to move up or down.

So, from what you have seen in Activities 26 and 27, it seems that for all values of \( a \), \( b \) and \( c \), the graph of \( y = ax^2 + bx + c \) is exactly the same shape as the graph of \( y = ax^2 \), but shifted horizontally and/or vertically to a different position relative to the axes. This is indeed the case, and you will see why later in the chapter.

This means that the only possible basic shapes of parabolas are the shapes of the graphs of the form \( y = ax^2 \). You have seen that these all have a vertical axis of symmetry and differ in how wide they are.

In particular, the graph of the equation \( y = ax^2 + bx + c \) is always a \( u \)-shaped or \( n \)-shaped parabola. If \( a \) is positive, then it is \( u \)-shaped; if \( a \) is negative, then it is \( n \)-shaped.

Here is a summary of what you have learned in this subsection about the graphs of quadratic functions.

The graph of the equation \( y = ax^2 + bx + c \)

If \( a \) is positive, then the graph is \( u \)-shaped.

If \( a \) is negative, then the graph is \( n \)-shaped.

The graph has the same shape as the graph of \( y = ax^2 \), but shifted.

The graph crosses the \( y \)-axis at \((0, c)\).

Figure 10  A smile-shaped parabola goes with a positive coefficient of \( x^2 \), and a frown-shaped parabola goes with a negative coefficient of \( x^2 \)

Figure 10 suggests a way to remember the relationship between the sign of the coefficient of \( x^2 \) and whether the graph is \( u \)-shaped or \( n \)-shaped.
4.16 The intercepts of a parabola

You met the idea of the intercepts of a graph in Chapter 3. An \( x \)-intercept of a graph is a value where it crosses or touches the \( x \)-axis. In other words, it is a value of \( x \) for which \( y = 0 \). Similarly, a \( y \)-intercept of a graph is a value where it crosses or touches the \( y \)-axis. That is, it is a value of \( y \) for which \( x = 0 \). For example, the \( x \)-intercepts of the parabola shown in Figure 11 are \(-2\) and \(5\), and the \( y \)-intercept is \(-10\).

![Figure 11](image.png)

**Figure 11** The graph of \( y = x^2 - 3x - 10 \)

The graph of a quadratic function can have two, one or zero \( x \)-intercepts, depending on its position relative to the \( x \)-axis. The three possibilities are shown in Figure 12, for u-shaped parabolas.

![Figure 12](image.png)

**Figure 12** The graph of \( y = ax^2 + bx + c \) can have two, one or zero \( x \)-intercepts

There is always exactly one \( y \)-intercept, because there is exactly one value of \( y \) for each value of \( x \), including \( x = 0 \).

You can use the equation of a parabola to find its intercepts.
Finding the \( y \)-intercept

To obtain the \( y \)-intercept of a parabola, you substitute \( x = 0 \) into its equation. For example, consider the parabola with equation

\[
y = x^2 - 3x - 10.
\]

Substituting \( x = 0 \) into this equation gives \( y = -10 \), so the \( y \)-intercept of this parabola is \(-10\), as shown in Figure 11.

Notice that whatever the values of \( a \), \( b \) and \( c \), if you substitute \( x = 0 \) into the equation \( y = ax^2 + bx + c \), then you obtain \( y = c \). So the \( y \)-intercept of the graph is always the value \( c \), as was observed in the previous subsection.

Finding the \( x \)-intercepts

To obtain the \( x \)-intercepts of a parabola, you substitute \( y = 0 \) into the equation. For example, consider again the parabola with equation

\[
y = x^2 - 3x - 10.
\]

Substituting \( y = 0 \) gives

\[
0 = x^2 - 3x - 10,
\]

so the \( x \)-intercepts of the parabola are the solutions of this quadratic equation. The quadratic expression on the right-hand side factorises as \((x + 2)(x - 5)\), so the \( x \)-intercepts are \(-2\) and \(5\), as shown in Figure 11.

Finding the \( x \)-intercepts of a parabola always involves solving a quadratic equation. If you cannot solve the equation by factorisation, then you may be able to solve it by using the quadratic formula, which you met earlier in this chapter.

If the quadratic equation has only one solution, then the parabola has only one \( x \)-intercept. If it has no solutions at all, then the parabola has no \( x \)-intercepts.

You will have a chance to practise finding the intercepts of parabolas in the next subsection.

4.17 Sketch graphs of quadratic functions

When you are dealing with a quadratic function, it can be useful to have an idea of what its graph looks like. Often you do not need an accurate plot, but just a quick sketch showing some of the main features. The features that you would usually show on such a sketch are as follows:

- whether it is u-shaped or n-shaped
- its intercepts
- its axis of symmetry
- its vertex.

You have already seen how to determine whether a parabola is u-shaped or n-shaped from its equation, and how to find its intercepts.
You can find the axis of symmetry by using the fact that this line lies halfway between the \(x\)-intercepts, or passes through the single \(x\)-intercept if there is only one. You will see later in this subsection how you can find the axis of symmetry if there are no \(x\)-intercepts.

The equation of the axis of symmetry tells you the \(x\)-coordinate of the vertex, and you can substitute that into the equation of the parabola to find the corresponding \(y\)-coordinate.

The next example shows you how you might go about producing a sketch graph of a quadratic function.

**Example 15  Sketching the graph of a quadratic function**

This question is about the parabola \(y = -x^2 + 2x + 8\).

(a) State whether the parabola is \(u\)-shaped or \(n\)-shaped, and find its intercepts.

(b) Find the equation of the axis of symmetry, and the coordinates of the vertex.

(c) Sketch the parabola.

**Solution**

(a) The coefficient of \(x^2\) is negative, so the graph is \(n\)-shaped.

Putting \(x = 0\) gives \(y = 8\), so the \(y\)-intercept is 8.

Putting \(y = 0\) gives

\[
0 = -x^2 + 2x + 8.
\]

Multiply through by \(-1\) to make factorising easier.

\[
0 = x^2 - 2x - 8
\]

\[
(x + 2)(x - 4) = 0
\]

\[
x + 2 = 0 \quad \text{or} \quad x - 4 = 0
\]

\[
x = -2 \quad \text{or} \quad x = 4
\]

So the \(x\)-intercepts are \(-2\) and 4.

(b) The axis of symmetry is halfway between the \(x\)-intercepts.

The number halfway between the \(x\)-intercepts is

\[
\frac{-2 + 4}{2} = \frac{2}{2} = 1,
\]

so the axis of symmetry is the line with equation \(x = 1\).

The vertex is on the axis of symmetry.

Hence the \(x\)-coordinate of the vertex is 1.

Substituting \(x = 1\) into the equation of the parabola gives

\[
y = -(1)^2 + 2 \times 1 + 8 = -1 + 2 + 8 = 9
\]

So the vertex is \((1,9)\).
(c) Plot the intercepts and the vertex, and draw the axis of symmetry. Hence sketch the parabola and label it with its equation. Indicate the values of the intercepts and the coordinates of the vertex.

Here is a summary of how to sketch the graph of a quadratic function.

**Strategy**  
To sketch the graph of a quadratic function  
1. Find whether the parabola is u-shaped or n-shaped.  
2. Find its intercepts, axis of symmetry and vertex.  
3. Plot the features found, and hence sketch the parabola.  
4. Label the parabola with its equation, and make sure that the values of the intercepts and the coordinates of the vertex are indicated.

You can practise drawing sketch graphs of quadratic equations in the next two activities. Try to draw each parabola smoothly through the points that you have plotted, and symmetrically on each side of the axis of symmetry. Sometimes finding and plotting one or two extra points on the parabola can help you to draw a good sketch.

As when you draw straight-line graphs, when you sketch parabolas it is usually best to use equal scales on the axes, unless that makes the graph hard to draw or interpret, in which case you should use different scales.

**Activity 28**  
**Sketching the graph of a quadratic function**

Use the strategy above to draw a neat sketch of the graph of the equation $y = x^2 + 5x - 6$.

As you have seen, to find the $x$-intercepts of a parabola you need to solve a quadratic equation. If you obtain just one solution, then the graph has just one $x$-intercept. In this case, the single point where the graph touches the $x$-axis is the vertex, and the vertical line through this point is the axis of
symmetry. You can use this information, together with the $y$-intercept, to sketch the graph in the usual way. Try this in the next activity.

**Activity 29  Sketching the graph of a quadratic function with one $x$-intercept**

Sketch the graph of the equation $y = 9x^2 - 6x + 1$.

You have seen that you can often find the axis of symmetry of a parabola by using the fact that it lies halfway between the $x$-intercepts. An alternative method is to use the formula below. This formula can be used when the parabola has no $x$-intercepts, and you might prefer to use it in other cases too.

**A formula for the axis of symmetry of a parabola**

The axis of symmetry of the parabola with equation $y = ax^2 + bx + c$ is the line with equation

$$x = -\frac{b}{2a}.$$

For example, to use the formula to work out the axis of symmetry of the parabola $y = -x^2 + 2x + 8$, which was considered in Example 15 on page 45, you substitute $a = -1$ and $b = 2$, which gives

$$x = -\frac{2}{2 \times (-1)}; \text{ that is, } x = 1.$$

This is the same line as was found in Example 15.

To see why the formula works, consider the equation

$$y = ax^2 + bx + c,$$

where $a$, $b$ and $c$ are constants with $a \neq 0$. You know that the graph of this equation is the same as the graph of

$$y = ax^2 + bx,$$

except that it is shifted vertically. So the two graphs have the same axis of symmetry. The $x$-intercepts of the second graph can be found by factorisation:

$$ax^2 + bx = 0$$

$$x(ax + b) = 0$$

$$x = 0 \text{ or } ax + b = 0$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

The value halfway between 0 and $-\frac{b}{a}$ is

$$\frac{1}{2} \left(0 + \left(-\frac{b}{a}\right)\right) = -\frac{b}{2a},$$

so the axis of symmetry is the line $x = -\frac{b}{2a}$.

In the next activity you are asked to sketch the graph of a quadratic equation that has no $x$-intercepts.
Activity 30  
**Sketching a quadratic function with no x-intercepts**

Find the $y$-intercept, axis of symmetry and vertex of the graph of the equation

$$y = x^2 + 2x + 3,$$

and hence sketch the graph.

In this section you have seen that the graph of an equation of the form

$$y = ax^2 + bx + c,$$

where $a$, $b$ and $c$ are constants with $a \neq 0$, is always a u-shaped or n-shaped parabola. The sign of the constant $a$ tells you whether the parabola is u-shaped or n-shaped, and the magnitude of $a$ determines how wide it is. The $y$-intercept of the graph is $c$. The position of the vertex depends on the values of all three coefficients.

You have also learned how to sketch the graphs of quadratic functions.
Solutions and comments on Activities

Activity 1
(a) \( a(2b + 3c) = 2ab + 3ac \)
(b) \(-r(2s - 3t) = -2rs + 3rt\)
(c) \((n - 1)n = n^2 - n\)

Activity 2
(a) \((x + 2)(x + 4) = x^2 + 4x + 2x + 8 = x^2 + 6x + 8\)
(b) \((a + 2b)(3x + 4y) = 3ax + 4ay + 6bx + 8by\)
(c) \((x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9\)
(d) \((a - 2b)(3c - d) = 3ac - ad - 6bc + 2bd\)
(e) \((p - 1)(-2 + 3q) = -2p + 3pq - 2 - 3q\)
(f) \((n - 2)(n + 2) = n^2 + 2n - 2n - 4 = n^2 - 4\)

Activity 3
\((2a - b)(c - 3d + 2e) = 2ac - 3ad + 2ae - bc + 3bd - 2be\)

Activity 4
(a) Writing down the answer directly gives \((x + 7)^2 = x^2 + 14x + 49\).
(b) Writing down the answer directly gives \((u - 10)^2 = u^2 - 20u + 100\).
(c) Writing down the answer directly gives \((t + \frac{1}{2})^2 = t^2 + 2 \times \frac{1}{2}t + \left(\frac{1}{2}\right)^2 = t^2 + t + \frac{1}{4}\).
(d) Multiplying out the brackets gives \((3x + 2)^2 = (3x + 2)(3x + 2) = (3x)^2 + 6x + 6x + 2^2 = 9x^2 + 12x + 4\).
(e) Multiplying out the brackets:
\[(2s - 5t)^2 = (2s - 5t)(2s - 5t) = (2s)^2 - 10st - 10st + (5t)^2 = 4s^2 - 20st + 25t^2\]

Activity 5
(a) By the difference of two squares identity, \((u - 12)(u + 12) = u^2 - 12^2 = u^2 - 144\).
(b) By the difference of two squares identity, \((x - 2y)(x + 2y) = x^2 - (2y)^2 = x^2 - 4y^2\).
(c) By the difference of two squares identity, \((10 - a)(10 + a) = 10^2 - a^2 = 100 - a^2\).

Activity 6
(a) \(9x^2 - 12x + 4\) is a quadratic, with \(a = 9\), \(b = -12\) and \(c = 4\).
(b) \(3y - 5\) is not a quadratic, as it has no squared term.
(c) \(-6 - 7s^2\) is a quadratic, with \(a = -7\), \(b = 0\) and \(c = -6\).
(d) \(2x^3 + x^2\) is not a quadratic, as it has a term in \(x^3\).

Activity 7
Substituting \(x = 2\) in the quadratic expression gives
\[2x^2 - x - 6 = 2 \times 2^2 - 2 - 6 = 8 - 2 - 6 = 0\]
Hence \(x = 2\) is a solution of the equation.

Activity 8
(a) The quadratic is \(x^2 + 3x + 2\).
The pair 1, 2 has sum 3 and product 2. Thus \(x^2 + 3x + 2 = (x + 1)(x + 2)\).
(Check: Multiplying out the brackets gives \((x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2\).)

(b) The quadratic is \(x^2 + 11x + 24\).
The positive factor pairs of 24 are
\[1, 24, \ 2, 12, \ 3, 8, \ 4, 6\]
The only pair whose sum is 11 is 3, 8. Thus \(x^2 + 11x + 24 = (x + 3)(x + 8)\).
(Check: Multiplying out the brackets gives \((x + 3)(x + 8) = x^2 + 8x + 3x + 24 = x^2 + 11x + 24\).)

Activity 9
(a) The quadratic is \(x^2 - 10x + 24\).
The negative factor pairs of 24 are
\[-1, -24, \ -2, -12, \ -3, -8, \ -4, -6\]
The only pair whose sum is -10 is -4, -6. Thus \(x^2 - 10x + 24 = (x - 4)(x - 6)\).
(Check: Multiplying out the brackets gives \((x - 4)(x - 6) = x^2 - 6x - 4x + 24 = x^2 - 10x + 24\).)

(b) The quadratic is \(t^2 - 4t + 3\).
The pair \(-1, -3\) has product 3 and sum \(-4\). Thus
\[ t^2 - 4t + 3 = (t - 1)(t - 3). \]

(Check: Multiplying out the brackets gives
\[ (t - 1)(t - 3) = t^2 - 3t - t + 3 = t^2 - 4t + 3. \]

(c) The quadratic is \(x^2 - 6x + 9\).

The negative factor pairs of 9 are
\(-1, 9, -9, 1\).

The only pair whose sum is \(-6\) is \(-3, -3\).

Thus \(x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2\).

(Check: Multiplying out the brackets gives
\[ (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9. \]

Activity 10

(a) The quadratic is \(x^2 - x - 2\).

The pair 1, \(-2\) has sum \(-1\) and product \(-2\). Thus
\[ x^2 - x - 2 = (x + 1)(x - 2). \]

(Check: Multiplying out the brackets gives
\[ (x + 1)(x - 2) = x^2 - 2x + x - 2 = x^2 - x - 2. \]

(b) The quadratic is \(u^2 + 4u - 12\).

The factor pairs of \(-12\) are
\(-1, 12, -2, 6, -3, 4, -1, 12, -2, 6, 3, -4, -1, 12, -2, 6, -3, 4\).

The only pair whose sum is \(-2\) is \(-6, 2\), Thus
\[ u^2 + 4u - 12 = (u - 6)(u + 6). \]

(Check: Multiplying out the brackets gives
\[ (u - 6)(u + 6) = u^2 + 6u - 2u - 12 = u^2 + 4u - 12. \]

Activity 11

(a) \(x^2 - x = x(x - 1)\)

(b) \(u^2 - 16 = u^2 - 4^2 = (u - 4)(u + 4)\)

(c) \(t^2 - 9t = t(t - 9)\)

(d) \(x^2 + 10x + 25 = (x + 5)^2\)

Activity 12

(a) The equation is: \(x^2 + 3x + 2 = 0\)

Factorise: \((x + 1)(x + 2) = 0\)

So:
\[ x + 1 = 0 \quad \text{or} \quad x + 2 = 0 \]

So:
\[ x = -1 \quad \text{or} \quad x = -2 \]

(Check: When \(x = -1\),
\[ x^2 + 3x + 2 = (-1)^2 + 3 \times (-1) + 2 = 1 - 3 + 2 = 0. \]

When \(x = -2\),
\[ x^2 + 3x + 2 = (-2)^2 + 3 \times (-2) + 2 = 4 - 6 + 2 = 0. \]

(b) The equation is: \(x^2 - 10x + 24 = 0\)

Factorise: \((x - 4)(x - 6) = 0\)

So:
\[ x - 4 = 0 \quad \text{or} \quad x - 6 = 0 \]

So:
\[ x = 4 \quad \text{or} \quad x = 6 \]

(Check: When \(x = 4\),
\[ x^2 - 10x + 24 = 4^2 - 10 	imes 4 + 24 = 16 - 40 + 24 = 0. \]

When \(x = 6\),
\[ x^2 - 10x + 24 = 6^2 - 10 \times 6 + 24 = 36 - 60 + 24 = 0. \]

(c) The equation is: \(t^2 - 16 = 0\)

Factorise: \((t - 4)(t + 4) = 0\)

So:
\[ t - 4 = 0 \quad \text{or} \quad t + 4 = 0 \]

So:
\[ t = 4 \quad \text{or} \quad t = -4 \]

(Check: When \(t = 4\),
\[ t^2 - 16 = 4^2 - 16 = 16 - 16 = 0. \]

When \(t = -4\),
\[ t^2 - 16 = (-4)^2 - 16 = 16 - 16 = 0. \]

(Factorising is not the best way to solve this equation. It is quicker to rearrange it as \(t^2 = 16\) and take square roots.)

(d) The equation is: \(u^2 - u - 12 = 0\)

Factorise: \((u - 4)(u + 3) = 0\)

So:
\[ u - 4 = 0 \quad \text{or} \quad u + 3 = 0 \]

So:
\[ u = 4 \quad \text{or} \quad u = -3 \]

(Check: When \(u = 4\),
\[ u^2 - u - 12 = 4^2 - 4 - 12 = 16 - 4 - 12 = 0. \]

When \(u = -3\),
\[ u^2 - u - 12 = (-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0. \]
(e) The equation is: $x^2 - 6x + 9 = 0$
Factorise: $(x - 3)(x - 3) = 0$
So: $x - 3 = 0$
So: $x = 3$

(Check: When $x = 3$,
$\begin{align*}
x^2 - 6x + 9 &= 3^2 - 6 \times 3 + 9 \\
&= 9 - 18 + 9 = 0.
\end{align*}$)

(f) The equation is: $x^2 - 9x = 0$
Factorise: $x(x - 9) = 0$
So: $x = 0$ or $x - 9 = 0$
So: $x = 0$ or $x = 9$

(Check: When $x = 0$,
$\begin{align*}
x^2 - 9x &= 0^2 - 9 \times 0 = 0 - 0 = 0.
\end{align*}$)
When $x = 9$,
$\begin{align*}
x^2 - 9x &= 9^2 - 9 \times 9 = 81 - 81 = 0.
\end{align*}$

**Activity 13**

(a) In this case 3 is a factor of each of the coefficients, so
\[3x^2 - 3x - 36 = 3(x^2 - x - 12) = 3(x - 4)(x + 3).\]

(b) In this case $-5$ is a factor of each of the coefficients, so
\[-5x^2 + 15x - 10 = -5(x^2 - 3x + 2) = -5(x - 1)(x - 2).\]

**Activity 14**

(a) The quadratic expression is $2x^2 - 5x + 3$.
You can use the first method to look for a possible factorisation of the form
$2x^2 - 5x + 3 = (2x\ \_\_\_\_)(x\ \_\_\_\_)$.
The integers in the gaps must multiply together to give 3, and the possible factor pairs of 3 are
1, 3, $-1, -3$.

These two factor pairs lead to four possible cases:
$(2x + 1)(x + 3)$ or $(2x + 3)(x + 1)$,
$(2x - 1)(x - 3)$ or $(2x - 3)(x - 1)$.

By multiplying out each of these pairs of brackets in turn, we find that one of these cases gives the required factorisation, specifically,
$(2x - 3)(x - 1) = 2x^2 - 5x + 3$.

(b) The quadratic expression is $6x^2 + 7x - 3$.
You can use the first method to look for possible factorisations of the form
$6x^2 + 7x - 3 = (6x\ \_\_\_\_)(x\ \_\_\_\_)$ or
$6x^2 + 7x - 3 = (3x\ \_\_\_\_)(2x\ \_\_\_\_)$.

The integers in the gaps must multiply together to give $-3$, and the possible factor pairs of $-3$ are
1, $-3$, $-1, 3$.

These two factor pairs and the two possible factorisations lead to 8 possible cases:
$(6x + 1)(x - 3)$ or $(6x - 3)(x + 1)$,
$(6x - 1)(x + 3)$ or $(6x + 3)(x - 1)$,
$(3x + 1)(2x - 3)$ or $(3x - 3)(2x + 1)$,
$(3x - 1)(2x + 3)$ or $(3x + 3)(2x - 1)$.

By multiplying out each of these pairs of brackets in turn, we find that one of these cases gives the required factorisation, namely,
$(3x - 1)(2x + 3) = 6x^2 + 7x - 3$.

(c) The quadratic expression is $8x^2 - 10x - 3$.
You can use the second method to find a factorisation.

First find two numbers whose product is $ac = 8 \times (-3) = -24$ and whose sum is $b = -10$.
The possible factor pairs of $-24$ (excluding those involving 24 or $-24$) are $-2, 12$, $2, -12$, $-3, 8$, $3, -8$, $-4, 6$, $4, -6$.
Now choose a factor pair whose sum is $-10$. The only such pair is $2, -12$.

Next rewrite the quadratic expression, splitting the middle term $-10x$ into two terms using the factor pair $2, -12$, as follows:
$8x^2 - 10x - 3 = 8x^2 + 2x - 12x - 3$.
The required factorisation of $8x^2 - 10x - 3$ can now be obtained by taking out common factors:
$\begin{align*}
8x^2 - 10x - 3 &= 8x^2 + 2x - 12x - 3 \\
&= 2x(4x + 1) - 3(4x + 1) \\
&= (2x - 3)(4x + 1).
\end{align*}$

So
$8x^2 - 10x - 3 = (2x - 3)(4x + 1)$.

**Activity 15**

(a) Since
$2x^2 - 5x + 3 = (2x - 3)(x - 1)$,
the solutions of this equation satisfy
$2x - 3 = 0$ or $x - 1 = 0$,
so they are
$x = \frac{3}{2}$ and $x = 1$.

(b) Since
$6x^2 + 7x - 3 = (3x - 1)(2x + 3)$,
the solutions of this equation satisfy
$3x - 1 = 0$ or $2x + 3 = 0$. 

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so they are
\[ x = \frac{3}{2} \quad \text{and} \quad x = -\frac{3}{4}. \]

(c) Since
\[ 8x^2 - 10x - 3 = (2x - 3)(4x + 1), \]
the solutions of this equation satisfy
\[ 2x - 3 = 0 \quad \text{or} \quad 4x + 1 = 0, \]
so they are
\[ x = \frac{3}{2} \quad \text{and} \quad x = -\frac{1}{4}. \]

Activity 16

(a) The equation is
\[ x^2 + 6x + 1 = 0, \]
so \(a = 1\), \(b = 6\) and \(c = 1\).

The quadratic formula gives
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 1}}{2 \times 1} \]
\[ = \frac{-6 \pm \sqrt{32}}{2} \]
\[ = \frac{-6 \pm 4\sqrt{2}}{2} \]
\[ = -3 \pm 2\sqrt{2}. \]

So the solutions are \( x = -3 + 2\sqrt{2} \) and \( x = -3 - 2\sqrt{2} \).

(b) The equation is
\[ 3x^2 - 8x - 2 = 0, \]
so \(a = 3\), \(b = -8\) and \(c = -2\).

The quadratic formula gives
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \]
\[ = \frac{8 \pm \sqrt{64 + 24}}{6} \]
\[ = \frac{8 \pm \sqrt{88}}{6} \]
\[ = \frac{8 \pm 2\sqrt{22}}{6} \]
\[ = \frac{4 \pm \sqrt{22}}{3}. \]

So the solutions are \( x = \frac{4 + \sqrt{22}}{3} \) and \( x = \frac{4 - \sqrt{22}}{3} \).

Activity 17

The equation is
\[ -x^2 = -x - \frac{3}{2}. \]
Clearing the fraction gives
\[ -2x^2 = -2x - 3. \]
This equation can be rearranged as follows.
\[ 0 = 2x^2 - 2x - 3 \]
\[ 2x^2 - 2x - 3 = 0 \]
So \( a = 2, \ b = -2 \) and \( c = -3 \).

The quadratic formula gives
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 2 \times (-3)}}{2 \times 2} \]
\[ = \frac{2 \pm \sqrt{4 + 24}}{4} \]
\[ = \frac{2 \pm \sqrt{28}}{4} \]
\[ = \frac{1 \pm \sqrt{7}}{2}. \]

So the solutions are \( x = \frac{1 + \sqrt{7}}{2} \) and \( x = \frac{1 - \sqrt{7}}{2} \).

Activity 18

(a) The width of the photograph is 12 inches, and there are \( x \) inches of white border on each side, so the width of the backboard is \( 12 + 2x \) inches.
Similarly, the height of the photograph is 18 inches, and there are \( x \) inches of white border both above and below, so the height of the backboard is \( 18 + 2x \) inches.

(b) The area of the photograph is \( \text{width} \times \text{height} = 12 \times 18 = 216 \text{ in}^2 \).

The area of the backboard is 150% of the area of the photograph, so its area is
\[ 150\% \times 216 = 1.5 \times 216 = 324 \text{ in}^2. \]

(c) The area of the backboard in terms of \( x \) is
\[ \text{width} \times \text{height} = (12 + 2x)(18 + 2x) \]
\[ = 216 + 24x + 36x + 4x^2 \]
\[ = 4x^2 + 60x + 216, \]
where the units are square inches.

(d) Using the answers to parts (b) and (c) gives
\[ 4x^2 + 60x + 216 = 324. \]
This equation can be rearranged as follows.
\[ 4x^2 + 60x - 108 = 0 \]
\[ x^2 + 15x - 27 = 0 \]
(e) The equation in part (d) cannot easily be factorised, so we use the quadratic formula.

We have \(a = 1\), \(b = 15\) and \(c = -27\).

Using the quadratic formula gives

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-15 \pm \sqrt{225 + 108}}{2}
\]

\[
= \frac{-15 \pm \sqrt{333}}{2}
\]

\[
= \frac{-15 \pm 3\sqrt{37}}{2}
\]

\[
= \frac{-15 + 3\sqrt{37}}{2} \quad \text{or} \quad \frac{-15 - 3\sqrt{37}}{2}
\]

\[
= 1.624 \ldots \quad \text{or} \quad -16.624 \ldots
\]

Only the positive solution makes sense in the context of the problem. So the width of the white border is 1.6 inches, to the nearest tenth of an inch.

The width of the backboard is

\[
12 + 2 \times 1.624 \ldots = 15.2 \text{ in},
\]

and its height is

\[
18 + 2 \times 1.624 \ldots = 21.2 \text{ in},
\]

to the nearest tenth of an inch.

**Activity 19**

(a) The parabola is u-shaped.

Its vertex is \((-1, 5)\).

(b) The parabola is n-shaped.

Its vertex is \((-3, 7)\).

(c) The parabola is u-shaped.

Its vertex is \((1, -4)\).

(d) The parabola is n-shaped.

Its vertex is \((-\frac{1}{2}, -1)\).

(e) The parabola is u-shaped.

Its vertex is \((0, 3)\).

(f) The parabola is u-shaped.

Its vertex is \((2, 0)\).

**Activity 20**

(a) \(x^2 + 16x = (x + 8)^2 - 8^2\)

\[
= (x + 8)^2 - 64
\]

(Check: \((x + 8)^2 - 64 = (x + 8)(x + 8) - 64\)

\[
= x^2 + 16x + 64 - 64
\]

\[
= x^2 + 16x.
\]

(b) \(x^2 - 12x = (x - 6)^2 - (-6)^2\)

\[
= (x - 6)^2 - 36
\]

(Check: \((x - 6)^2 - 36 = (x - 6)(x - 6) - 36\)

\[
= x^2 - 12x + 36 - 36
\]

\[
= x^2 - 12x.
\]

(c) \(t^2 - 2t = (t - 1)^2 - (-1)^2\)

\[
= (t - 1)^2 - 1
\]

(Check: \((t - 1)^2 - 1 = (t - 1)(t - 1) - 1\)

\[
= t^2 - 2t + 1 - 1
\]

\[
= t^2 - 2t.\]

(d) \(x^2 + 3x = (x + \frac{3}{2})^2 - \left(\frac{3}{2}\right)^2\)

\[
= (x + \frac{3}{2})^2 - \frac{9}{4}
\]

(Check: \((x + \frac{3}{2})^2 - \frac{9}{4} = (x + \frac{3}{2})(x + \frac{3}{2}) - \frac{9}{4}\)

\[
= x^2 + \frac{3}{2}x + \frac{3}{2}x + \frac{9}{4} - \frac{9}{4}
\]

\[
= x^2 + 3x.\]

**Activity 21**

(a) \(x^2 + 6x - 3 = (x + 3)^2 - 9 - 3\)

\[
= (x + 3)^2 - 12
\]

(Check: \((x + 3)^2 - 12 = x^2 + 6x + 9 - 12\)

\[
= x^2 + 6x - 3.\)

(b) \(x^2 - 4x + 9 = (x - 2)^2 - 4 + 9\)

\[
= (x - 2)^2 + 5
\]

(Check: \((x - 2)^2 + 5 = x^2 - 4x + 4 + 5\)

\[
= x^2 - 4x + 9.\)

(c) \(p^2 - 12p - 5 = (p - 6)^2 - 36 - 5\)

\[
= (p - 6)^2 - 41
\]

(Check: \((p - 6)^2 - 41 = p^2 - 12p + 36 - 41\)

\[
= p^2 - 12p - 5.\)

(d) \(x^2 + x + 1 = (x + \frac{1}{2})^2 - \frac{1}{4} + 1\)

\[
= (x + \frac{1}{2})^2 + \frac{3}{4}
\]

(Check: \((x + \frac{1}{2})^2 + \frac{3}{4} = x^2 + x + \frac{1}{4} + \frac{3}{4}\)

\[
= x^2 + x + 1.\)

**Activity 22**

(a) \(x^2 + 6x - 5 = 0\)

\[
(x + 3)^2 - 9 - 5 = 0\)

\[
(x + 3)^2 - 14 = 0\)

\[
(x + 3)^2 = 14\)

\[
x + 3 = \pm \sqrt{14}\)

\[
x = -3 \pm \sqrt{14}\)

The solutions are \(x = -3 + \sqrt{14}\) and \(x = -3 - \sqrt{14}\).
(b) \(2x^2 - 12x - 5 = 0\)
\[x^2 - 6x - \frac{5}{2} = 0\]
\[(x - 3)^2 - 9 - \frac{5}{2} = 0\]
\[(x - 3)^2 - \frac{18}{2} - \frac{5}{2} = 0\]
\[(x - 3)^2 - \frac{23}{2} = 0\]
\[x - 3 = \pm \sqrt{\frac{23}{2}}\]
\[x = 3 \pm \sqrt{\frac{23}{2}}\]
The solutions are \(x = 3 + \sqrt{\frac{23}{2}}\) and \(x = 3 - \sqrt{\frac{23}{2}}\).

**Activity 23**

(a) \(2x^2 - 4x - 1 = 2(x^2 - 2x) - 1\)
\[= 2((x - 1)^2 - 1) - 1\]
\[= 2(x - 1)^2 - 2 - 1\]
\[= 2(x - 1)^2 - 3\]

(Check: \(2(x - 1)^2 - 3 = 2(x^2 - 2x + 1) - 3\)
\[= 2x^2 - 4x + 2 - 3\]
\[= 2x^2 - 4x - 1.\])

The vertex of the parabola with equation \(y = 2x^2 - 4x - 1\) is \((1, -3)\).

(b) \(-x^2 - 8x - 18 = -(x^2 + 8x) - 18\)
\[= -((x + 4)^2 - 16) - 18\]
\[= -(x + 4)^2 + 16 - 18\]
\[= -(x + 4)^2 - 2\]

(Check: \(-(x + 4)^2 - 2 = -(x^2 + 8x + 16) - 2\)
\[= -x^2 - 8x - 16 - 2\]
\[= -x^2 - 8x - 18.\])

The vertex of the parabola with equation \(y = -x^2 - 8x - 18\) is \((-4, -2)\).

**Activity 24**

(a) A table of values for the equation \(y = -x^2\) is given below.

<table>
<thead>
<tr>
<th>(x)</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-9</td>
<td>-4</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-4</td>
<td>-9</td>
</tr>
</tbody>
</table>

(For example, substituting \(x = -3\) into the equation gives \(y = -(3)^2 = -9\).

The resulting graph is shown below.

(b) The graph of \(y = -x^2\) is a mirror image of the graph of \(y = x^2\), reflected in the \(x\)-axis.

**Activity 25**

(a) Increasing the value of \(a\) within the positive values seems to make the parabola more narrow, while decreasing the value of \(a\) within the positive values seems to make it wider. The axis of symmetry is always the \(y\)-axis.

(b) Pairs of values of \(a\) that are negatives of each other give graphs that are reflections of each other in the \(x\)-axis, as expected.

So, as expected, increasing the size of \(a\) within the negative values seems to make the parabola narrower, while decreasing the size of \(a\) within the negative values seems to make it wider.

**Activity 26**

There are comments in the text after this activity.

**Activity 27**

(b) Changing \(c\) from 0 to 1 moves the graph up by 1 unit. The new graph crosses the \(y\)-axis at \((0, 1)\).

(c) The graph of \(y = x^2 + c\) seems to be exactly the same as the graph of \(y = x^2\), but shifted up or down the \(y\)-axis. It crosses the \(y\)-axis at \((0, c)\) (whether \(c\) is positive or negative).

(d) In general, whatever the values of \(a, b\) and \(c\), the graph of \(y = ax^2 + bx + c\) seems to be exactly the same as the graph of \(y = ax^2 + bx\), but shifted vertically up or down, so that it crosses the \(y\)-axis at \((0, c)\).
**Activity 28**

The equation is \( y = x^2 + 5x - 6 \).

The coefficient of \( x^2 \) is positive, so the graph is \( u \)-shaped.

Putting \( x = 0 \) gives \( y = -6 \), so the \( y \)-intercept is \(-6\).

Putting \( y = 0 \) gives
\[
x^2 + 5x - 6 = 0
\]
\[
(x - 1)(x + 6) = 0
\]
\[
x - 1 = 0 \quad \text{or} \quad x + 6 = 0
\]
\[
x = 1 \quad \text{or} \quad x = -6.
\]

So the \( x \)-intercepts are 1 and \(-6\).

The value halfway between the \( x \)-intercepts is
\[
\frac{1 + (-6)}{2} = -\frac{5}{2} = -2.5.
\]

So the axis of symmetry is \( x = -2.5 \).

Substituting \( x = -2.5 \) into the equation of the parabola gives
\[
y = x^2 + 5x - 6
\]
\[
= (-2.5)^2 + 5 \times (-2.5) - 6
\]
\[
= -12.25.
\]

So the vertex is \((-2.5, -12.25)\).

A sketch of the graph is shown below.

(If you feel uncomfortable basing your sketch on just two points, then you can calculate a third point on the parabola. For example, substituting \( x = 1 \) into the equation gives
\[
y = 9 \times 1^2 - 6 \times 1 + 1 = 4,
\]
so you can plot the point \((1, 4)\) to make the shape of the parabola clearer.)

**Activity 29**

The equation is \( y = 9x^2 - 6x + 1 \).

The coefficient of \( x^2 \) is positive, so the graph is \( u \)-shaped.

Putting \( x = 0 \) gives \( y = 1 \), so the \( y \)-intercept is 1.

Putting \( y = 0 \) gives
\[
9x^2 - 6x + 1 = 0
\]
\[
(3x - 1)(3x - 1) = 0
\]
\[
3x - 1 = 0
\]
\[
x = \frac{1}{3}.
\]

So the only \( x \)-intercept is \( \frac{1}{3} \).

Therefore the axis of symmetry is \( x = \frac{1}{3} \), and the vertex is \((\frac{1}{3}, 0)\).

A sketch of the graph is shown below.
**Activity 30**

The equation is \( y = x^2 + 2x + 3 \).

The coefficient of \( x^2 \) is positive, so the graph is u-shaped.

Putting \( x = 0 \) gives \( y = 3 \), so the \( y \)-intercept is 3.

The equation of the axis of symmetry is \( x = -b/(2a) \), where \( a = 1 \) and \( b = 2 \), so it is

\[
x = -\frac{2}{2 \times 1}; \quad \text{that is,} \quad x = -1.
\]

Substituting \( x = -1 \) into the equation of the parabola gives

\[
y = x^2 + 2x + 3
= (-1)^2 + 2 \times (-1) + 3
= 1 - 2 + 3
= 2.
\]

So the vertex is \((-1, 2)\).

A sketch of the graph is shown below.