**Practice Integration – set 1**

**Exercise 1.1**

Differentiate the following functions

(a) \( F(x) = 2x^3 - x \)  (b) \( F(x) = 2x^3 - x + \pi \)

(c) \( F(x) = 2x^3 - x + e^2 \)

**Exercise 1.2**

(a) Show that \( F(x) = x^2 + \frac{1}{2}e^{2x} \) is an antiderivative for \( f(x) = 2x + e^{2x} \).

(b) What is the indefinite integral of \( f(x) = 2x + \frac{1}{2}e^{2x} \)?

(c) Write down an antiderivative for \( f(x) = 2x + \frac{1}{2}e^{2x} \) other than the one found in part (a).

**Exercise 1.3**

Find the indefinite integrals of the following functions. All the functions are defined on a domain that only includes positive values.

(a) \( f(x) = x^{-7} \)  (b) \( g(s) = \sqrt{s} \)  (c) \( h(v) = \frac{1}{\sqrt{v}} \)

(d) \( i(\omega) = \sqrt[3]{\omega} \)  (e) \( j(\delta) = \frac{1}{\sqrt[3]{\delta}} \)  (f) \( k(z) = z^{-3/5} \)

**Exercise 1.4**

Find the indefinite integrals of the following functions. All the functions are defined on a domain that only includes positive values.

(a) \( f(x) = x^3 - 3x^2 \)  (b) \( g(s) = \sqrt{s} + \frac{2}{\sqrt{s}} \)  (c) \( h(y) = y \left( \frac{y}{2} - 2y^{-3} \right) \)

(d) \( l(r) = 2r^{-3/4} - 4r^{-2} \)  (e) \( m(z) = (z + 1)(z - 3) + 2z \)

(f) \( n(t) = \frac{t^3 + t^2}{t} \)  (g) \( p(v) = v(v - 1) - 2v^2 \)

(h) \( q(w) = 3w^2 + 2w + 1 \)

**Exercise 1.5**

Which one of the following statements follows from the fundamental theorem of calculus?

(a) All functions \( f(x) \) can be integrated

(b) A definite integral gives the area between the curve and the \( x \) axis

(c) An indefinite integral gives the area between the curve and the \( x \) axis

(d) A definite integral of \( f(x) \) can be found using any antiderivative \( F(x) \)

(e) A definite integral needs an arbitrary constant

**Exercise 1.6**

Evaluate the following expressions.

(a) \( \int_{-1}^{2} -2x^2 \)  (b) \( \int_{1}^{2} \frac{3}{x} \)  (c) \( \int_{-1}^{\pi/4} \sin(2x) \)
**Exercise 1.7**

Consider the graph shown below showing a curve $f(x)$. The area of each shaded region is shown as a fraction.

Using the diagram find the following definite integrals.

(a) $\int_{-4}^{1} f(x) \, dx$  (b) $\int_{-4}^{-3} f(x) \, dx$  (c) $\int_{2}^{3} f(x) \, dx$

(d) $\int_{2}^{-3} f(x) \, dx$  (e) $\int_{2}^{4} f(x) \, dx$  (f) $\int_{1}^{-3} f(x) \, dx$

(g) $\int_{2}^{-3} f(x) \, dx$  (h) $\int_{1}^{2} f(x) \, dx$  (i) $\int_{1}^{-4} f(x) \, dx$

**Exercise 1.8**

Find the following definite integrals.

(a) $\int_{1}^{2} x^3 \, dx$  (b) $\int_{0}^{\pi/2} \cos(x) \, dx$  (c) $\int_{1}^{e} 2 \, dr$

(d) $\int_{1}^{2} \csc^2(x) \, dx$  (e) $\int_{1}^{1/\sqrt{2}} 1/\sqrt{1-y^2} \, dy$  (f) $\int_{0}^{1} \sec(z) \tan(z) \, dz$

**Exercise 1.9**

Find the following indefinite integrals by substitution.

(a) $\int 2x \sec^2(x^2) \, dx$  (b) $\int x^{-2}e^{1/x} \, dx$ for $x > 0$

(c) $\int 8x^3(x^4 - 5)^8 \, dx$  (d) $\int \tan(x) \sec^2(x) \, dx$

(e) $\int \frac{4x}{1+4x^2} \, dx$. Use $u = 2x^2$ so that $u^2 = 4x^4$.

(f) $\int \frac{3x^4}{4x^2 + 7} \, dx$

**Exercise 1.10**

Find the following integrals by substituting for the linear expression.

(a) $\int \sin(2x - 3) \, dx$  (b) $\int e^{4x+1} \, dx$

(c) $\int \frac{1}{x+7} \, dx$ for $x > 1/7$  (d) $\int \sec^2(1 - 3x) \, dx$

**Exercise 1.11**

Find the following indefinite integrals by integrating by parts.

(a) $\int xe^x \, dx$  (b) $\int 3x \ln x \, dx$ for $x > 0$

(c) $\int 2x \sin \left( \frac{1}{5}x \right) \, dx$  (d) $\int (4 - 3x) \cos(5x) \, dx$
Exercise 1.12
Find the following indefinite integrals by using trigonometric identities.
(a) \( \int (\sin^2(x) - \cos^2(x)) \, dx \)  
(b) \( \int (\sin(x)(\cos(x) - \sin(x))) \, dx \)
(c) \( \int (\cos^2(x)(1 + \tan^2(x))) \, dx \)  
(d) \( \int (\sin(2x) + 2\sin(x)\cos(x)) \, dx \)

Exercise 1.13
For each of the integrals below suggest a method of integration and perform the integration. This section uses all of the methods covered in these exercises.
(a) \( \int 6x^2e^{x^3} \, dx \)  
(b) \( \int x^2 \ln(x) \, dx \) for \( x > 0 \)
(c) \( \int \frac{2x-2}{x^2-2x+1} \, dx \)  
(d) \( \int \sin(x)\csc(x) - \cot(x) \, dx \)
(e) \( \int \arctan(x) \, dx \)  
(f) \( \int 6x^2(x^3 - 9)^{12} \, dx \)
(g) \( \int \sin^2 \left( \frac{x}{4} \right) - \cos^2 \left( \frac{x}{4} \right) \, dx \)  
(h) \( \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx \)
(i) \( \int \frac{1}{\sqrt{1-16y^2}} \, dy \)  
(j) \( \int \frac{1}{1+3x^2} \, dx \)
(k) \( \int e^{2x} (e^{-x} + e^{-3x}) \, dx \)  
(l) \( \int \sec(x) \left( \frac{1}{\cos(x)} + \cos^2(x) \right) \, dx \)
Solutions to Exercises

Solution 1.1
(a) When \( F(x) = 2x^3 - x \) then 
\[
f(x) = F'(x) = 6x^2 - 1.
\]
(b) When \( F(x) = 2x^3 - x + \pi \) then 
\[
f(x) = F'(x) = 6x^2.
\]
(c) When \( F(x) = 2x^3 - x + e^2 \) then 
\[
f(x) = F'(x) = 6x^2 - 1.
\]

Solution 1.2
(a) \( F'(x) = 2x^2 - 1 + 2 \cdot \frac{1}{4} e^{2x} = 2x + e^{2x} = f(x) \)
(b) The indefinite integral is 
\[
F(x) + c = x^3 + \frac{1}{4} e^{2x} + c
\]
(c) Choosing \( c \) to be any non-zero real number will give a valid anti-derivative. For example \( c = 1 \) gives  
\[
x^2 + \frac{1}{4} e^{2x} + 1
\]

Solution 1.3
(a) \( f(x) = x^{-7} \) \( \Rightarrow \) \( F(x) = \frac{1}{(-7+1)} x^{-7+1} + c \)
\[
= \frac{1}{6} x^{-6} + c
\]
(b) \( g(s) = \sqrt{s} \) \( \Rightarrow \) \( G(s) = \frac{1}{(7/2+1)} s^{7/2+1} + c = \frac{2}{3} s^{3/2} + c \)
(c) \( h(v) = \frac{1}{\sqrt{2}} \) \( \Rightarrow \) \( H(v) = \frac{1}{(7/2+1)} v^{-7/2+1} + c = \frac{2}{3} v^{1/2} + c \)
(d) \( i(\omega) = \frac{1}{\sqrt{2}} \) \( \Rightarrow \) \( I(\omega) = \frac{1}{(7/2+1)} \omega^{-7/2+1} + c = \frac{2}{3} \omega^{1/2} + c \)
(e) \( j(\delta) = \frac{1}{\sqrt{2}} \) \( \Rightarrow \) \( J(\delta) = \frac{1}{(7/2+1)} \delta^{-7/2+1} + c = \frac{2}{3} \delta^{1/2} + c \)
(f) \( k(z) = z^{-3/5} \) \( \Rightarrow \) \( K(z) = \frac{1}{(7/2+1)} z^{-3/5+1} + c = \frac{2}{3} z^{2/5} + c \)

Solution 1.4
(a) \( \int f(x) dx = \int x^3 dx - 3 \int x^2 dx = \frac{1}{4} x^4 - \frac{3}{2} x^2 + c \)
(b) \( g(s) = \int s^{1/2} ds + 2 \int s^{3/2} + \frac{2}{3} s^2 + c \)
(c) \( h(y) = \frac{1}{y^2} y^2 dy - 2 \int y^{-2} dy = \frac{1}{y} + 2 y^{-1} + c \)
(d) \( l(r) = 2 \int r^{-3/4} dr - 4 \int r^{-2} dr = 8 r^{1/4} + 4 r^{-1} + c \)
(e) \( m(z) = (z + 1)(z - 3) + 2 z = z^2 - 2 z - 3 + 2 z = z^2 - 3 \)
\[
\Rightarrow \int m(z) dz = \int z^2 dz - 3 \int dz = \frac{1}{3} z^3 - 3 z + c
\]
(f) \( n(t) = t^2 dt + \int t^2 dt = \int t^2 dt + \int t dt = \frac{t^3}{3} + \frac{t^2}{2} + c \)

(g) \( p(v) = v(v - 1) - 2v^2 = v^3 - v - 2v^2 = -v^3 - v \)
\[
\Rightarrow \int p(v) dv = - \int v^3 dv - \int v dv - \int \frac{v^2}{v} - \int \frac{v^2}{v} = \frac{v^2}{2} - \frac{v^3}{3} + c
\]
(h) \( \int q(w) dw = 3 \int w^2 dw + 2 \int w dw + \int dw = w^3 + w^2 + w + c \)

Solution 1.5
(a) All functions \( f(x) \) can be integrated. False as discontinuous functions may not have an antiderivative.
(b) A definite integral gives the area between the curve and the \( x \) axis. False. Integration gives the signed area.
(c) An indefinite integral gives the area between the curve and the \( x \) axis False. An indefinite integral produces a function and not a number.
(d) A definite integral of \( f(x) \) can be found using any antiderivative \( F(x) \). True.
(e) A definite integral needs an arbitrary constant. False. An indefinite integral requires an arbitrary constant.

Solution 1.6
(a) \( [\text{2x}^2]_{-1}^{-1} = (-2(2)^2) - (-2(-1)^2) = -8 - 2 = -6 \)
(b) \( \left[ \frac{x}{x} \right]_1^2 = \left( \frac{2}{2} \right) - \left( \frac{1}{1} \right) = 2 \left( \frac{1}{2} \right) - 1 \)
(c) \( \sin(2x) \mid_1^\frac{\pi}{4} = \sin(2(\pi/4)) - \sin(2(-1/2)) = \sin(\pi/2) - \sin(-1) = 1 + \sin(1) \)

Solution 1.7
Some of the solutions here can be worked out from previous parts.
(a) \( \int_1^3 f(x) dx = \int_1^3 f(x) dx + f_1 f(x) dx \)
\[
= \frac{187}{12} + \frac{76}{12} = \frac{263}{12}
\]
(b) \( \int_{-4}^3 f(x) dx = - \int_{-4}^3 f(x) dx = - \left( -\frac{187}{12} \right) = \frac{187}{12} \)
(c) \( \int_{-2}^1 f(x) dx = - \int_{-2}^1 f(x) dx \)
\[
= \left( -\int_{-2}^1 f(x) dx + f_1 f(x) dx + f_1 f(x) dx \right) = -\left( \frac{187}{12} + \frac{76}{12} - \frac{17}{12} \right) = -\frac{20}{3}
\]
(d) \( \int_{-3}^1 f(x) dx = - \int_{-3}^1 f(x) dx = - \left( \frac{187}{12} + \frac{76}{12} - \frac{17}{12} \right) = -\frac{89}{4} \)
(e) \( \int_{-3}^1 f(x) dx = - \int_{-3}^1 f(x) dx = - \left( \frac{89}{4} \right) = \frac{22}{3} \)
(f) \( \int_{-3}^1 f(x) dx = - \int_{-3}^1 f(x) dx = - \left( \frac{22}{3} \right) = \frac{22}{3} \)
(g) \( \int_{-3}^1 f(x) dx = - \int_{-3}^1 f(x) dx = \frac{22}{3} \)
The 'inside' function is always a linear expression in

\[ \]  

Solution 1.8

(a) \[ \int_{-1}^{1} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{-1}^{1} = 0 \]

(b) \[ \int_{0}^{\pi} \cos(x) \, dx = \sin(x) \Big|_{0}^{\pi} = 1 \]

(c) \[ \int_{1}^{2} x^2 \, dx = \left[ \frac{2}{3} x^3 \right]_{1}^{2} = \frac{8}{3} - \frac{2}{3} = 2 \]

(d) \[ \int_{1}^{2} \csc^2(x) \, dx = \left[ -\cot(x) \right]_{1}^{2} = -\cot(2) + \cot(1) \]

(e) \[ \int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{1-y^2}} \, dy = \left[ \sin(y) \right]_{1/\sqrt{2}}^{1} = \sin(1) - \frac{1}{\sqrt{2}} \]

(f) \[ \int_{0}^{1} \sec(z) \, dz = \left[ \sec(z) \right]_{0}^{1} = \sec(1) - \sec(0) \]

Solution 1.9

(a) \[ \int 2x \sec^2(x^2) \, dx \]

(b) \[ \int x^{-2} e^{1/x} \, dx \]

(c) \[ \int 8x^3(x^4 - 5)^8 \, dx \]

(d) \[ \int \tan(x) \sec^2(x) \, dx \]

(e) \[ \int \frac{1}{1+4x} \, dx \]

(f) \[ \int \frac{1}{4x^3 + 1} \, dx \]

Solution 1.10

The 'inside' function is always a linear expression in this section. Therefore the reciprocal of the coefficient of the x term is the constant multiple.

(a) \[ \int \sin(2x - 3) \, dx \]

(b) \[ \int e^{4x+1} \, dx \]

Solution 1.11

(a) \[ \int e^x \, dx \]

(b) \[ \int 3x \ln(x) \, dx \]

(c) \[ \int \frac{1}{x} \, dx \]

(d) \[ \int 4 - 3x \cos(5x) \, dx \]

Solution 1.12

(a) \[ \int \sin^2(x) - \cos^2(x) \, dx \]

Rewrite \[ \sin^2(x) = \frac{1}{2} (1 - \cos(2x)) \] and \[ \cos^2(x) = \frac{1}{2} (1 + \cos(2x)) \]. Then

\[ \int \sin^2(x) - \cos^2(x) \, dx = \frac{1}{2} \left( 1 - \cos(2x) - (1 + \cos(2x)) \right) \]

(b) \[ \int \sin(x) \cos(x) - \sin(x) \, dx \]

The first step is to expand out the brackets so the integrand becomes \[ \sin(x) \cos(x) - \sin^2(x) \]. An identity from the previous part is used for \[ \sin^2(x) \] and \[ \frac{1}{2} \sin(2x) = \sin(x) \cos(x) \].

\[ \int \sin(x) \cos(x) - \sin(x) \, dx = \frac{1}{4} \left( \sin(2x) - (1 - \cos(2x)) \right) \]

\[ = \frac{1}{4} \cos(2x) - \frac{1}{2} \sin(2x) + c \]

\[ = \frac{1}{4} \sin(2x) - \cos(2x) - 2x + c \]
(c) \( \int \cos^2(x)(1 + \tan^2(x)) \, dx \).
Since \( \tan(x) = \frac{\sin(x)}{\cos(x)} \) then \( \tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} \).
\[
\cos^2(x)(1 + \tan^2(x)) = \cos^2(x) \left(1 + \frac{\sin^2(x)}{\cos^2(x)}\right)
= \cos^2(x) + \sin^2(x) = 1
\]
Therefore the integrand simplifies to 1 and \( \int \cos^2(x)(1 + \tan^2(x)) \, dx = \int 1 \, dx = x + c \)

(d) \( \int \sin(2x) + 2 \sin(x) \cos(x) \, dx \).
The first step is to use \( 2 \sin(x) \cos(x) = \sin(2x) \) and then simplify the integrand.
\[
\int \sin(2x) + 2 \sin(x) \cos(x) \, dx = 2 \int \sin(2x)x \, dx = -\cos(2x) + c
\]

**Solution 1.13**

(a) \( \int 6x^2 e^3 \, dx \)
This integral can be solved by substitution by choosing \( u = x^3 \). It follows that \( \frac{du}{dx} = 3x^2 \) and the integral becomes
\[
2 \int e^u du = 2e^u + c = 2e^{x^3} + c
\]

(b) \( \int x^2 \ln(x) \, dx \) for \( x > 0 \)
This integral can be solved by using integration by parts. Let \( f(x) = \ln(x) \) and \( g(x) = x^2 \). Then \( f'(x) = \frac{1}{x} \) and \( G(x) = \frac{1}{3}x^3 \).
\[
\int x^2 \ln(x) \, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3} \int x^2 \, dx
= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + c
\]

(c) \( \int \frac{2x^2-2}{x^2+2x+17} \, dx \)
This integral can be solved by substitution by choosing \( u = x^2 + 2x + 17 \). It follows that \( \frac{du}{dx} = 2x + 2 \) and the integral becomes
\[
\int \frac{2x^2-2}{x^2+2x+17} \, dx = \int \frac{2x+2}{u} \, du
= \int \frac{1}{u} \, du = \ln |u| + c = \ln(x^2 + 2x + 17) + c.
\]
Since \( x^2 + 2x + 17 = (x - 1)^2 + 16 > 0 \) for all real values of \( x \) the modulus sign in the logarithm function can be safely removed.

(d) \( \int \sin(x) \cos(x) - \cot(x) \, dx \).
The integrand can be simplified by using trigonometric identities. Write \( \cos(x) = \frac{1}{\sin(x)} \) and \( \cot(x) = \frac{\cos(x)}{\sin(x)} \).
\[
\sin(x)(\cos(x) - \cot(x)) = \sin(x) \left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right) = 1 + \cos(x)
\]
so, \( \int (1 + \cos(x)) \, dx = x + \sin(x) + c \)

(e) \( \int \arctan(x) \, dx = \int 1 \times \arctan(x) \, dx \)
This integral can be solved by using integration by parts. Let \( f(x) = \arctan(x) \) and \( g(x) = 1 \).
Then \( f'(x) = \frac{1}{1+x^2} \) and \( G(x) = x \).
\[
\int \arctan(x) \, dx = x \arctan(x) - \int x \frac{1}{1+x^2} \, dx
\]
The integral on the right hand side can now be solved by substitution.
Let \( u = 1 + x^2 \) then \( du = 2x \, dx \).
\[
\int \frac{1}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |u| + c = \frac{1}{2} \ln(1 + x^2) + c
\]
Substituting into this into the original integral gives
\[
\int \arctan(x) \, dx = x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + c
\]

(f) \( \int 6x^2(x^3 - 9)^{12} \, dx \)
This integral can be solved by substitution by choosing \( u = x^3 - 9 \). It follows that \( \frac{du}{dx} = 3x^2 \) and the integral becomes
\[
\int 6x^2(x^3 - 9)^{12} \, dx = 2 \int u^{12} \, du = 2 \int u^{12} \, du = \frac{u^{13}}{13} + c = \frac{(x^3-9)^{13}}{13} + c
\]

(g) \( \int \sin^2 \left(\frac{x}{2}\right) - \cos^2 \left(\frac{x}{2}\right) \, dx \)
This integral can be rewritten using \( \sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)) \) and \( \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)). \)
Therefore \( \sin^2(x/4) = \frac{1}{2}(1 - \cos(x/2)) \) and \( \cos^2(x/4) = \frac{1}{2}(1 + \cos(x/2)). \)
\[
\int \sin^2 \left(\frac{x}{2}\right) - \cos^2 \left(\frac{x}{2}\right) \, dx
= \frac{1}{2} \int 1 - \cos(x/2) - 1 - \cos(x) \, dx
= -\frac{1}{2}(2 \sin(x/2) + \sin(x)) + c
\]

(h) \( \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx \)
This integral can be solved by substitution by choosing \( u = \cos(x) \). It follows that \( \frac{du}{dx} = -\sin(x) \) and the integral becomes
\[
\int \tan(x) \, dx = -\int \frac{1}{u} \, du = -\ln |u| + c
= -\ln |\cos(x)| + c
\]

(i) \( \int \frac{1}{\sqrt{1-16y^2}} \, dy \)
This integral can be solved by substitution. Let \( u = 4y \) then \( \frac{du}{dy} = 4 \).
\[
\int \frac{1}{\sqrt{1-16y^2}} \, dy = \frac{1}{4} \int \frac{1}{\sqrt{1-u^2}} \, du = \frac{1}{4} \int \frac{1}{\sqrt{1-u^2}} \, du
= \frac{1}{4} \arcsin(u) + c = \frac{1}{4} \arcsin(4y) + c
\]

(j) \( \int \frac{1}{x+2} \, dx \)
This integral can be solved by substitution. Let \( u = \sqrt{x+2} \) then \( \frac{du}{dx} = \frac{1}{\sqrt{x+2}} \).
\[
\int \frac{1}{x+2} \, dx = \frac{1}{\sqrt{x+2}} \int \frac{1}{u} \, du = \frac{1}{\sqrt{x+2}} \int \frac{1}{u} \, du
= \frac{1}{\sqrt{5}} \arctan(u) + c = \frac{1}{\sqrt{5}} \arctan(\sqrt{x+2}) + c
\]

(k) \( \int e^{2x} (e^{-x} + e^{-3x}) \, dx \)
The integrand should be expanded to obtain \( e^x + e^{-x} \).
\[
\int e^{2x} (e^{-x} + e^{-3x}) \, dx = \int e^x \, dx + \int e^{-x} \, dx
= e^x - e^{-x} + c
\]

(l) \( \int \sec(x) \left(\frac{1}{\cos(x)} + \cos^2(x)\right) \, dx \)
The integrand needs to be rearranged using \( \sec(x) = \frac{1}{\cos(x)} \).
\[
\sec(x) \left(\frac{1}{\cos(x)} + \cos^2(x)\right)
= \frac{1}{\cos(x)} \left(\frac{1}{\cos(x)} + \cos^2(x)\right)
= \frac{1}{\cos^2(x)} + \cos(x) = \sec^2(x) + \cos(x)
\]
Therefore the integral can be rewritten as \( \int \sec^2(x) + \cos(x) \, dx = \tan(x) + \sin(x) + c. \)