Unit 1

Algebra
Welcome to MST124

In this module you’ll learn the essential ideas and techniques that underpin university-level study in mathematics and mathematical subjects such as physics, engineering and economics. You’ll also develop your skills in communicating mathematics.

Here are some of the topics that you’ll meet.

Vectors are quantities that have both a size and a direction. For example, a ship on the ocean moves not only with a particular speed, but also in a particular direction. Speed in a particular direction is a vector quantity known as velocity.

Calculus is a fundamental topic in mathematics that’s concerned with quantities that change continuously. If you know that an object is moving at a constant speed, then it’s straightforward to work out how much distance it covers in any given period of time. It’s not so easy to do this if the object’s speed is changing – for example, if it’s accelerating, as a falling object does. Calculus can be used to deal with situations like this.

Matrices are rectangular arrays of numbers – for example, any rectangular table of numbers forms a matrix. Matrices have many applications, which involve performing operations on them that are similar to the operations that you perform on individual numbers. For example, you can add, subtract and multiply matrices.

Sequences are lists of numbers. Sequences whose numbers have a connecting mathematical relationship arise in many different contexts. For example, if you invest £100 at a 5% rate of interest paid annually, then the value in pounds of your investment at the beginning of each year forms the sequence 100, 105, 110.25, 115.76, 121.55, . . .

The complex numbers include all the real numbers that you know about already, and also many ‘imaginary’ numbers, such as the square root of −1. Amazingly, they provide a simple way to deal with some types of complicated mathematics that arise in practical problems.

You’ll see that not only do the topics above have important practical applications, but they’re also intriguing areas of study in their own right.

One of the main aims of the first few units of MST124 is to make sure that you’re confident with the basic skills in algebra, graphs, trigonometry, indices and logarithms that you’ll need. The mathematics in the later units of the module depends heavily on these basic skills, and you’ll find it much easier and much quicker to study and understand if you can work with all the basic skills fluently and correctly.

To help you attain confidence with these skills, the first few units of the module include many revision topics, as well as some new ones. Which parts, and how much, of the revision material you’ll need to study will depend on your mathematical background – different students start MST124 with widely differing previous mathematical experiences. When you’re deciding which revision topics you need to study, remember that
even though you’ll have met most of them before, you won’t necessarily have acquired the ‘at your fingertips’ fluency in working with them that you’ll need. Where that’s the case, you’ll benefit significantly from working carefully through the revision material.

Information for joint MST124 and MST125 students

If you are studying Essential mathematics 2 (MST125) with the same start date as MST124, then you should not study the MST124 units on the dates shown on the main MST124 study planner. Instead, you should follow the MST124 and MST125 joint study planner, which is available from the MST124 and MST125 websites. This is important because you will not be prepared to study many of the topics in MST125 if you have not already studied the related topics in MST124. The MST124 and MST125 joint study planner ensures that you study the units of the two modules interleaved in the correct order.

The MST124 assignment cut-off dates shown in the MST124 and MST125 joint study planner are the same as those shown on the MST124 study planner.

Introduction

The main topic of this first unit is basic algebra, the most important of the essential mathematical skills that you’ll need. You’ll find it difficult to work through many of the units in the module, particularly the calculus units, if you’re not able to manipulate algebraic expressions and equations fluently and accurately. So it’s worth spending some time now practising your algebra skills. This unit gives you the opportunity to do that.

The unit covers a lot of topics quite rapidly, in the expectation that you’ll be fairly familiar with much of the material. You should use it as a resource to help you make sure that your algebra skills are as good as they can be. You may not need to study all the topics – you should concentrate on those in which you need practice. For many students these will be the topics in Sections 3 to 6. A good strategy might be to read through the whole unit, doing the activities on the topics in which you know you need practice. For the topics in which you think you don’t need practice, try one or two of the later parts of each activity to make sure – there may be gaps and rustiness in your algebra skills of which you’re unaware. Remember to check all your answers against the correct answers provided (these are at the end of the unit in the print book, and can be obtained by pressing the ‘show solution’ buttons in some screen versions).

As with all the units in this module, further practice questions are available in both the online practice quiz and the exercise booklet for the unit.
Working through the revision material in this unit should also help you to clarify your thinking about algebra. For example, you might know what to do with a particular type of algebraic expression or equation, but you might not know, or might have forgotten, why this is a valid thing to do. If you can clearly understand the ‘why’, then you’ll be in a much better position to decide whether you can apply the same sort of technique to a slightly different situation, which is the sort of thing that you’ll need to do as you study more mathematics.

Some of the topics in the unit may seem very easy – basic algebra is revised starting from the simplest ideas. Others may seem quite challenging – some of the algebraic expressions and equations that you’re asked to manipulate may be more complicated than those that you’ve dealt with before, particularly the ones involving algebraic fractions and indices.

The final section of the unit, Section 6, describes some basic principles of communicating mathematics in writing. This will be important throughout your study of this module and in any further mathematical modules that you study.

If you find that much of the content of this unit (and/or Unit 2) is unfamiliar to you, then contact your tutor and/or Learner Support Team as soon as possible, to discuss what to do.

The word ‘algebra’ is derived from the title of the treatise al-Kitāb al-mukhtāṣar fī ḥiṣāb al-jabr wa’l-muqābala (Compendium on calculation by completion and reduction), written by the Islamic mathematician Muḥammad ibn Mūsā al-Khwārizmī in around 825. This treatise deals with solving linear and quadratic equations, but it doesn’t use algebra in the modern sense, as no letters or other symbols are used to represent numbers. Modern, symbolic algebra emerged in the 1500s and 1600s.

1 Numbers

In this section you’ll revise different types of numbers, and some basic skills associated with working with numbers.
1.1 Types of numbers

We’ll make a start by briefly reviewing some different types of numbers.

Remember that all the definitions given here, and all the other definitions and important facts and techniques given in the module, are also set out in the Handbook, so you can refer to them easily.

The **natural numbers**, also known as the **positive integers**, are the counting numbers,

\[1, 2, 3, \ldots\]

(The symbol ‘…’ here is called an *ellipsis* and is used when something has been left out. You can read it as ‘dot, dot, dot’. In some texts the natural numbers are defined to be 0, 1, 2, 3, ….)

The natural numbers, together with their negatives and zero, form the **integers**:

\[\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\]

The Latin word *integer* consists of the prefix *in*, meaning ‘not’, attached to the root of *tangere*, meaning ‘to touch’. So it literally means ‘untouched’, in the sense of ‘whole’.

The **rational numbers** are the numbers that can be written in the form

\[
\frac{\text{integer}}{\text{integer}};
\]

that is, as an integer divided by an integer.

For example, all the following numbers are rational numbers:

\[
\frac{3}{7}, \quad 2\frac{1}{3}, \quad 4, \quad -4, \quad -\frac{8}{7}, \quad 0.16, \quad 7.374.
\]

You can check this by writing them in the form above, as follows:

\[
\frac{3}{7}, \quad \frac{7}{3}, \quad \frac{4}{1}, \quad \frac{-4}{1}, \quad \frac{-8}{7}, \quad \frac{16}{100}, \quad \frac{7374}{1000}.
\]

The **real numbers** include all the rational numbers, and many other numbers as well. A useful way to think of the real numbers is to envisage them as lying along a straight line that extends infinitely far in each direction, called the **number line** or the **real line**. Every point on the number line corresponds to a real number, and every real number corresponds to a point on the line. Some points on the line correspond to rational numbers, while others correspond to numbers that are not rational, which are known as **irrational** numbers. Figure 1 shows some numbers on the number line.
1 Numbers

Figure 1  Some numbers on the number line

Four of the numbers marked in Figure 1 are irrational, namely $-\sqrt{2}$, $\sqrt{2}$, $e$ and $\pi$. The number $\sqrt{2}$ is the positive square root of 2, that is, the positive number that when multiplied by itself gives the answer 2. Its value is approximately 1.41. The number $-\sqrt{2}$ is the negative of this number. The numbers $e$ and $\pi$ are two important constants that occur frequently in mathematics. You probably know that $\pi$ is the number obtained by dividing the circumference of any circle by its diameter (see Figure 2). Its value is approximately 3.14. (The symbol $\pi$ is a lower-case Greek letter, pronounced ‘pie’.) The constant $e$ has value approximately 2.72, and you’ll learn more about it in this module, starting in Unit 3.

To check that the numbers $-\sqrt{2}$, $\sqrt{2}$, $e$ and $\pi$ are irrational, you have to prove that they can’t be written as an integer divided by an integer. If you’d like to see how this can be done for $\sqrt{2}$, then look at the document *A proof that $\sqrt{2}$ is irrational* on the module website. Proving that $e$ and $\pi$ can’t be written as an integer divided by an integer is more difficult, and outside the scope of this module.

Every rational number can be written as a decimal number. To do this, you divide the top number of the fraction of the form $\frac{\text{integer}}{\text{integer}}$ by the bottom number. For example,

$$\frac{1}{8} = 0.125,$$

$$\frac{2}{3} = 0.666 666 666 \ldots ,$$

$$\frac{83}{74} = 1.121 621 621 621 \ldots .$$

As you can see, the decimal form of $\frac{1}{8}$ is terminating: it has only a finite number of digits after the decimal point. The decimal forms of both $\frac{2}{3}$ and $\frac{83}{74}$ are recurring: each of them has a block of one or more digits after the decimal point that repeats indefinitely. There are two alternative notations for indicating a recurring decimal: you can either put a dot above the first and last digit of the repeating block, or you can put a line above the whole repeating block. For example,

$$\frac{2}{3} = 0.\overline{6} = 0.666 666 \ldots = 0.\overline{6},$$

$$\frac{83}{74} = 1.\overline{121 6} = 1.121 6 216 216 216 \ldots = 1.\overline{121 6}.$$ 

In fact, the decimal form of every rational number is either terminating or recurring. Also, every terminating or recurring decimal can be written as an integer divided by an integer and is therefore a rational number. If you’d like to know why these facts hold, then look at the document *Decimal forms of rational numbers* on the module website.
The decimal numbers that are neither terminating nor recurring – that is, those that are infinitely long but have no block of digits that repeats indefinitely – are the irrational numbers. This gives you another way to distinguish between the rational and irrational numbers, summarised below.

### Decimal forms of rational and irrational numbers

- **Rational numbers** are the decimal numbers that terminate or recur.
- **Irrational numbers** are the decimal numbers with an infinite number of digits after the decimal point but with no block of digits that repeats indefinitely.

So, for example, the irrational number \( \pi \) has a decimal expansion that is infinitely long and has no block of digits that repeats indefinitely. Here are its first 40 digits:

\[
\pi = 3.141592653589793238462643383279502884197 \ldots
\]

You might like to watch the one-minute video clip entitled *The decimal expansion of \( \pi \)*, available on the module website.

In 2006, a Japanese retired engineer and mental health counsellor, Akira Haraguchi, recited the first 100,000 digits of \( \pi \) from memory. It took him 16 hours.

Figure 3 is a summary of the types of numbers mentioned in this subsection. It illustrates that all the natural numbers are also integers, all the integers are also rational numbers, and all the rational numbers are also real numbers.

In Unit 12 you’ll learn about yet another type of number. The **complex numbers** include all the numbers in Figure 3, and also many ‘imaginary’ numbers, such as the square root of \(-1\). The idea of imaginary numbers might seem strange, but these numbers are the foundation of a great deal of interesting and useful mathematics. They provide a natural, elegant way to work with seemingly complicated mathematics, and have many practical applications.
1.2 Working with numbers

In this subsection, you’ll revise some basic skills associated with working with numbers. It’s easy to make mistakes with these particular skills, and people often do! So you should find it helpful to review and practise them.

Before doing so, notice the label ‘(1)’ on the right of the next paragraph. It’s used later in the text to refer back to the contents of the line in which it appears. Labels like this are used throughout the module.

The BIDMAS rules

When you evaluate (find the value of) an expression such as

\[ 200 - 3 \times (1 + 5 \times 2^3) + 7, \]  

(1)

it’s important to remember the following convention for the order of the operations, so that you get the right answer.

### Order of operations: BIDMAS

Carry out mathematical operations in the following order.

- **B** brackets
- **I** indices (powers and roots)
- **D** divisions { same precedence
- **M** multiplications { same precedence
- **A** additions
- **S** subtractions { same precedence

Where operations have the same precedence, work from left to right.

As you can see, the I in the BIDMAS rules refers to ‘indices (powers and roots)’. Remember that raising a number to a power means multiplying it by itself a specified number of times. For example, \( 2^3 \) (2 to the power 3) means three 2s multiplied together:

\[ 2^3 = 2 \times 2 \times 2. \]

In particular, **squaring** and **cubing** a number mean raising it to the powers 2 and 3, respectively. When you write an expression such as \( 2^3 \), you’re using **index notation**. Taking a root of a number means taking its square root, for example, or another type of root. Roots are revised in Subsection 4.1.

If you type an expression like expression (1) into a calculator of the type recommended in the *MST124 guide*, then it will be evaluated according to the BIDMAS rules. However, it’s essential that you understand and remember the rules yourself. For example, you’ll need to use them when you work with algebra.

Example 1 reminds you how to use the BIDMAS rules. It also illustrates another feature that you’ll see throughout the module. Some of the worked examples include lines of blue text, marked with the following icons 🌈 🎉.
This text tells you what someone doing the mathematics might be thinking, but wouldn’t write down. It should help you understand how you might do a similar calculation yourself.

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**Example 1  Using the BIDMAS rules**

Evaluate the expression

\[
200 - 3 \times (1 + 5 \times 2^3) + 7
\]

without using your calculator.

**Solution**

1. The brackets have the highest precedence, so start by evaluating what’s inside them. Within the brackets, first deal with the power, then do the multiplication, then the addition.

\[
200 - 3 \times (1 + 5 \times 2^3) + 7 = 200 - 3 \times (1 + 5 \times 8) + 7
\]

\[
= 200 - 3 \times (1 + 40) + 7
\]

\[
= 200 - 3 \times 41 + 7
\]

2. Now do the multiplication, then the addition and subtraction from left to right.

\[
= 200 - 123 + 7
\]

\[
= 77 + 7
\]

\[
= 84
\]

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You can practise using the BIDMAS rules in the next activity. Remember that where division is indicated using fraction notation, the horizontal line not only indicates division but also acts as brackets for the expressions above and below the line. For example,

\[
\frac{1 + 2}{1 + 3^2} \quad \text{means} \quad \frac{(1 + 2)}{(1 + 3^2)}, \quad \text{that is,} \quad (1 + 2) \div (1 + 3^2).
\]

In a line of text, this expression would normally be written as \((1 + 2)/(1 + 3^2)\), with a slash replacing the horizontal line. The brackets are needed here because \(1 + 2/1 + 3^2\) would be interpreted as \(1 + (2/1) + 3^2\).

Part (b) of the activity involves algebraic expressions. Remember that multiplication signs are usually omitted when doing algebra – quantities that are multiplied are usually just written next to each other instead (though, for example, \(3 \times 4\) can’t be written as \(34\)).
Activity 1  Using the BIDMAS rules

(a) Evaluate the following expressions without using your calculator.

(i) $23 - 2 \times 3 + (4 - 2)$  (ii) $2 - \frac{1}{2} \times 4$  (iii) $4 \times 3^2$

(iv) $2 + 2^2$  (v) $\frac{1 + 2}{1 + 3^2}$  (vi) $1 - 2/3^2$

(b) Evaluate the following expressions when $a = 3$ and $b = 5$, without using your calculator.

(i) $3(b - a)^2$  (ii) $a + b(2a + b)$  (iii) $a + 9 \left( \frac{b}{a} \right)$  (iv) $30/(ab)$

Rounding

When you use your calculator to carry out a calculation, you often need to round the result. There are various ways to round a number. Sometimes it’s appropriate to round to a particular number of decimal places (often abbreviated to ‘d.p.’). The decimal places of a number are the positions of the digits to the right of the decimal point. You can also round to the nearest whole number, or to the nearest 10, or to the nearest 100, for example. More often, it’s appropriate to round to a particular number of significant figures (often abbreviated to ‘s.f.’ or ‘sig. figs.’). The first significant figure of a number is the first non-zero digit (from the left), the next significant figure is the next digit along (whether zero or not), and so on.

Once you’ve decided where to round a number, you need to look at the digit immediately after where you want to round. You round up if this digit is 5 or more, and round down otherwise. When you round a number, you should state how it’s been rounded, in brackets after the rounded number, as illustrated in the next example.

Notice the ‘play button’ icon next to the following example. It indicates that the example has an associated tutorial clip – a short video in which a tutor works through the example and explains it. You can watch the clip, which is available on the module website, instead of reading through the worked example. Many other examples in the module have tutorial clips, indicated by the same icon.
Example 2  
_Rounding numbers_

Round the following numbers as indicated.

(a) 0.0238 to three decimal places
(b) 50 629 to three significant figures
(c) 0.002 958 2 to two significant figures

_Solution_

(a) Look at the digit after the first three decimal places: 0.023_8_.
   It’s 8, which is 5 or more, so round up. 
   0.0238 = 0.024 (to 3 d.p.)

(b) Look at the digit after the first three significant figures:
    506_29_. It’s 2, which is less than 5, so round down.
    50629 = 50 600 (to 3 s.f.)

(c) Look at the digit after the first two significant figures:
    0.00_29582_. It’s 5, which is 5 or more, so round up.
    0.002 958 2 = 0.0030 (to 2 s.f.)

Notice that in Example 2(c), a 0 is included after the 3 to make it clear that the number is rounded to *two* significant figures. You should do likewise when you round numbers yourself.

Activity 2  
_Rounding numbers_

Round the following numbers as indicated.

(a) 41.394 to one decimal place
(b) 22.325 to three significant figures
(c) 80 014 to three significant figures
(d) 0.056 97 to two significant figures
(e) 0.006 996 to three significant figures
(f) 56 311 to the nearest hundred
(g) 72 991 to the nearest hundred
The use of the digit 0 to indicate an empty place in the representation of a number seems essential nowadays. For example, the digit 0 in 3802 distinguishes it from 382. However, many civilisations managed to use place-value representations of numbers for hundreds of years with no symbol for the digit zero. Instead, they distinguished numbers by their context. Evidence from surviving clay tablets shows that the Babylonians used place-value representations of numbers from at least 2100 BC, and used a place-holder for zero from around 600 BC.

When you need to round a negative number, you should round the part after the minus sign in the same way that you would round a positive number. For example,

\[ -0.25 = -0.3 \text{ (to 1 d.p.)} \]

When you’re rounding an answer obtained from your calculator, it’s often useful to write down a more precise version of the answer before you round it. You can do this by using the ‘...’ symbol, like this:

\[ 9.86960440... = 9.87 \text{ (to 2 d.p.)}. \]

Also, as an alternative to writing in brackets how you rounded a number, you can replace the equals sign by the symbol \( \approx \), which means, and is read as, ‘is approximately equal to’. For example, you can write

\[ 9.86960440... \approx 9.87. \]

The activities and TMA questions in this module will sometimes tell you what rounding to use in your answers. In other situations where you need to round answers, a useful rule of thumb is to round to the number of significant figures in the least precise number used in your calculation. For example, suppose that you’re asked to calculate how long it would take to travel 11 400 metres at a speed of 8.9 metres per second. The first and second numbers here seem to be given to three and two significant figures, respectively, so you would round your answer to two significant figures. Note, however, that there are situations where this rule of thumb is not appropriate. Note also that if an activity or TMA question includes a number with no units, such as ‘120’, then you should usually assume that this number is exact, whereas if it includes a number with units, such as ‘120 cm’, then you should usually assume that this is a measurement and has been rounded.

Now try the following activity. Don’t skip it: it might look easy, but it illustrates an important point about rounding – one that’s a common source of errors. To do the activity, you need to use the facts that the radius \( r \), circumference \( c \) and area \( A \) of any circle (see Figure 4) are linked by the formulas

\[ c = 2\pi r \quad \text{and} \quad A = \pi r^2. \]

Remember to use the \( \pi \) button on your calculator to obtain a good approximation for \( \pi \).
Activity 3  Rounding in a multi-stage calculation

The circumference of a circle is 77.2 cm.

(a) Find the radius of the circle, in cm to three significant figures.

(b) Find the area of the circle, in cm$^2$ to two significant figures.

The correct answer to part (b) of Activity 3 is 470 cm$^2$. If you obtained the answer 480 cm$^2$, then this was probably because you carried out your calculation in part (b) using the rounded answer for the radius that you found in part (a). To obtain the correct answer in part (b), you need to use a more precise value for the radius, such as the value that you obtained on your calculator before you rounded it. (Alternatively, you could combine the two formulas $c = 2\pi r$ and $A = \pi r^2$ to obtain the formula $A = c^2/(4\pi)$ for $A$ in terms of $c$, and use that to obtain the answer to part (b).)

Errors of this sort are known as rounding errors. To avoid them, whenever you carry out a calculation using an answer that you found earlier, you should use the full-calculator-precision version of the earlier answer. One way to do this is to write down the full value and re-enter it in your calculator, but a more convenient way is to store it in your calculator’s memory. Another convenient way to avoid rounding errors is to carry out your calculations using a computer algebra system. You’ll start to learn how to do this in Unit 2.

Sometimes people who work with numbers, such as statisticians, use slightly different rounding conventions to those described above. These alternative conventions usually differ only in how they deal with cases in which the digit immediately after where you want to round is the last non-zero digit of the unrounded number, and it’s a 5.

For example, with some conventions, $3.65 = 3.6$ (to 1 d.p.), and there are conventions for which $-0.25 = -0.2$ (to 1 d.p.).

Negative numbers

Negative numbers occur frequently in mathematics, so it’s important that you’re confident about working with them.

When you’re carrying out calculations that involve negative numbers, it’s sometimes helpful to mention the sign of a number. This is either + or −, that is, plus or minus, according to whether the number is positive or negative, respectively. The number zero doesn’t have a sign.
Here’s a reminder of how to deal with addition and subtraction when negative numbers are involved.

When you have a number (positive, negative or zero), and you want to add or subtract a positive number, you simply increase or decrease the number that you started with by the appropriate amount. For example, as shown in Figure 5, to add 3 to −5 you increase −5 by 3, and to subtract 3 from −5 you decrease −5 by 3. This gives:

\[ -5 + 3 = -2 \quad \text{and} \quad -5 - 3 = -8. \]

![Figure 5](image)

**Figure 5** Increasing −5 by 3 and decreasing −5 by 3

When you have a number (positive, negative or zero), and you want to add or subtract a negative number, you use the rules below.

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**Adding and subtracting negative numbers**

Adding a negative number is the same as subtracting the corresponding positive number.

Subtracting a negative number is the same as adding the corresponding positive number.

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For example,

\[ 7 + (-3) = 7 - 3 = 4 \quad \text{and} \quad -9 - (-3) = -9 + 3 = -6. \]

Now here’s a reminder of how to deal with multiplication and division when you’re working with negative numbers.

To multiply or divide two negative or positive numbers, you multiply or divide them without their signs in the usual way, and use the rules below to find the sign of the answer.
Multiplying and dividing negative numbers

When two numbers are multiplied or divided:
- if the signs are different, then the answer is negative
- if the signs are the same, then the answer is positive.

For example,

- $2 \times (-3) = -6$
- $\frac{-8}{-2} = 4$
- $2 \times (-3) \times (-5) = (-6) \times (-5) = 30$

Notice that some of the negative numbers in the calculations above are enclosed in brackets. This is because no two of the mathematical symbols $+, -, \times$ and $\div$ should be written next to each other, as that would look confusing (and is often meaningless). So if you want to show that you’re adding $-2$ to $4$, for example, then you shouldn’t write $4 + -2$, but instead you should put brackets around the $-2$ and write $4 + (-2)$.

If you’d like to know more about why negative numbers are added, subtracted, multiplied and divided in the way that they are, then have a look at the document Arithmetic of negative numbers on the module website.

You can practise working with negative numbers in the next activity. Remember that the BIDMAS rules apply in the usual way.

Activity 4 Working with negative numbers

Evaluate the following expressions without using a calculator.

(a) $-3 + (-4)$  
(b) $2 + (-3)$  
(c) $2 - (-3)$  
(d) $-1 - (-5)$  
(e) $5 \times (-4)$  
(f) $\frac{-15}{-3}$  
(g) $(-2) \times (-3) \times (-4)$  
(h) $6(-3 - (-1))$  
(i) $20 - (-5) \times (-2)$  
(j) $-5 + (-3) \times (-1) - 2 \times (-2)$  
(k) $\frac{-2 - (-1) \times (-2)}{-8}$

When you’re working with negative numbers, there’s an extra operation that you have to deal with, as well as the usual operations of addition, subtraction, multiplication and division. When you put a minus sign in front of a number, the new number that you get is called the negative of the original number. For example, the negative of $4$ is $-4$. The operation of putting a minus sign in front of a number is called taking the negative of the number.
You can take the negative of a number that’s already negative. This changes its sign to plus. For example,
\[-(-7) = +7 = 7.\]

To see why this is, notice that taking the negative of a positive number is the same as subtracting it from zero: for example, 
\[-3 = 0 - 3.\] It’s just the same for negative numbers: 
\[-(-7) = 0 - (-7) = 0 + 7 = 7.\]

You can also take the negative of zero. This leaves it unchanged:
\[-0 = 0 - 0 = 0.\]

In general, taking the negative of a positive or negative number changes its sign to the opposite sign. Taking the negative of zero leaves it unchanged. Another helpful way to think about negatives is that a number and its negative always add up to zero.

The operation of taking a negative has the same precedence in the BIDMAS rules as subtraction. For example, in the expression 
\[-3^2,\] the operation of taking the power is done before the operation of taking the negative, by the BIDMAS rules. So 
\[-3^2\] is equal to 
\[-9,\] not 9, as you might have expected. However, 
\[-(3)^2\] is equal to 9.

**Activity 5  Practice with taking negatives**

Evaluate the following expressions without using a calculator. Then check that your calculator gives the same answers.

(a) \(-5^2\)  
(b) \((-5)^2\)  
(c) \(-(-8)\)  
(d) \(-(-8)^2\)  
(e) \(-2^2 + 7\)  
(f) \(-(-5) - (-1)\)  
(g) \(-4^2 - (-4)^2\)  
(h) \(-3 \times (-2^2)\)

When you substitute a negative number into an algebraic expression, you usually have to enclose it in brackets, to make sure that you evaluate the expression correctly. For example, to evaluate the expression 
\[x^2 - 2x\] when 
\[x = -3,\] you proceed as follows:
\[
x^2 - 2x = (-3)^2 - 2 \times (-3) \\
= 9 - (-6) \\
= 9 + 6 = 15.
\]

**Activity 6  Substituting negative numbers into algebraic expressions**

Evaluate the following expressions when \(a = -2\) and \(b = -3\), without using a calculator.

(a) \(-b\)  
(b) \(-a - b\)  
(c) \(-b^2\)  
(d) \(a^2 + ab\)  
(e) \(\frac{3 - a^2}{b}\)  
(f) \(a^2 - 2a + 5\)  
(g) \((6 - a)(2 + b)\)  
(h) \(a^3\)  
(i) \(-b^3\)
Units of measurement

Most of the units of measurement used in this module come from the standard metric system known as the Système International d’Unités (SI units). This system is used by the scientific community generally and is the main system of measurement in nearly every country in the world.

There are seven SI base units, from which all the other units are derived. The base units (and their abbreviations) used in this module are the metre (m), the kilogram (kg) and the second (s). Prefixes are used to indicate smaller or larger units. Some common prefixes are shown in Table 1.

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Abbreviation</th>
<th>Meaning</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>nano</td>
<td>n</td>
<td>a billionth ( \left( \frac{1}{1\text{,}000} \right) )</td>
<td>1 nanometre (nm) = ( \frac{1}{1\text{,}000} ) metre</td>
</tr>
<tr>
<td>micro</td>
<td>µ</td>
<td>a millionth ( \left( \frac{1}{1\text{,}000} \right) )</td>
<td>1 micrometre (µm) = ( \frac{1}{1\text{,}000} ) metre</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>a thousandth ( \left( \frac{1}{1000} \right) )</td>
<td>1 millimetre (mm) = ( \frac{1}{1000} ) metre</td>
</tr>
<tr>
<td>centi</td>
<td>c</td>
<td>a hundredth ( \left( \frac{1}{100} \right) )</td>
<td>1 centimetre (cm) = ( \frac{1}{100} ) metre</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>a thousand (1000)</td>
<td>1 kilometre (km) = 1000 metres</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>a million ( (10^6) )</td>
<td>1 megametre (Mm) = 10^6 metres</td>
</tr>
</tbody>
</table>

The metric system was founded in France after the French Revolution. Its modern form, the SI, was founded in 1960 and continues to evolve, keeping pace with the increasing precision of measurement. At the time of writing, only three countries have chosen not to adopt SI units as their sole/primary system of measurement.

Four mathematical words

Finally in this subsection, here’s a reminder of four standard mathematical words that are used frequently throughout the module.

- The sum of two or more numbers is the result of adding them.
- The product of two or more numbers is the result of multiplying them.
- A difference of two numbers is the result of subtracting one from the other.
- A quotient of two numbers is the result of dividing one by the other.

Each pair of numbers has two differences and (provided neither of the numbers in the pair is zero) two quotients. For example, the numbers 2
and 8 have the two differences $8 - 2 = 6$ and $2 - 8 = -6$, and the two quotients $\frac{8}{2} = 4$ and $\frac{2}{8} = \frac{1}{4}$. When we say the difference of a pair of positive numbers, we mean their positive difference. For example, the difference of 2 and 8 is 6.

### 1.3 Integers

In this subsection, you’ll revise some properties of the integers

$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

If an integer $a$ divides exactly into another integer $b$, then we say that

- $b$ is a **multiple** of $a$, or
- $b$ is **divisible** by $a$, or
- $a$ is a **factor** or **divisor** of $b$.

For example, 15 is a multiple of 5; also 15 is divisible by 5; and 5 is a factor of 15. Similarly, $-15$ is a multiple of 5, and so on.

Notice that every integer is both a multiple and a factor of itself: for example, 5 is a multiple of 5, and 5 is a factor of 5.

A **factor pair** of an integer is a pair of its factors that multiply together to give the integer. For example, the factor pairs of 12 are

$1, 12; 2, 6; 3, 4; -1, -12; -2, -6; -3, -4$;

and the factor pairs of $-4$ are

$1, -4; 2, -2; -1, 4; -2, 2$.

(The order of the two numbers within a factor pair doesn’t matter – for example, the factor pair 1, 12 is the same as the factor pair 12, 1.)

A **positive factor pair** of a positive integer is a factor pair in which both factors are positive. For example, the positive factor pairs of 12 are

$1, 12; 2, 6; 3, 4$.

An integer that has a factor pair in which the same factor is repeated – in other words, an integer that can be written in the form $a^2$ for some integer $a$ – is called a **square number** or a **perfect square**. For example, 9 is a square number, since $9 = 3^2$. Here are the first few square numbers.

### The square numbers up to $15^2$

$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225$
You can use the following strategy to help you find all the positive factor pairs (and hence all the positive factors) of a positive integer.

**Strategy:**
To find the positive factor pairs of a positive integer
- Try dividing the integer by each of the numbers 1, 2, 3, 4, ... in turn. Whenever you find a factor, write it down along with the other factor in the factor pair.
- Stop when you get a factor pair that you have already.

**Activity 7  Finding factor pairs of integers**

(a) For each of the following positive integers, find all its positive factor pairs and then list all its positive factors in increasing order.

(i) 28  (ii) 25  (iii) 36

(b) Use your answer to part (a)(i) to find all the factor pairs of the following integers.

(i) 28  (ii) −28

An integer greater than 1 whose only factors are 1 and itself is called a prime number, or just a **prime**. For example, 3 is a prime number because its only factors are 1 and 3, but 4 is not a prime number because its factors are 1, 2 and 4. Note that 1 is not a prime number.

Here are the first 25 prime numbers.

**The prime numbers under 100**


An integer greater than 1 that isn’t a prime number is called a **composite number**. The first ten composite numbers are

4, 6, 8, 9, 10, 12, 14, 15, 16, 18.
You can break down any composite number into a product of prime factors. For example, here’s how you can break down the composite number 504:

\[
504 = 2 \times 252 \\
= 2 \times 2 \times 126 \\
= 2 \times 2 \times 2 \times 63 \\
= 2 \times 2 \times 2 \times 3 \times 21 \\
= 2 \times 2 \times 2 \times 3 \times 3 \times 7.
\]

The product of prime numbers in the last line of this working is the *only* product of prime numbers that’s equal to 504 (except that you can change the order of the primes in the product, of course: for example, \(504 = 2 \times 7 \times 2 \times 2 \times 3 \times 3\)). In general, the following important fact holds.

**The fundamental theorem of arithmetic**

Every integer greater than 1 can be written as a product of prime factors in just one way (except that you can change the order of the factors).

The **prime factorisation** of an integer greater than 1 is any expression that shows it written as a product of prime factors. For example, the working above shows that the prime factorisation of 504 is

\[
504 = 2^3 \times 3^2 \times 7.
\]

Here the prime factors are written using index notation, with the prime factors in increasing order. This is the usual way that prime factorisations are written.

Notice that the fundamental theorem of arithmetic applies to all integers greater than 1, not just composite numbers. The prime factorisation of a prime number is just the prime number itself (a ‘product of one prime’!).

The strategy that was used above to find the prime factorisation of 504 is summarised below. You can use it to find the prime factorisation of any integer greater than 1 (though for a large integer it can take a long time).
Strategy:
To find the prime factorisation of an integer greater than 1

- Repeatedly ‘factor out’ the prime 2 until you obtain a number that isn’t divisible by 2.
- Repeatedly factor out the prime 3 until you obtain a number that isn’t divisible by 3.
- Repeatedly factor out the prime 5 until you obtain a number that isn’t divisible by 5.

Continue this process with each of the successive primes, 2, 3, 5, 7, 11, 13, 17, ..., in turn. Stop when you have a product of primes.

Write out the prime factorisation with the prime factors in increasing order, using index notation for any repeated factors.

Your calculator may be able to find prime factorisations of integers, but it’s useful to know how to do this yourself, to improve your understanding of numbers and algebra.

Activity 8  Finding prime factorisations

Find the prime factorisations of the following integers, without using your calculator.
(a) 594  (b) 525  (c) 221  (d) 223

We finish this section with a reminder about common multiples and common factors.

A common multiple of two or more integers is a number that is a multiple of all of them. For example, the common multiples of 4 and 6 are \ldots, −36, −24, −12, 0, 12, 24, 36, \ldots.

The lowest (or least) common multiple (LCM) of two or more integers is the smallest positive integer that is a multiple of all of them. For example, the lowest common multiple of 4 and 6 is 12.

Similarly, a common factor of two or more integers is a number that is a factor of all of them. For example, the common factors of 24 and 30 are −6, −3, −2, −1, 1, 2, 3 and 6.

The highest common factor (HCF) (or greatest common divisor (GCD)) of two or more integers is the largest positive number that is a factor of all of them. For example, the highest common factor of 24 and 30 is 6.

Notice that the lowest common multiple and highest common factor of two or more integers are the same as the lowest common multiple and highest common factor of the corresponding positive integers. For example, the
lowest common multiple of $-4$ and $6$ is the same as the lowest common multiple of $4$ and $6$, which is $12$. So you only ever need to find the lowest common multiple and the highest common factor of two or more positive integers.

You can often find the lowest common multiple or highest common factor of two or more fairly small positive integers simply by thinking about their common multiples or common factors. For larger integers, or more tricky cases, you can use their prime factorisations, as illustrated in the next example.

---

**Example 3** Using prime factorisations to find LCMs and HCFs

Find the lowest common multiple and highest common factor of $594$ and $693$.

**Solution**

1. Find and write out the prime factorisations, lining up each prime with the same prime in the other factorisation(s), where possible.

   $594 = 2 \times 3^3 \times 11$

   $693 = 3^2 \times 7 \times 11$

2. To find the lowest common multiple, identify the highest power of the prime in each column, and multiply all these numbers together.

   The LCM of $594$ and $693$ is

   $2 \times 3^3 \times 7 \times 11 = 4158$.

3. To find the highest common factor, identify the lowest power of the prime in each column, and multiply all these numbers together. (Omit any primes, such as 2 and 7 here, that are missing from one or more rows.)

   The HCF of $594$ and $693$ is

   $3^2 \times 11 = 99$.

---

The methods used in Example 3 can be summarised as follows.
Strategy: To find the lowest common multiple or highest common factor of two or more integers greater than 1

- Find the prime factorisations of the numbers.
- To find the LCM, multiply together the highest power of each prime factor occurring in any of the numbers.
- To find the HCF, multiply together the lowest power of each prime factor common to all the numbers.

Activity 9 Using prime factorisations to find LCMs and HCFs

Find the prime factorisations of 9, 18 and 24, and use them to find the lowest common multiple and highest common factor of each of the following sets of numbers.

(a) 18 and 24  (b) 9, 18 and 24  (c) −18 and −24

There’s an efficient method, known as Euclid’s algorithm, for finding lowest common multiples and highest common factors, without having to find prime factorisations first. You can learn about it in the module Essential Mathematics 2 (MST125).

2 Algebraic expressions

In this section you’ll revise some basic skills that you need when working with algebraic expressions.

2.1 Algebraic terminology

In mathematics, an expression is an arrangement of letters, numbers and/or mathematical symbols (such as +, −, ×, ÷, brackets, and so on), which is such that if numbers are substituted for any letters present, then you can work out the value of the arrangement. So, for example, \(3x + 4\) is an expression, but \(3x + ÷4\) isn’t, because ‘+ ÷’ doesn’t make sense.
An expression that contains letters (usually as well as numbers and mathematical symbols) is an **algebraic expression**. An expression, such as $10 - 2 \times 4$, that contains only numbers and mathematical symbols is a **numerical expression**. An expression that’s part of a larger expression, and can be enclosed in brackets without affecting the meaning of the larger expression, is called a **subexpression** of the larger expression. For example, the expression $3x$ is a subexpression of the expression $3x + 4$, but $x + 4$ is not, because $3(x + 4)$ has a different meaning.

Letters representing numbers in algebraic expressions can be any of three different types. A letter may be a **variable**: this means that it represents any number, or any number of a particular kind, such as any positive number, or any integer. Alternatively, it may be an **unknown**: this means that it represents a particular number that you don’t know, but usually you want to discover, perhaps by solving an equation. Or it may represent a **constant**, a particular number whose value is specified, or regarded as unchanging for a particular calculation.

For example, as you saw earlier, there’s a mathematical constant, whose value is approximately 2.72, that’s denoted by the letter $e$. The word ‘variable’ is sometimes used as a catch-all for both variables and unknowns. When you’re working with algebraic expressions, you don’t usually need to think about which types of letter they contain – you treat all letters that represent numbers in a similar way.

As you know, multiplication signs are usually omitted in algebraic expressions (unless they are between two numbers) – things that are multiplied are just written next to each other instead. For example, $3 \times x$ is written as $3x$. Similarly, division signs are not normally used – fraction notation is used instead. For example, $3 \div x$ is usually written as

$$\frac{3}{x},$$

or as $3/x$, within a line of text. However, sometimes it’s helpful to include multiplication signs or division signs in algebraic expressions.

You use equals signs when you’re working with expressions, but expressions don’t **contain** equals signs. For example, the statement

$$x + 2x = 3x$$

isn’t an expression – it’s an equation. An **equation** is made up of **two** expressions, with an equals sign between them.
The equals sign was introduced by the Tudor mathematician Robert Recorde. It first appeared in his algebra book *The Whetstone of Witte* of 1577, where he described his invention as a useful abbreviation: ‘And to avoide the tedioufe repetition of these woordes: is equalle to: I will fette as I doe often in woorke vfe, a paire of paralleles, or Gemowe [twin] lines of onel enghte, thus: =, bicaufe noe .2. thynges can be more equalle.’ This statement is immediately followed by the first equations to be written using his new notation, which are reproduced in Figure 8.

If an expression is a list of quantities that are all added or subtracted, then it’s often helpful to think of it as a list of quantities that are all *added*. For example, the expression

\[ 5ab - 2a^2 + 3b + 6\sqrt{a} - 4 \]  

means the same as

\[ 5ab + (-2a^2) + 3b + 6\sqrt{a} + (-4). \]

The quantities that are added are called the **terms** of the expression. For example, the terms of expression (2) are

\[ 5ab, \quad -2a^2, \quad 3b, \quad 6\sqrt{a} \quad \text{and} \quad -4. \]

A term like \(-4\), that doesn’t contain any variables, is known as a **constant term** (because its value doesn’t change when the values of the variables in the expression are changed). On the other hand, if a term has the form

\[ \text{a fixed value} \times \text{a combination of variables}, \]

then the fixed value is called the **coefficient** of the term, and we say that the term is a term *in* whatever the combination of variables is. For example, in expression (2),

- \(5ab\) has coefficient 5 and is a term in \(ab\);
- \(-2a^2\) has coefficient \(-2\) and is a term in \(a^2\);
- \(3b\) has coefficient 3 and is a term in \(b\);
- \(6\sqrt{a}\) has coefficient 6 and is a term in \(\sqrt{a}\);
- \(-4\) is a constant term.

It’s possible for a term to have coefficient 1 or \(-1\). For example, in the expression \(x^2 - x\), the two terms \(x^2\) and \(-x\) have coefficients 1 and \(-1\), respectively. This is because these terms can also be written as \(1x^2\) and \(-1x\), though normally we wouldn’t write them like that, as the forms \(x^2\) and \(-x\) are simpler. It’s also possible for a coefficient to include constants. For example, in the expression \(8r^3 - \frac{4}{3}\pi r^3\) the term \(-\frac{4}{3}\pi r^3\) is a term in \(r^3\) and its coefficient is \(-\frac{4}{3}\pi\).

Sometimes two different expressions ‘mean the same thing’. For example, \(x + x\) and \(2x\) are two ways of saying that there are ‘two lots of \(x\’\). Expressions like these, that mean the same, are said to be **equivalent**.
More precisely, we say that two expressions are **equivalent**, or different **forms** of the same expression, if they have the same value as each other whatever values are chosen for their variables. We usually indicate this by writing an equals sign between them. (In some texts, the symbol \(\equiv\) is used instead.) We also say that either expression can be **rearranged**, **manipulated** or **rewritten** to give the other. **Simplifying** an expression means rewriting it as a simpler expression.

One way to change an expression into an equivalent expression is to change the order of the terms. This doesn’t change the meaning of the expression, because the order in which you add quantities doesn’t affect the overall result. For example, the expressions

\[5ab - 2a^2 + 3b + 6\sqrt{a} - 4 \quad \text{and} \quad -2a^2 - 4 + 5ab + 6\sqrt{a} + 3b\]

are equivalent, because they are each the sum of the same five terms

\[5ab, \quad -2a^2, \quad 3b, \quad 6\sqrt{a} \quad \text{and} \quad -4.\]

### 2.2 Simplifying algebraic expressions

When you’re working with an algebraic expression, you should usually try to write it in as simple a form as you can. One way in which some expressions can be simplified is by **collecting like terms**. If two or more terms of an expression differ only in the value of their coefficients, then you can combine them into a single term by adding the coefficients. For example:

\[3h + 9h - h \quad \text{can be simplified to} \quad 11h;\]

\[a^2 - 3ab - 5a^2 + ab \quad \text{can be simplified to} \quad -4a^2 - 2ab.\]

You can also often simplify the individual terms in an expression. For example, the term

\[ab(-a) \quad \text{can be simplified to} \quad -a^2b.\]

Usually a term in an expression consists of a product of numbers and letters (and possibly other items like square roots). If such a term isn’t in its simplest form, then you can simplify it by using the following strategy.

**Strategy:**

**To simplify a term**

1. Find the overall sign and write it at the front.
2. Simplify the rest of the coefficient and write it next.
3. Write any remaining parts of the term in an appropriate order; for example, letters are usually ordered alphabetically. Use index notation to avoid writing letters (or other items) more than once.

In Step 1 of the strategy, you find the overall sign of the term by using the following rules, which you saw for numbers on page 16.
When multiplying or dividing:
  two signs the same give a plus sign;
  two different signs give a minus sign.

Remember that a plus sign or a minus sign at the start of a term has the
same effect as multiplying the term by 1 or $-1$, respectively, so you can
apply the rule in the box above to such signs.

---

**Example 4  Simplifying single terms**

Simplify the following single-term expressions.

(a) $+(-2x)$  (b) $-(-7y)$  (c) $-3a^3b \times (-2a^2b)$
(d) $-c \times (-c) \times d \times (-d)$  (e) $(2x)^3$

**Solution**

(a) $\bullet$ A positive times a negative gives a negative, so the overall
    sign is minus. $\bullet$
    $+(−2x) = −2x$

(b) $\bullet$ A negative times a negative gives a positive, so the overall
    sign is plus. $\bullet$
    $−(−7y) = +7y = 7y$

(c) $\bullet$ First find the overall sign. A negative times a negative gives a
    positive, so the overall sign is plus. The rest of the coefficient is
    $3 \times 2 = 6$. The last part of the term is $a^3b \times a^2b = a^5b^2$. $\bullet$
    $−3a^3b \times (-2a^2b) = +6a^5b^2 = 6a^5b^2$

(d) $\bullet$ First find the overall sign. A negative times a negative gives a
    positive, then this positive times a positive gives a positive, then
    this positive times a negative gives a negative. So the overall sign
    is minus. The rest of the term is $c \times c \times d \times (-d) = c^2d^2$. $\bullet$
    $−c \times (-c) \times d \times (-d) = −c^2d^2$

(e) $\bullet$ Use the fact that the cube of a number is three copies of the
    number multiplied together. Then proceed in the same way as in
    the earlier parts. $\bullet$
    $(2x)^3 = (2x) \times (2x) \times (2x) = 8x^3$

Parts (a) and (b) of Example 4 illustrate the following rules, which you
saw for numbers on page 15.
• Adding the negative of something is the same as subtracting the something.
• Subtracting the negative of something is the same as adding the something.

Notice also that in Example 4(c) the final power of $a$ was found by calculating $a^3 \times a^2 = a^5$. That’s because

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5.$$

You can see that, in general, for any natural numbers $m$ and $n$, we have $a^m \times a^n = a^{m+n}$. This is an example of an index law. There’s much more about index laws in Subsection 4.3.

**Activity 10  Simplifying single terms**

Simplify the following single-term expressions.

(a) $-(-uv)$  (b) $+(-9p)$  (c) $-(-4r^2)$  (d) $-(8z)$
(e) $2x^2y^2 \times 5xy^4$  (f) $-P(-PQ)$  (g) $5m \times (-\frac{2}{5}n)$
(h) $(-a^3)(-2b^3)(-2a^3)$  (i) $(cd)^2$  (j) $(-3x)^2$  (k) $-(3x)^2$
(l) $-(-3x)^2$  (m) $(-2x)^3$  (n) $-(-2x)^3$

You can simplify an expression that has more than one term by using the strategy below. In the first step you have to identify where each term begins and ends. You can do that by scanning through the expression from left to right – each time you come across a plus or minus sign that isn’t inside brackets, that’s the start of the next term.

**Strategy:**

**To simplify an expression with more than one term**

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn’t inside brackets.
2. Simplify each term, using the strategy on page 27. Include the sign (plus or minus) at the start of each term (except the first term, of course, if it has a plus sign).
3. Collect any like terms.
Example 5  *Simplifying an expression with more than one term*

Simplify the expression $3x \times 2x - 4y(-3x) - 10x^2$.

**Solution**

1. First identify the terms: you might find it helpful to mark them in the way shown below. Then simplify each term individually. Finally, collect any like terms.

   
   
   \[
   \begin{align*}
   3x \times 2x - 4y(-3x) - 10x^2 &= 6x^2 + 12xy - 10x^2 \\
   &= 12xy - 4x^2
   \end{align*}
   \]

2. The answer could also be written as $-4x^2 + 12xy$, but $12xy - 4x^2$ is slightly shorter and tidier.

**Activity 11  *Simplifying expressions with more than one term***

Simplify the following expressions.

(a) $3a \times 3b - 2b \times 3b$
(b) $5x \times 8x - 3x(-3x)$
(c) $3x^2 - (-3y^2) + (-x^2) + (2y^2)$
(d) $-3cd + (-5c \times 2d^2) - (-cd^2)$
(e) $-6p(-\frac{1}{2}p) + (-5p \times p) - 2(-\frac{1}{2}p^2)$
(f) $A(-B) + (-AB) - (-AB) + (-A)(-B)$

2.3  *Multiplying out brackets*

In this subsection you’ll revise how to rewrite an expression that contains a pair of brackets, such as

\[3b(1 + 2a),\]

as an expression that doesn’t contain brackets. This process is called multiplying out the brackets, expanding the brackets, or simply removing the brackets. The new form of the expression, with no brackets, is called the expansion of the original expression.

In an expression like the one above, the subexpression that multiplies the brackets is called the multiplier. For example, the multiplier in the expression above is $3b$. The basic rule for multiplying out brackets is as follows.
Strategy: To multiply out brackets
Multiply each term inside the brackets by the multiplier.

For example,
\[3b(1 + 2a) = 3b \times 1 + 3b \times 2a = 3b + 6ab.\]

Remember that you have to multiply each term inside the brackets by the multiplier. Multiplying only the first term is a common mistake!

When you multiply out brackets, it’s usually best not to write down an expression that contains multiplication signs, and then simplify it, as was done above. Instead, you should simplify the terms in your head as you multiply out. This leads to tidier expressions and fewer errors, and it’s particularly helpful when minus signs are involved. Here’s an example.

**Example 6**  Multiplying out brackets

Multiply out the brackets in the expression

\[-x(2x - y + 5).\]

**Solution**

The first term in the expanded form of the expression is \(-x \times 2x\). Simplify this in your head (using the strategy of first finding the overall sign, then the rest of the coefficient, then the letters) and write it down. Do likewise for the other terms, which are \(-x \times (-y)\) and \(-x \times 5.\)

\[-x(2x - y + 5) = -2x^2 + xy - 5x\]

When you multiply out brackets, it doesn’t matter whether the multiplier is before or after the brackets. Here’s an example of multiplying out where the multiplier is after the brackets:

\[(5g - 1)h = 5gh - h.\]
Activity 12  Multiplying out brackets

Multiply out the brackets in the following expressions.
(a)  $a(a^4 + b)$  (b)  $-x(6x - x^2)$  (c)  $3pq(2p + 3q - 1)$
(d)  $(C^3 - C^2 - C)C^2$  (e)  $-\frac{1}{2}x\left(\frac{1}{3}x^2 + \frac{2}{3}x\right)$

Sometimes you have to remove brackets that have just a minus or plus sign in front, such as

$$-(p + 5q - s) \quad \text{or} \quad + (a - b).$$

You do this by using the following strategy.

**Strategy:**

To remove brackets with a plus or minus sign in front

- If the sign is plus, keep the sign of each term inside the brackets the same.
- If the sign is minus, change the sign of each term inside the brackets.

For example,

$$-(p + 5q - s) = -p - 5q + s \quad \text{and} \quad + (a - b) = a - b.$$

The strategy comes from the fact that a minus sign in front of brackets is just the same as multiplying by $-1$, and a plus sign in front is just the same as multiplying by $1$.

Activity 13  Plus and minus signs in front of brackets

Remove the brackets in the following expressions.
(a)  $-(-2x^2 + x - 1)$  (b)  $+(2x - 3y + z)$  (c)  $-(p - 2q)$

Notice that, for any expressions $A$ and $B$,

$$-(A - B) = -A + B = B - A.$$

So we have the following useful fact.

For any expressions $A$ and $B$,

the negative of $A - B$ is $B - A$. 

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It’s helpful to remember this fact. For example, it tells you immediately that
\[-(x - y) = y - x \quad \text{and} \quad -(n - 3n^2) = 3n^2 - n,
\]
and so on.

The strategy for multiplying out brackets leads to two further useful facts. You’ve seen how to use the strategy to write, for example,
\[(a - b + 2)c = ac - bc + 2c.\]

In effect this says that when you want to multiply the expression \(a - b + 2\) by \(c\), you simply multiply each of its terms by \(c\) to obtain \(ac - bc + 2c\).

Similarly, since dividing by \(c\) is the same as multiplying by \(1/c\),
\[
\frac{a - b + 2}{c} = \frac{1}{c}(a - b + 2) = \frac{1}{c} \times a - \frac{1}{c} \times b + \frac{1}{c} \times 2 = \frac{a}{c} - \frac{b}{c} + \frac{2}{c}.
\]

So when you want to divide the expression \(a - b + 2\) by \(c\), you simply divide each of its terms by \(c\) to obtain the result \((a/c) - (b/c) + (2/c)\).

In general, you can use the following facts.

---

**Multiplying an expression that contains several terms by a second expression** is the same as multiplying each term of the first expression by the second expression.

Similarly, dividing an expression that contains several terms by a second expression is the same as dividing each term of the first expression by the second expression.

---

**Brackets in expressions with more than one term**

Sometimes you have to multiply out brackets in an expression that contains more than one term. For example, the expression
\[2c(c + d) + 5c^2 - d(c - d)\]
has three terms, two of which contain brackets, as follows:
\[2c(c + d) + 5c^2 - d(c - d).
\]

If you multiply out the brackets in both the first and last terms, then you obtain a new expression with five terms. You can then collect any like terms. The approach is summarised in the following strategy.
**Strategy:**
To multiply out brackets in an expression with more than one term

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn’t inside brackets.
2. Multiply out the brackets in each term. Include the sign (plus or minus) at the start of each resulting term.
3. Collect any like terms.

---

**Example 7  Multiplying out brackets when there’s more than one term**

Multiply out the brackets in the expression

\[ 2c(c + d) + 5c^2 - d(c - d), \]

and simplify your answer.

**Solution**

Identify the terms; you might find it helpful to mark them. Multiply out the brackets in each term individually to obtain a new expression with five terms. Finally, collect any like terms.

\[
\frac{2c(c + d)}{c + 5c^2} - d(c - d) = 2c^2 + 2cd + 5c^2 - cd + d^2
\]

\[
= 7c^2 + cd + d^2
\]

---

**Activity 14  Multiplying out brackets when there’s more than one term**

Multiply out the brackets in the following expressions, simplifying where possible.

(a) \( x + x^2(1 + 3x) \)  
(b) \( 7ab - b(a + 2b) \)  
(c) \( -6(c + d) + 3(c - d) \)  
(d) \( 2X - 5Y(-4X + 2Y) \)  
(e) \( (1 - p^4)p + p^2 - p \)
2 Algebraic expressions

Multiplying out two pairs of brackets

Some expressions, such as the one in Example 8 below, contain two pairs of brackets multiplied together. You can multiply out brackets like these by choosing one of the two bracketed expressions to be the multiplier, and multiplying out the other pair of brackets in the usual way. This gives you an expression with several terms each containing a pair of brackets, which again you can multiply out in the usual way.

It doesn’t matter which of the two bracketed expressions you choose to be the multiplier to start with, but it’s probably slightly easier to go for the second one, as illustrated in the next example.

Example 8  Multiplying out two pairs of brackets

Multiply out the brackets in the expression

\[(x - xy + 3y)(x - 2y),\]

and simplify your answer.

Solution

1. Take the second bracketed expression to be the multiplier.
2. Multiply each term in the first pair of brackets by this multiplier.

\[(x - xy + 3y)(x - 2y) = x(x - 2y) - xy(x - 2y) + 3y(x - 2y)\]

3. Multiply out the brackets in each term, then collect any like terms.

\[= x^2 - 2xy - x^2 y + 2xy^2 + 3xy - 6y^2\]
\[= x^2 + xy - x^2 y + 2xy^2 - 6y^2\]

Activity 15  Multiplying out two pairs of brackets

Use the method described above to multiply out the brackets in the following expressions. Simplify your answers where possible.

(a) \((a + b)(c + d + e)\)  (b) \((x + 3)(x + 5)\)
(c) \((x^2 - 2x + 3)(3x^2 - x - 1)\)

You can see from Example 8 and Activity 15 that the effect of multiplying out two pairs of brackets is that each term in the first pair of brackets is multiplied by each term in the second pair of brackets, and the resulting terms are added. So, for example, if there are three terms in the first pair of brackets and two terms in the second pair (as there are in Example 8,
for instance), then altogether there will be $3 \times 2 = 6$ multiplications, which gives six terms in the multiplied-out expression before any like terms are collected.

You can use this fact to give you an alternative method for multiplying out two pairs of brackets, though you have to be careful not to miss out any of the multiplications! It’s a good way to multiply out two pairs of brackets that each contain only two terms. In this case there are only four multiplications to be done, and you can use the acronym FOIL (first terms, outer terms, inner terms, last terms) to help you remember them. Here’s an example.

**Example 9  Using FOIL to multiply out brackets**

Multiply out the brackets in the expression

$$(x + 2)(3x - 5),$$

and simplify your answer.

**Solution**

$$\begin{align*}
(x + 2)(3x - 5) &= 3x^2 - 5x + 6x - 10 \\
&= 3x^2 + x - 10.
\end{align*}$$

You can practise using FOIL in the next activity.

**Activity 16  Multiplying out brackets containing two terms each**

Multiply out the brackets in the following expressions, and simplify your answers.

(a) $(x + 5)(x - 7)$  
(b) $(x - 3)(x - 1)$  
(c) $(2x - 1)(8x + 3)$  
(d) $(2 - 5x)(x - 9)$  
(e) $(c - 2d)(1 + c)$  
(f) $(A - B)(2A - 3B^2)$  
(g) $(a - 1)(a + 1)$  
(h) $(2 + 3x)(2 - 3x)$  
(i) $x(1 + x) + (x - 1)(2 - x)$

Make sure that you’ve done parts (g) and (h) of Activity 16, in particular, as they illustrate a useful fact that’s discussed next.
**Differences of two squares**

In each of parts (g) and (h) of Activity 16, you’ll have noticed that the product of the inner terms and the product of the outer terms added to zero. This happens whenever you multiply out an expression of the form

\[(A + B)(A - B),\]

where \(A\) and \(B\) are subexpressions. In fact, as you can check by multiplying out the brackets, the following holds.

---

**Difference of two squares**

For any expressions \(A\) and \(B\),

\[(A + B)(A - B) = A^2 - B^2.\]

---

Any expression that has the form of the right-hand side of the equation in the box above is known as a **difference of two squares**. The fact in the box will be useful in Subsection 4.2, and also in Unit 2.

**Squared brackets**

Sometimes you have to multiply out squared brackets, such as \((x - 5)^2\). You can do this by first writing the squared brackets as two pairs of brackets multiplied together, and then multiplying them out in the usual way. For example,

\[(x - 5)^2 = (x - 5)(x - 5) = x^2 - 5x - 5x + 25 = x^2 - 10x + 25.\]

Notice that \((x - 5)^2\) is *not* equal to \(x^2 - 5^2\).

---

**Activity 17  Multiplying out squared brackets**

Multiply out the brackets in the following expressions, and simplify your answers.

(a) \((x + 1)^2\)  
(b) \((3x - 2)^2\)  
(c) \((2p + 3q)^2\)

---

A quick way to multiply out squared brackets that contain two terms, like those in Activity 17, is to use the fact that they all follow a similar pattern.
To see this pattern, notice what happens when you multiply out the expressions \((A + B)^2\) and \((A - B)^2\):

\[
(A + B)^2 = (A + B)(A + B) = A^2 + AB + AB + B^2 = A^2 + 2AB + B^2,
\]

\[
\]

So the following useful facts hold.

---

**Squaring brackets**

For any expressions \(A\) and \(B\),

\[
(A + B)^2 = A^2 + 2AB + B^2, \quad \text{and} \quad (A - B)^2 = A^2 - 2AB + B^2.
\]

---

Here’s an example of how you can use these facts to quickly multiply out squared brackets.

---

**Example 10  Multiplying out squared brackets efficiently**

Multiply out the brackets in the following expressions, and simplify your answers.

(a) \((x + 3y)^2\)  \quad (b) \((4h - 1)^2\)

**Solution**

(a) The answer is the square of \(x\), plus twice the product of \(x\) and \(3y\), plus the square of \(3y\).

\[
(x + 3y)^2 = x^2 + 2 \times x \times (3y) + (3y)^2 = x^2 + 6xy + 9y^2
\]

(b) The answer is the square of \(4h\), minus twice the product of \(4h\) and \(1\), plus the square of \(1\).

\[
(4h - 1)^2 = (4h)^2 - 2 \times (4h) \times 1 + 1^2 = 16h^2 - 8h + 1
\]
Activity 18  Multiplying out squared brackets, again

Use the facts in the box above to multiply out the brackets in the following expressions, and simplify your answers.

(a) \((x + 6)^2\)  
(b) \((x - 2)^2\)  
(c) \((1 + m)^2\)  
(d) \((1 - 2u)^2\)  
(e) \((2x - 3)^2\)  
(f) \((3c + d)^2\)

Simplest forms of expressions

As you know, you should usually write expressions in the simplest way you can. For example, you should write

\[ x + 2x + 3x \text{ as } 6x. \]

The second form of this expression is clearly simpler than the first:

- it’s shorter and easier to understand, and
- it’s easier to evaluate for any particular value of \(x\).

These are the attributes to aim for when you try to write an expression in a simpler way.

However, sometimes it’s not so clear that one way of writing an expression is better than another. For example,

\[ x(x + 1) \text{ is equivalent to } x^2 + x. \]

Both of these forms are reasonably short, and both are reasonably easy to evaluate. So this expression doesn’t have a simplest form.

The same is true of many other expressions. You should try to write each expression that you work with in a reasonably simple way, but often there’s no ‘right answer’ for the simplest form. One form might be better for some purposes, and a different form might be better for other purposes.

In particular, multiplying out the brackets in an expression doesn’t always simplify it. You should multiply out brackets only if you think that this is likely to make the expression simpler, or if you think that it will help you with the problem that you’re working on.

3  Algebraic factors, multiples and fractions

In this section you’ll revise what are meant by factors and multiples of algebraic expressions, how to take out common factors and how to work with algebraic fractions.
3.1 Factors and multiples of algebraic expressions

Algebraic expressions have *factors* and *multiples* in a similar way to integers, though these words are often used a little more loosely for algebraic expressions than they are for integers. Roughly speaking, if an algebraic expression can be written in the form

\[ \text{something} \times \text{something}, \]

then both ‘somethings’ are *factors* of the expression, and the expression is a *multiple* of both ‘somethings’. For example, the equation

\[ a^2b = a \times ab \]

shows that both \( a \) and \( ab \) are factors of \( a^2b \), and it also shows that \( a^2b \) is a multiple of both \( a \) and \( ab \).

Every algebraic expression is both a factor and a multiple of itself. For example, the equation

\[ a^2b = 1 \times a^2b \]

shows that \( a^2b \) is both a factor and a multiple of itself.

Two or more algebraic expressions also have *common factors* and *common multiples*, in a similar way to two or more integers.

As you’d expect, a *common factor* of two or more algebraic expressions is an expression that’s a factor of all of them. For example, the expression \( a \) is a common factor of the two expressions

\[ a^2b \text{ and } abc, \]

because

\[ a^2b = a \times ab \text{ and } abc = a \times bc. \]

Similarly, a *common multiple* of two or more algebraic expressions is an expression that’s a multiple of all of them. For example, the expression \( abcd \) is a common multiple of the two expressions

\[ ab \text{ and } bc, \]

because

\[ abcd = ab \times cd \text{ and } abcd = bc \times ad. \]
Activity 19  Checking common factors and common multiples

Show that:
(a) (i) $2a$ is a common factor of $4a^2$ and $2ab$;
 (ii) $x^2$ is a common factor of $x^3$ and $x^5$;
 (iii) $6z$ is a common factor of $18z^2$, $6z^2$ and $6z$;
(b) (i) $10cd^2$ is a common multiple of $5c$ and $2cd$;
 (ii) $p^7$ is a common multiple of $p^2$ and $p^3$;
 (iii) $9y^2$ is a common multiple of $3$, $9y^2$ and $3y$.

Two or more algebraic expressions with integer coefficients have a *highest common factor* and a *lowest common multiple*, just as two or more integers do, though again these words are often used more loosely for algebraic expressions than they are for integers. In the context of algebraic expressions, *highest common factor* means a common factor that is a multiple of all other common factors. Similarly, *lowest common multiple* means a common multiple that is a factor of all other common multiples.

The next example shows you how to find highest common factors and lowest common multiples of algebraic expressions.

Example 11  Finding HCFs and LCMs of algebraic expressions

Consider the expressions
10$a^6$ and 15$a^8b^3$.

(a) Find the highest common factor of the expressions, and write each expression in the form

highest common factor $\times$ something.

(b) Find the lowest common multiple of the expressions, and, for each expression, write the lowest common multiple in the form

the expression $\times$ something.
Solution

(a)

First consider the coefficients. The largest integer that is a factor of both 10 and 15 (that is, their HCF) is 5.

Then consider the powers of \( a \). The largest power of \( a \) that is a factor of both \( a^6 \) and \( a^8 \) is \( a^6 \). (Note that \( a^6 \) is a factor of both \( a^6 \) and \( a^8 \) because \( a^8 = a^6 \times 1 \) and \( a^8 = a^6 \times a^2 \).)

Finally, consider the powers of \( b \). There is no power of \( b \) in \( 10a^6 \), so there is no power of \( b \) in the highest common factor.

The highest common factor of the two terms is \( 5a^6 \).

The expressions can be written as

\[
10a^6 = 5a^6 \times 2 \quad \text{and} \quad 15a^8b^3 = 5a^6 \times 3a^2b^3.
\]

(b)

First consider the coefficients. The smallest positive integer that is a multiple of both 10 and 15 (that is, their LCM) is 30.

Then consider the powers of \( a \). The smallest power of \( a \) that is a multiple of both \( a^6 \) and \( a^8 \) is \( a^8 \).

Finally, consider the powers of \( b \). The smallest power of \( b \) that is a multiple of both ‘no power of \( b \)’ and \( b^3 \) is \( b^3 \).

The lowest common multiple of the two terms is \( 30a^8b^3 \).

It can be written as

\[
30a^8b^3 = 10a^6 \times 3a^2b^3 \quad \text{and} \quad 30a^8b^3 = 15a^8b^3 \times 2.
\]

Activity 20  Finding HCFs and LCMs of algebraic expressions

(a) Consider the expressions

\[ 3x^2 \quad \text{and} \quad 9xy. \]

(i) Find the highest common factor of the expressions, and write each expression in the form

\[
\text{highest common factor} \times \text{something}.
\]
(ii) Find the lowest common multiple of the expressions, and, for each expression, write the lowest common multiple in the form the expression $\times$ something.

(b) Repeat part (a) for the expressions $6p^2q^3$, $4pq^2$ and $2pq$.

### 3.2 Taking out common factors

**Factorising** an expression means writing it as the product of two or more expressions, neither of which is 1 (and, usually, neither of which is $-1$).

If all of the terms of an expression have a common factor other than 1, then the expression can be factorised. For example, consider the expression $x^3y + xy$.

The terms of this expression, $x^3y$ and $xy$, have $xy$ as a common factor. So the expression can be written as $xy \times x^2 + xy \times 1$.

From your work on multiplying out brackets, you know that this is the same as $xy(x^2 + 1)$.

The original expression has now been factorised. We say that we have taken out the common factor $xy$. This example illustrates the thinking behind the following general strategy for taking out common factors.

**Strategy:**

To take out a common factor from an expression

1. Find a common factor of the terms (usually the highest common factor).
2. Write the common factor in front of a pair of brackets.
3. Write what’s left of each term inside the brackets.
Example 12  Taking out a common factor

Factorise the expression \(-8g^5 + 4g^2h^2 - 2g^2\).

Solution

The highest common factor of the three terms is \(2g^2\).

\[-8g^5 + 4g^2h^2 - 2g^2 = 2g^2(-4g^3 + 2h^2 - 1)\]

Notice that if you’re taking out a common factor, and it’s the same as, or the negative of, one of the terms, then ‘what’s left’ of the term is 1 or \(-1\). This is illustrated in Example 12 above, and also in the example in the text at the beginning of this subsection.

Remember that you can check whether a factorisation that you’ve carried out is correct by just multiplying out the brackets again. It’s often a good idea to carry out some sort of check on an answer that you’ve found to a mathematical problem, where this is possible, especially when you’re learning a new technique, or when you’re answering an assignment question.

If you’re trying to take out the highest common factor from an expression, then it’s worth checking the expression inside the brackets when you’ve finished, to make sure that you haven’t missed any common factors.

Activity 21  Taking out common factors

Factorise the following expressions. Take out the highest common factor in each case.

(a) \(pq + 12qr\)  
(b) \(14cd - 7cd^2\)  
(c) \(m^3 - m^7 - 8m^2\)  
(d) \(-6AB + 3A^2B - 12A^3B^2\)  
(e) \(\sqrt{T} - s\sqrt{T}\)  
(f) \(5x^2 - 10x\)  
(g) \(18y^2 + 6\)

Sometimes it’s convenient to take out a common factor with a minus sign. When you do this, ‘what’s left’ of each term will have the opposite sign to the original sign. For example, an alternative way to factorise the expression in Example 12 is

\[-8g^5 + 4g^2h^2 - 2g^2 = -2g^2(4g^3 - 2h^2 + 1).\]

You can also take out just a minus sign from an expression. For example,

\[-a - b + c = -(a + b - c).\]

It can be helpful to do these things when all or most of the terms in an expression have minus signs.
Activity 22  Taking out common factors with minus signs

Factorise each of the following expressions by taking out the negative of the highest common factor of the terms. (If the highest common factor of the terms is 1, then take out just a minus sign.)

(a) \(-2x^3 + 3x^2 - x - 5\)  (b) \(-ab - a - b\)  (c) \(5cd^2 - 10c^2d - 5cd\)

If the coefficients of the terms of an expression aren’t integers, then you can often still factorise the expression in a helpful way. For example, sometimes you can simplify an expression by taking out a fraction as a ‘common factor’, leaving terms with integer coefficients inside the brackets. To achieve this, usually the fraction that you take out must have a denominator that is a common multiple of the denominators of the coefficients. Here’s an example:

\[
\frac{1}{2}n^2 - \frac{2}{3}n + 1 = \frac{1}{6}(3n^2 - 4n + 6).
\]

Sometimes the expression in the brackets has further common factors that can also be taken out, as in Activity 23(a) and (d) below.

Activity 23  Working with non-integer coefficients

Simplify each of the following expressions by factorising them.

(a) \(\frac{1}{2}a^2 + \frac{3}{2}a\)  (b) \(\frac{1}{4}x - \frac{1}{6}\)  (c) \(2x^2 - \frac{1}{2}x + \frac{1}{4}\)  (d) \(\frac{2}{3}u^2v^2 + \frac{1}{2}u^3v\)

3.3  Algebraic fractions

As you know, in algebraic expressions, division is normally indicated by fraction notation rather than by division signs. For example, the expression \(2x \div (x + 1)\) is written as

\[
\frac{2x}{x + 1}.
\]

An algebraic expression written in the form of a fraction, such as the expression above, is called an algebraic fraction. As with a numerical fraction, the top and bottom of the algebraic fraction are called the numerator and the denominator, respectively.

When you want to write an algebraic fraction as part of a line of text, you can replace the horizontal line by a slash symbol, but remember that you may need to enclose the numerator and/or denominator in brackets to make it clear what’s divided by what. For example, you can write the algebraic fraction above as \((2x)/(x + 1)\) or as \(2x/(x + 1)\), but not as \(2x/x + 1\), which means \((2x/x) + 1\).
There are other special situations where you may find the slash notation helpful. For example, it’s convenient when you’re dealing with a fraction whose numerator and denominator are themselves fractions, or when a fraction appears in a power, such as $2^{x/5}$. However, in most situations, and especially when you’re manipulating fractions, you should use the horizontal line notation, as it’s simpler to work with and helps you avoid mistakes.

You can manipulate algebraic fractions using the same rules that you use for numerical fractions. However, there’s an extra issue that you need to take into account: an algebraic fraction is valid only for the values of the variables that make the denominator non-zero. This is because division by zero isn’t possible. For example, the algebraic fraction $2x/(x + 1)$ is valid for every value of $x$ except $-1$.

On 21 September 1997, the computer system of the US Navy cruiser USS Yorktown failed because of a division by zero error, and the ship was incapacitated.

In the rest of this subsection you’ll revise and practise manipulating algebraic fractions.

**Equivalence of algebraic fractions**

Two numerical fractions are equal or *equivalent* if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same (non-zero) number. For example,

\[
\frac{2}{10} \text{ and } \frac{1}{5} \text{ are equivalent, because } \frac{2}{10} = \frac{1 \times 2}{5 \times 2}.
\]

Similarly, two algebraic fractions are equal or *equivalent* if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same expression. For example,

\[
\frac{a(b - 2)}{b(b - 2)} \text{ and } \frac{a}{b} \text{ are equivalent.}
\]

Notice that the first of these two algebraic fractions is valid for all values of its variables except $b = 0$ and $b = 2$, whereas the second is valid for all values of its variables except just $b = 0$. So the two fractions aren’t *exactly* the same. This sort of thing doesn’t usually matter, but sometimes it does. For example, it can occasionally cause problems when you’re solving equations. There’s more about this in Subsections 5.3 and 5.4.
Simplifying algebraic fractions

A numerical fraction whose numerator and denominator have a common factor can be simplified by dividing both the numerator and the denominator by this factor. For example, 2 is a common factor of 2 and 10, so

$$\frac{2}{10} = \frac{1 \times 2}{5 \times 2} = \frac{1}{5}.$$  

You can indicate this sort of working by crossing out the numbers on the numerator and denominator of the fraction, and replacing them with the results of the divisions, like this:

$$\frac{2}{10} = \frac{\cancel{2}}{\cancel{10}} = \frac{1}{5}.$$  

In the same way, an algebraic fraction whose numerator and denominator have a common factor can be simplified by dividing both the numerator and the denominator by the factor. For example, $x$ is a common factor of $xy$ and $x^3$, so

$$\frac{xy}{x^3} = \frac{y \times x}{x^2 \times x} = \frac{y}{x^2},$$

which you can write as

$$\frac{xy}{x^3} = \frac{\cancel{x}y}{\cancel{x}^2} = \frac{y}{x^2}.$$  

Notice that this calculation uses the fact that $x^3 \div x = x^2$. This is because $x^3 = x \times x \times x$, so dividing $x^3$ by $x$ gives $x^2$. You can see that for any natural numbers $m$ and $n$ with $m$ larger than $n$, we have $a^m \div a^n = a^{m-n}$. This is another index law. It’s discussed more thoroughly in Subsection 4.3.

The process of dividing the numerator and denominator of a fraction, whether numerical or algebraic, by a common factor is known as cancelling the factor, or cancelling down the fraction.

Usually it’s best to simplify a numerical or algebraic fraction as much as possible. To do this, you need to cancel the highest common factor of the numerator and denominator. It’s often easiest to do this by cancelling in stages: first you cancel one common factor, then another, and so on, until eventually the overall effect is that you’ve cancelled the highest common factor. This is illustrated in the next example.
Example 13  Simplifying algebraic fractions

Simplify the following algebraic fractions.

(a) \( \frac{12a^2b^4}{9a^3b} \)

(b) \( \frac{x^3(x + 1)}{x(x + 1)^2} \)

Solution

(a) Divide top and bottom by 3 (the highest common factor of the coefficients), then by \( a^2 \) (the highest common factor of the powers of \( a \)), and finally by \( b \) (the highest common factor of the powers of \( b \)).

\[
\frac{12a^2b^4}{9a^3b} = \frac{4}{3a} \frac{12a^2b^4}{3a} = \frac{4}{3a} \frac{41b^3}{3a} = \frac{4b^3}{3a} \quad (b \neq 0)
\]

(b) Divide top and bottom by \( x \) (the highest common factor of the powers of \( x \)), then by \( x + 1 \) (the highest common factor of the powers of \( x + 1 \)).

\[
\frac{x^3(x + 1)}{x(x + 1)^2} = \frac{x^2}{x(x + 1)^2} = \frac{x^2}{x + 1} \quad (x \neq 0)
\]

In order to show the cancelling-down process clearly, Example 13 displays each cancellation on a fresh copy of the original fraction. Normally, however, you should be able to show all stages of the cancellation on a single copy of the fraction.

Another feature of Example 13 is that restrictions, \( b \neq 0 \) and \( x \neq 0 \), are used to indicate that the original and the final fraction in each part are equivalent only if the restriction holds. (The symbol ‘\( \neq \)’ means, and is read as, ‘is not equal to’.) Often we won’t note such restrictions explicitly, but you should keep in mind that they apply.
Activity 24  Simplifying algebraic fractions

Simplify the following algebraic fractions.

(a) \( \frac{8xy^3}{6x^2y^2} \)  (b) \( \frac{2(3x - 1)}{10(3x - 1)^3} \)  (c) \( \frac{(x - 1)^2(x - 2)}{(x - 1)(x - 2)^2} \)

As you become familiar with cancelling down algebraic fractions, you can make your working tidier by not actually crossing out factors. Instead, you can just write down the original form of the fraction, followed by an equals sign and then the simplified form, as you would with other types of algebraic simplification. If you do want to cross out factors in a fraction, perhaps because it’s a complicated one, then you should make sure that your working also includes a copy of the unsimplified fraction with nothing crossed out.

The next example illustrates that even if there aren’t any obvious common factors of the numerator and the denominator of an algebraic fraction, you may still be able to find some by factorising the numerator and/or the denominator.

Example 14  Simplifying more algebraic fractions

Simplify the following algebraic fractions.

(a) \( \frac{x^2 + 2x}{5x^3 - 3x} \)  (b) \( \frac{u^2 - 2u}{3u - 6} \)

Solution

\( \text{American} \)  Factorise the numerator and denominator to check for common factors. Cancel any common factors. \( \text{American} \)

(a) \( \frac{x^2 + 2x}{5x^3 - 3x} = \frac{x(x + 2)}{x(5x^2 - 3)} = \frac{x + 2}{5x^2 - 3} \quad (x \neq 0) \)

(b) \( \frac{u^2 - 2u}{3u - 6} = \frac{u(u - 2)}{3(u - 2)} = \frac{u}{3} \quad (u \neq 2) \)

In part (a) of Example 14, you can avoid having to write down the middle expression in the working if you notice that each term in both the numerator and denominator of the original fraction has \( x \) as a common factor. In general, if all the terms in both the numerator and the denominator of a fraction have a common factor, then you can simplify the fraction by dividing each of these terms individually by this common factor. That’s because, as stated in a box on page 33, if an expression is a
sum of several terms, then dividing it by a second expression is the same as dividing each term of the expression by the second expression.

For example, each term in both the numerator and denominator of the fraction below has \( c^2 \) as a common factor, so you can simplify it by dividing each term individually by \( c^2 \):

\[
\frac{c^4 + c^3 + c^2}{2c^3 + c^2} = \frac{c^2 + c + 1}{2c + 1}.
\]

**Activity 25  Simplifying more algebraic fractions**

Simplify the following algebraic fractions.

(a) \( \frac{ab + a}{a^2 + a} \)  
(b) \( \frac{3}{9 + 6y^2} \)  
(c) \( \frac{u^2 - u^3}{u^5} \)  
(d) \( \frac{2x^2 - 4x^3}{6x^4 - 2x^2} \)  
(e) \( \frac{x^3 + x^2}{2x + 2} \)  
(f) \( \frac{1 - n}{n^2 - n} \)

**Adding and subtracting algebraic fractions**

As you know, if the denominators of two or more numerical fractions are the same, then to add or subtract them you just add or subtract the numerators. For example,

\[
\frac{5}{7} - \frac{2}{7} = \frac{3}{7}.
\]

If the denominators are different, then before you can add or subtract the fractions, you need to write them with the same denominator. This denominator has to be a common multiple of the denominators of all the fractions to be added or subtracted. Here’s an example:

\[
\frac{5}{4} + \frac{2}{3} - \frac{1}{6} = \frac{15}{12} + \frac{8}{12} - \frac{2}{12} = \frac{21}{12} = \frac{7}{4}.
\]

In this calculation, a common multiple of the original denominators 4, 3 and 6 is 12. So each fraction was written with denominator 12, by multiplying top and bottom by an appropriate number (namely 3, 4 and 2, respectively). Then the numerators were added and subtracted in the usual way, and the resulting answer was simplified by cancelling.

You can use the same method to add or subtract algebraic fractions, as follows.
Strategy:
To add or subtract algebraic fractions
1. Make sure that the fractions have the same denominator – if necessary, rewrite each fraction as an equivalent fraction to achieve this.
2. Add or subtract the numerators.
3. Simplify the answer if possible.

Example 15  Adding and subtracting algebraic fractions
Write each of the following expressions as a single algebraic fraction.

(a) \( \frac{x + 2}{x^2} - \frac{2}{x^2} \)  (b) \( \frac{1}{ab} + \frac{2}{a} - \frac{3}{b} \)  (c) \( \frac{x}{1-x} - \frac{1}{x} \)  (d) \( \frac{5}{d} + c \)

Solution

(a) The denominators are the same, so subtract the second numerator from the first. Simplify the answer.

\[
\frac{x + 2}{x^2} - \frac{2}{x^2} = \frac{x + 2 - 2}{x^2} = \frac{x}{x^2} = \frac{1}{x}
\]

(b) First write the fractions with the same denominator, by multiplying the top and bottom of each fraction by an appropriate expression. Then add and subtract the numerators.

\[
\frac{1}{ab} + \frac{2}{a} - \frac{3}{b} = \frac{1}{ab} + \frac{2b}{ab} - \frac{3a}{ab} = \frac{1 + 2b - 3a}{ab}
\]

(c) Proceed in a similar way to part (b). Simplify the answer.

\[
\frac{x}{1-x} - \frac{1}{x} = \frac{x^2}{x(1-x)} - \frac{x}{x(1-x)} = \frac{x^2 - x}{x(1-x)} = \frac{x^2 + x - 1}{x(1-x)}
\]

(d) First write c as a fraction, then proceed as before.

\[
\frac{5}{d} + c = \frac{5}{d} + \frac{c}{1} = \frac{5}{d} + \frac{cd}{d} = \frac{5 + cd}{d}
\]
Activity 26  Adding and subtracting algebraic fractions

Write each of the following expressions as a single algebraic fraction, simplifying your answer if possible.

(a) \( \frac{1}{x} + \frac{y}{x} \)  
(b) \( \frac{c+2}{c^2+c} - \frac{1}{c^2+c} \)  
(c) \( \frac{1}{ab} - \frac{1}{bc} \)  
(d) \( \frac{2}{3a} + \frac{1}{2a} \)

(e) \( \frac{1}{x^2} - \frac{2}{x} + 3 \)  
(f) \( 5 - \frac{1}{x} + \frac{2}{y} \)  
(g) \( A - \frac{A^2 - 1}{2A} \)  
(h) \( \frac{3}{2u-1} + u \)

(i) \( \frac{x+2}{x+1} + \frac{1}{x} \)  
(j) \( \frac{2}{p+2} - \frac{1}{p-3} \)  
(k) \( \frac{x+1}{x(x-1)} + \frac{1}{x} - x \)

Expanding algebraic fractions

You saw on page 33 that if an expression is a sum of several terms, then dividing it by a second expression is the same as dividing each term of the expression by the second expression.

This means that if you have an algebraic fraction whose numerator is a sum of terms, then you can rewrite it so that each term is individually divided by the denominator. For example,

\[ \frac{2x^2 - y + 3}{x} = \frac{2x^2}{x} - \frac{y}{x} + \frac{3}{x} = 2x - \frac{y}{x} + \frac{3}{x}. \]

This procedure is known as expanding the algebraic fraction. Like expanding brackets, it’s sometimes a useful thing to do, and sometimes not. Expanding an algebraic fraction is essentially the reverse procedure to adding or subtracting algebraic fractions.

Activity 27  Expanding algebraic fractions

Expand the following algebraic fractions.

(a) \( \frac{a^2 + a^5 - 1}{a^2} \)  
(b) \( \frac{2 - 5cd}{c} \)  
(c) \( \frac{2a + 3a^2}{6} \)

Remember that an algebraic fraction can be expanded only if it has a sum of terms in the numerator. The following fraction, which has a sum of terms in the denominator, can’t be expanded:

\[ \frac{a^2}{a^2 + a^5 - 1}. \]
Multiplying and dividing algebraic fractions

To multiply numerical fractions, you multiply the numerators together and multiply the denominators together. For example,

\[
\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}.
\]

The rule for dividing numerical fractions can be conveniently described using the idea of the reciprocal of a number. A number and its reciprocal multiply together to give 1. So, for example, the reciprocal of \(\frac{2}{3}\) is \(\frac{3}{2}\), since \(\frac{2}{3} \times \frac{3}{2} = 1\). Similarly, the reciprocal of \(\frac{1}{2}\) is \(\frac{2}{1} = 2\), and the reciprocal of \(-\frac{3}{5}\) is \(-\frac{5}{3}\). Every number except 0 has a reciprocal.

Since a number and its reciprocal multiply together to give 1, another way to think about the reciprocal of a number is that it is 1 divided by the number.

To find the reciprocal of a fraction, you just swap the numerator and the denominator. For example, the reciprocal of \(\frac{3}{5}\) is \(\frac{5}{3}\), since \(\frac{3}{5} \times \frac{5}{3} = \frac{15}{15} = 1\).

The rule for dividing numerical fractions is as follows: to divide by a numerical fraction, you multiply by its reciprocal. For example,

\[
\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8}.
\]

If you don’t know why fractions are multiplied and divided using the rules described here, have a look at the document *Fraction arithmetic* on the module website.

Algebraic fractions are multiplied and divided in the same way as numerical ones.

**Strategy:**

To multiply or divide algebraic fractions

- To multiply two or more algebraic fractions, multiply the numerators together and multiply the denominators together.
- To divide by an algebraic fraction, multiply by its reciprocal.

Simplify the answer if possible.

Here’s an example of multiplying algebraic fractions:

\[
\frac{4}{7a^3} \times \frac{7a}{2b^3} = \frac{28a}{14a^3b^3} = \frac{2}{a^2b^3}.
\]

In this manipulation the two numerators were multiplied and the two denominators were multiplied, and then the answer was simplified by cancelling common factors. However, it’s often quicker and easier to cancel
any common factors before you multiply the numerators and denominators of fractions, like this:

\[
\frac{4}{7a^3} \times \frac{7a}{2b^3} = \frac{2}{a} \times \frac{1}{b} = \frac{2}{a^2b^3}.
\]

This technique is known as cross-cancelling. As with ordinary cancelling, there is no need to show every, or indeed any, stage of the cancelling process.

**Example 16  Multiplying and dividing algebraic fractions**

Simplify the following expressions.

(a) \( \frac{2x - 5}{(x - 1)^2} \times \frac{x - 1}{4} \)
(b) \( \frac{9P^2}{Q^8} \div \frac{3P^3}{Q^9} \)

**Solution**

(a) To multiply the fractions, multiply the numerators and multiply the denominators. Cross-cancel any common factors first.

\[
\frac{2x - 5}{(x - 1)^2} \times \frac{x - 1}{4} = \frac{2x - 5}{(x - 1)^2} \times \frac{1}{4} = \frac{2x - 5}{4(x - 1)}
\]

(b) To divide by a fraction, multiply by the reciprocal. Cross-cancel any common factors before doing the multiplication.

\[
\frac{9P^2}{Q^8} \div \frac{3P^3}{Q^9} = \frac{9P^2}{Q^8} \times \frac{Q^9}{3P^3} = \frac{3}{P} \times \frac{Q^9}{3P^3} = \frac{3Q}{P}
\]
Activity 28  
*Multiplying and dividing algebraic fractions*

Simplify the following expressions.

(a) \[
\frac{40A}{B} \times \frac{BC}{16A^4}
\]
(b) \[
\frac{b}{c^2} \div c^3
\]
(c) \[
\left(\frac{6y}{x^7}\right) \div \left(\frac{15y^{10}}{x^4}\right)
\]
(d) \[
\frac{a/(a+1)}{a^6/(a+1)^2}
\]
(e) \[
\frac{3x}{y} \div \frac{6}{y^2}
\]
(f) \[
g \times \frac{5}{k}
\]

4  
**Roots and powers**

In this section you’ll revise how to work with roots and powers of numbers and algebraic expressions.

4.1  
**Roots of numbers**

As you know, a *square root* of a number is a number that when squared (raised to the power 2) gives the original number. For example, both 6 and \(-6\) are square roots of 36, since

\[6^2 = 6 \times 6 = 36\] and \[(-6)^2 = (-6) \times (-6) = 36\].

Similarly, a *cube root* of a number is a number that when cubed (raised to the power 3) gives the original number. For example, 4 is a cube root of 64, and \(-4\) is a cube root of \(-64\), because

\[4^3 = 4 \times 4 \times 4 = 64\] and \[(-4)^3 = (-4) \times (-4) \times (-4) = -64\].

As you’d expect, a *fourth root* of a number is a number that when raised to the power 4 gives the original number, and *fifth roots*, *sixth roots* and so on are defined in a similar way.

Every positive number has two square roots, a positive one and a negative one. For example, as you saw above, the two square roots of 36 are 6 and \(-6\). When we say the *square root* of a positive number, we mean the positive square root. Every negative number has no real square roots – in other words, it has no square roots that are real numbers. For example, there’s no real number that when squared gives \(-36\). (You’ll see in Unit 12 that negative numbers do have ‘imaginary’ square roots.) The number 0 has one square root, namely itself.

The situation for cube roots is simpler: every real number, whether positive, negative or zero, has exactly one real cube root.

The situation for fourth roots, sixth roots and all even-numbered roots is similar to that for square roots, and the situation for fifth roots, seventh roots and all odd-numbered roots is similar to that for cube roots. That is,
the following facts hold. For every even natural number \( n \), every positive number has exactly two real \( n \)th roots (a positive one and a negative one), every negative number has no real \( n \)th roots, and the number 0 has exactly one \( n \)th root (itself). For every odd natural number \( n \), every real number has exactly one real \( n \)th root. For example,

- 64 has two real sixth roots, 2 and −2;
- −64 has no real sixth roots;
- 243 has one real fifth root, 3;
- −243 has one real fifth root, −3.

You can use the symbol ‘±’, which means ‘plus or minus’, to write a positive and a negative root together. For example, the two square roots of 36 are ±6, and the two real sixth roots of 64 are ±2.

The symbol \( \sqrt{\ } \) is used to denote the positive square root of a positive real number, or the square root of zero. For example,

\[ \sqrt{36} = 6 \quad \text{and} \quad \sqrt{0} = 0. \]

Because \( \sqrt{\ } \) means a positive or zero square root, it’s incorrect to write, for example,

‘if \( x^2 = 4 \), then \( x = \sqrt{4} = \pm 2 \).’

What you should write is

‘if \( x^2 = 4 \), then \( x = \pm \sqrt{4} = \pm 2 \).’

Similarly, the symbol \( \sqrt[3]{\ } \) denotes the cube root of a positive or zero real number, the symbol \( \sqrt[4]{\ } \) denotes the positive or zero fourth root of a positive or zero real number, and so on. For example, \( \sqrt[4]{81} = 3 \) and \( \sqrt[4]{0} = 0 \). Note in particular that the number under any of the symbols \( \sqrt{\ } \), \( \sqrt[3]{\ } \), \( \sqrt[4]{\ } \), . . . must be positive or zero, and the resulting root is always positive or zero.

Most roots of numbers are irrational numbers. In particular, the square root of any natural number that isn’t a perfect square is irrational. So, for example, \( \sqrt{2} \), \( \sqrt{3} \), \( \sqrt{5} \) and \( \sqrt{6} \) are all irrational numbers.

Because numbers like these can’t be written down exactly as fractions or terminating decimals, we often leave them just as they are in calculations and in the answers to calculations. For example, we might say that the answer to a calculation is \( 1 - 2\sqrt{5} \). The advantage of this approach is that it allows us to work with exact numbers, rather than approximations. This can help to simplify calculations.
An expression such as $1 - 2\sqrt{5}$ is called a surd. That is, a surd is a numerical expression that contains one or more irrational roots of numbers. Here are some more surds:

$$\sqrt{2}, \quad -\sqrt{2}, \quad 2 + \frac{3}{\sqrt{5}}, \quad \sqrt{6} + \sqrt{7}, \quad \frac{\sqrt{7}}{3}.$$ 

Note that in some texts, the word ‘surd’ means ‘irrational root’ rather than ‘numerical expression containing one or more irrational roots’.

Surds are usually written concisely, in a similar way to algebraic expressions. Multiplication signs are usually omitted – for example, we write $2\sqrt{5}$ rather than $2 \times \sqrt{5}$. However, sometimes it’s necessary or helpful to include multiplication signs in a surd. Also, where a number and an irrational root are multiplied together, the number is written first – for example, we write $2\sqrt{5}$ rather than $\sqrt{5}2$. This is because, for example, $\sqrt{5}2$ could easily be misread as $\sqrt{52}$.

The word ‘surd’ is derived from the same Latin word as ‘absurd’. The original Latin word is ‘surdus’, which means deaf or silent.

### 4.2 Manipulating surds

You sometimes have to manipulate surds, in a similar way to algebraic expressions. In particular, you sometimes have to simplify them. Many calculators can simplify surds, but you should also know how to do this yourself. This is because it is useful to be able to simplify straightforward surds without having to resort to a calculator, and because you’ll have to use similar methods to simplify algebraic expressions involving roots.

You can manipulate surds using the usual rules of algebra – you treat the irrational roots in the same way that you treat variables. For example, you can collect like terms. In the expression below, there are two terms with $\sqrt{3}$ in them, which can be collected, and two terms with $\sqrt{7}$ in them, which can also be collected:

$$\sqrt{3} + 2\sqrt{7} + 4\sqrt{3} - \sqrt{7} = (1 + 4)\sqrt{3} + (2 - 1)\sqrt{7}$$

$$= 5\sqrt{3} + \sqrt{7}.$$ 

There are also several ways to manipulate surds that don’t involve treating irrational roots in the same way that you treat variables. In particular, you should keep in mind that an expression of the form $(\sqrt{a})^2$ can be simplified to $a$. For example,

$$\sqrt{8}\sqrt{8} = (\sqrt{8})^2 = 8.$$ 

Two further useful rules are given below. The reason why they hold is explained in the next subsection. These rules apply to all appropriate numbers – for example, in the second rule $b$ must be non-zero, because division by zero isn’t possible.
The square root of a product of numbers is the same as the product of the square roots of the numbers, and similarly for a quotient:

\[ \sqrt{ab} = \sqrt{a} \sqrt{b}, \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \]

Analogous rules apply to cube roots, fourth roots and so on.

For example,

\[ \sqrt{5} \sqrt{3} = \sqrt{5 \times 3} = \sqrt{15} \quad \text{and} \quad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}. \]

Similarly,

\[ \frac{3\sqrt{7}}{\sqrt{2}} = \frac{3 \sqrt{7 \times 2}}{\sqrt{2}} = \frac{3 \sqrt{14}}{2} \quad \text{and} \quad \frac{\sqrt{8}}{\sqrt{2}} = \sqrt{\frac{8}{2}} = \sqrt{4}. \]

It’s not true that, in general, \( \sqrt{a + b} = \sqrt{a} + \sqrt{b} \). For example, \( \sqrt{9 + 16} = \sqrt{25} = 5 \), whereas \( \sqrt{9} + \sqrt{16} = 3 + 4 = 7 \).

**Activity 29  Simplifying surds**

Simplify the following surds.

(a) \( \sqrt{5} \sqrt{6} \)  (b) \( \frac{\sqrt{75}}{\sqrt{15}} \)  (c) \( 3 + \sqrt{10} \sqrt{10} \)  (d) \( \sqrt{8} \sqrt{2} \)  (e) \( \frac{3\sqrt{5}}{\sqrt{15}} \)

Here are some more complicated examples.

**Example 17  Manipulating surds**

In parts (a) and (b), multiply out the brackets. In part (c), write the expression as a single fraction. Simplify your answers.

(a) \( \sqrt{7}(\sqrt{7} - \sqrt{2}) \)  (b) \( (3 + \sqrt{2})(1 + 5\sqrt{2}) \)  (c) \( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \)

**Solution**

Use the usual rules of algebra, combining them with the extra rules for manipulating surds. In part (a), start by multiplying each term in the brackets by the multiplier. In part (b), start by using FOIL. In part (c), start by writing the fractions with the same denominator.
(a) \(\sqrt{7}(\sqrt{7} - \sqrt{2}) = (\sqrt{7})^2 - \sqrt{7}\sqrt{2} = 7 - \sqrt{14}\)

(b) \((3 + \sqrt{2})(1 + 5\sqrt{2}) = 3 + 15\sqrt{2} + \sqrt{2} + 5(\sqrt{2})^2 = 3 + 16\sqrt{2} + 10 = 13 + 16\sqrt{2}\)

(c) \(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{3}} + \frac{\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{3} + \sqrt{2}}{\sqrt{6}} = \frac{\sqrt{2} + \sqrt{3}}{\sqrt{6}}\)

**Activity 30  Manipulating surds**

In parts (a)–(e), multiply out the brackets. In parts (f) and (g), write the expression as a single fraction. Simplify your answers.

(a) \(\sqrt{3}(2\sqrt{2} + \sqrt{3})\)  
(b) \(\sqrt{2}(1 + \sqrt{3}) + 9\sqrt{2}\)  
(c) \((1 - \sqrt{5})(6 - 2\sqrt{5})\)  
(d) \((6 - \sqrt{7})(6 + \sqrt{7})\)  
(e) \((\sqrt{3} + 2\sqrt{2})(\sqrt{3} - 2\sqrt{2})\)  
(f) \(3 - \frac{1}{\sqrt{2}}\)  
(g) \(\frac{4}{\sqrt{5}} + \frac{\sqrt{3}}{\sqrt{2}}\)

Here’s another useful way to simplify surds. If, in an irrational square root (such as \(\sqrt{48}\) or \(\sqrt{10}\)) in a surd, the number under the square root sign has a factor that’s a perfect square, then you can simplify this square root.

To do this, you write the number under the square root sign as the product of the perfect square and another number, then you use the rule \(\sqrt{ab} = \sqrt{a}\sqrt{b}\). This is illustrated in the next example.
Example 18  Simplifying square roots in surds

Simplify the following surds, where possible.
(a) \( \sqrt{48} \)  \hspace{1cm} (b) \( \sqrt{10} \)

Solution
(a) \( \sqrt{48} \) has a factor that’s a perfect square, namely 16.
\[
\sqrt{48} = \sqrt{16 \times 3}
\]

Use the rule \( \sqrt{ab} = \sqrt{a} \sqrt{b} \).
\[
= \sqrt{16} \sqrt{3} = 4\sqrt{3}.
\]
(b) \( \sqrt{10} \) doesn’t have a factor that’s a perfect square, so it can’t be simplified.

Note that, as mentioned earlier, the first 15 perfect squares are:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225.

Activity 31  Simplifying square roots in surds

(a) Simplify the following surds, where possible.
(i) \( \sqrt{8} \)  \hspace{1cm} (ii) \( \sqrt{150} \)  \hspace{1cm} (iii) \( \sqrt{22} \)  \hspace{1cm} (iv) \( \sqrt{32} \)  \hspace{1cm} (v) \( 5 + \sqrt{108} \)
(vi) \( \sqrt{12} + \sqrt{4} \)  \hspace{1cm} (vii) \( \sqrt{27} - \sqrt{3} \)

(b) Multiply out the brackets in the following surds, and simplify your answers.
(i) \( \sqrt{6}(\sqrt{3} + \sqrt{2}) \)  \hspace{1cm} (ii) \( (\sqrt{10} - \sqrt{5})(2\sqrt{5} + 1) \)

Here’s a final way to simplify surds. If a surd contains an irrational root in the denominator of a fraction, then it’s sometimes useful to rewrite it so that the denominator no longer contains this irrational root. This is called rationalising the denominator, and it can make the surd easier to work with. It can often be achieved by multiplying the top and bottom of the fraction by a suitable surd. (As you know, multiplying the top and bottom of a fraction by the same number doesn’t change the value of the fraction.)
For example, you can rationalise the denominator of the surd $1/\sqrt{2}$ by multiplying the top and bottom by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$ 

In more complicated cases, the surd might include a fraction such as

$$\frac{5}{4\sqrt{7} + \sqrt{2}} \text{ or } \frac{\sqrt{2}}{5 - 2\sqrt{3}},$$

in which the denominator is a sum of two terms, either or both of which is a rational number multiplied by an irrational square root. You can rationalise a denominator like this by multiplying the top and bottom of the fraction by a *conjugate* of the expression in the denominator. This is the expression that you get when you change the sign of one of the two terms – it’s usual to choose the second term.

For example, the expressions in the denominators of the two fractions above have conjugates

$$4\sqrt{7} - \sqrt{2} \text{ and } 5 + 2\sqrt{3},$$

respectively.

Part (b) of the next example shows you how to rationalise a denominator by multiplying top and bottom by a conjugate of the denominator. To see why this method works, notice that when you multiply top and bottom of the fraction by the conjugate, in the denominator you obtain a product of the form

$$(A + B)(A - B),$$

where $A$ and $B$ are the terms of the denominator of the original fraction.

As you saw on page 37, when you multiply out this product you get the expression

$$A^2 - B^2.$$ 

The squaring of the terms $A$ and $B$ gets rid of the irrational square roots, as you can see in the example.
Example 19  Rationalising denominators
Rationalise the denominators of the following surds.

(a) \( \frac{1}{3\sqrt{2}} \)  (b) \( \frac{\sqrt{2}}{5 - 2\sqrt{3}} \)

Solution

(a) \( \frac{1}{3\sqrt{2}} = \frac{1}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{3 \times 2} = \frac{\sqrt{2}}{6} \)

(b) A conjugate of \( 5 - 2\sqrt{3} \) is \( 5 + 2\sqrt{3} \).
\[
\frac{\sqrt{2}}{5 - 2\sqrt{3}} = \frac{\sqrt{2}}{5 - 2\sqrt{3}} \times \frac{5 + 2\sqrt{3}}{5 + 2\sqrt{3}}
= \frac{\sqrt{2}(5 + 2\sqrt{3})}{(5 - 2\sqrt{3})(5 + 2\sqrt{3})}
= \frac{5\sqrt{2} + 2\sqrt{3}\sqrt{2}}{5^2 - (2\sqrt{3})^2}
= \frac{5\sqrt{2} + 2\sqrt{6}}{25 - 4 \times 3}
= \frac{5\sqrt{2} + 2\sqrt{6}}{13}
\]

Activity 32  Rationalising denominators
Rationalise the denominators of the following surds.

(a) \( \frac{3}{\sqrt{7}} \)  (b) \( \frac{\sqrt{2}}{\sqrt{6}} \)  (c) \( \frac{5}{\sqrt{5}} \)  (d) \( \frac{2}{1 + \sqrt{17}} \)  (e) \( \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \)
4 Roots and powers

In general, if the final answer that you’ve found to a question is a surd, then you should use the methods in this subsection to simplify it as much as possible, except that it isn’t always necessary to rationalise a denominator that contains an irrational root. For example, it’s acceptable to leave the surd $1/\sqrt{2}$ as it is. However, many calculators rationalise denominators: for example, they display $1/\sqrt{2}$ as $\sqrt{2}/2$.

4.3 Working with powers

As you know, raising a number to a power means multiplying the number by itself a specified number of times. For example, raising 2 to the power 3 gives

$$2^3 = 2 \times 2 \times 2 = 8.$$ 

Here the number 2 is called the base number or just base, and the superscript 3 is called the power, index or exponent. The word ‘power’ is also used to refer to the result of raising a number to a power – for example, we say that $2^3$, or 8, is a power of 2. These two alternative meanings of the word ‘power’ don’t normally cause any confusion, as it’s usually clear from the context which of them is meant.

Note that the plural of ‘index’, in the context here, is ‘indices’. (With some other meanings of ‘index’, such as the type of index that you get at the back of a book, the plural is ‘indexes’.)

Activity 33 Calculating powers

Evaluate the following expressions without using your calculator.

(a) $2^4$  
(b) $(-2)^4$  
(c) $-2^4$  
(d) $(-3)^3$  
(e) $(\frac{1}{2})^2$  
(f) $(\frac{1}{3})^3$

There are some useful rules that you can use to manipulate expressions that contain indices. These are known as index laws. In this subsection you’ll revise these rules, and practise using them.

We’ll start with the three most basic index laws. To see where the first of them comes from, suppose that you want to simplify the expression $a^5 \times a^2$.

You could do this as follows:

$$a^5 \times a^2 = a \times a \times a \times a \times a \times a \times a$$

5 copies of $a$  
2 copies of $a$

$$= a \times a \times a \times a \times a \times a \times a$$

7 copies of $a$

$$= a^7.$$
You can see that, in general, to multiply two powers with the same base, you add the indices. This gives the first index law in the box below.

Now suppose that you want to simplify the expression \( \frac{a^5}{a^2} \). You could do this as follows:

\[
\frac{a^5}{a^2} = \frac{\underbrace{a \times a \times a \times a \times a}}{\underbrace{a \times a}} = \underbrace{a \times a \times a} \times \underbrace{a \times a} = a \times a = a^3.
\]

You can see that, in general, to divide a power by another power with the same base, you subtract the ‘bottom’ index from the ‘top’ one. This gives the second index law in the box below.

Finally, suppose that you want to simplify the expression \( (a^2)^3 \). You could do this as follows:

\[
(a^2)^3 = a^2 \times a^2 \times a^2 = (a \times a) \times (a \times a) \times (a \times a) = a^6.
\]

You can see that, in general, to raise a power to a power you multiply the indices. This gives the third index law in the box below.

### Index laws for a single base

To multiply two powers with the same base, add the indices:

\[ a^m a^n = a^{m+n}. \]

To divide two powers with the same base, subtract the indices:

\[ \frac{a^m}{a^n} = a^{m-n}. \]

To find a power of a power, multiply the indices:

\[ (a^m)^n = a^{mn}. \]

If you think about where the first index law comes from, you’ll see that it extends to products of more than two powers of the same base. For example,

\[ a^m a^n a^r = a^{m+n+r}. \]

Another way to think about such extensions of the first index law is as two or more applications of the law. For example,

\[ a^m a^n a^r = (a^m a^n) a^r = a^{m+n} a^r = a^{m+n+r}. \]
Activity 34  Using the three basic index laws

Simplify the following expressions.

(a) \( \frac{a^{20}}{a^{5}} \)  
(b) \( (y^4)^5 \)  
(c) \( \frac{b^{4}b^{7}}{b^{3}} \)  
(d) \( p^{3}p^{5}p^{3}p \)  
(e) \( (x^{3})^{3}x^{2} \)

(f) \( (m^{3}m^{2})^{5} \)  
(g) \( \left( \frac{c^{5}}{c^{2}} \right)^{2} \)

Activity 35  Using the three basic index laws again

Multiply out the brackets or expand the fraction, as appropriate, and simplify your answers.

(a) \( (x^{3} - 1)(2x^{3} + 5) \)  
(b) \( (a^{4} + b^{4})^{2} \)  
(c) \( \frac{p^{8} + p^{2}}{p^{2}} \)

All the indices in Activities 34 and 35 are positive integers, but you can also have zero or negative indices. It might not be immediately clear to you what it means to raise a number to the power 0, or to the power \(-3\), for example, but if you think about the fact that we want the basic index laws in the box above to work for powers like these, then it soon becomes clear what they must mean.

For example, consider the power \(2^0\). If the first index law in the box above works for this power, then, for instance,

\[ 2^{3}2^{0} = 2^{3+0} = 2^{3} \]

This calculation tells you that if you multiply \(2^3\) by \(2^0\), then you get \(2^3\) again. So the value of \(2^0\) must be 1. You can see that, in general, a non-zero number raised to the power 0 must be 1. Usually we don’t give a meaning to \(0^0\), but in some contexts it’s convenient to take it to be equal to 1.

There’s no obvious, correct meaning for \(0^0\). This is because on the one hand you’d expect that if you raise a number to the power 0 then the answer will be 1, but on the other hand you’d expect that if you raise 0 to a power then the answer will be 0. Mathematicians have debated the meaning of \(0^0\) for several centuries, and in 1821 Augustin-Louis Cauchy included it in a list of undefined forms, along with expressions like \(0/0\). Cauchy was a French mathematician, one of the pioneers of mathematical analysis, the theory that underlies calculus.
Now let’s consider $2^{-3}$, for example. Again, if the first index law in the box above works for this power, then, for example,

$$2^3 2^{-3} = 2^{3+(-3)} = 2^0 = 1.$$ 

This calculation tells you that if you multiply $2^3$ by $2^{-3}$ then you get 1. So $2^{-3}$ must be the reciprocal of $2^3$; that is, $2^{-3} = 1/2^3$. You can see that, in general, a non-zero number raised to a negative index must be the reciprocal of the number raised to the corresponding positive index.

In summary, the meanings of zero and negative indices are given by the following index laws.

### More index laws for a single base

A number raised to the power 0 is 1:

$$a^0 = 1.$$ 

A number raised to a negative power is the reciprocal of the number raised to the corresponding positive power:

$$a^{-n} = \frac{1}{a^n}.$$ 

These index laws hold for all appropriate numbers. So, for example, in the second law above, $a$ can be any number except 0; it can’t be 0 because division by 0 isn’t possible.

The second index law in the box above tells you that, in particular,

$$a^{-1} = \frac{1}{a}.$$ 

So raising a number to the power $-1$ is the same as finding its reciprocal. For example,

$$2^{-1} = \frac{1}{2}, \quad \left(\frac{1}{2}\right)^{-1} = 2 \quad \text{and} \quad \left(\frac{2}{3}\right)^{-1} = \frac{3}{2}.$$ 

This fact can help you to evaluate numerical expressions that contain negative indices. For example,

$$\left(\frac{1}{2}\right)^{-3} = \left(\frac{1}{2}\right)^{-1 \times 3} = \left(\left(\frac{1}{2}\right)^{-1}\right)^3 = 2^3 = 8.$$
Activity 36  Understanding zero and negative indices

Evaluate the following powers without using your calculator.
(a) $4^{-2}$  (b) $3^{-1}$  (c) $5^0$  (d) $(\frac{2}{7})^{-1}$  (e) $(\frac{1}{3})^{-1}$  (f) $(\frac{1}{2})^{-2}$

Indices can also be fractions, or any real numbers at all – the meaning of such indices is discussed later in this subsection. All the index laws given in this subsection hold for indices and base numbers that are any real numbers (except that the numbers must be appropriate for the operations – for example, you can’t divide by zero).

The index laws that you’ve seen so far in this subsection involve just one base number, but there are also two index laws that involve two different base numbers. To see where they come from, consider the following algebraic manipulations:

$$(ab)^3 = ab 	imes ab 	imes ab = ababab = aabbba = a^3b^3,$$

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{aaa}{bbb} = \frac{a^3}{b^3}.$$ 

In general, the following rules hold.

**Index laws for two bases**

A power of a product of numbers is the same as the product of the same powers of the numbers, and similarly for a power of a quotient:

$$(ab)^n = a^n b^n,$$

$$\left(\frac{a}{b}\right)^n = a^n b^{-n}.$$ 

The first index law above extends to a product of any number of factors. For example,

$$(abc)^n = a^n b^n c^n.$$ 

You can see why this is if you think about where the law comes from, or you can think of it as more than one application of the law:

$$(abc)^n = (ab)^n c^n = a^n b^n c^n.$$ 

The next example illustrates how you can use all the index laws that you’ve seen so far in this subsection to simplify expressions that contain indices. There are several alternative ways to do this. Whichever you use, you should check that you have combined all powers of the same base. For example, you should change $a^2a^3$ to $a^5$, and $a^2/a^6$ to $1/a^4$. It’s often best for your final version to include only positive indices, especially if the expression has a denominator. You can change negative indices into
positive indices by using the rule $a^{-n} = 1/a^n$. For example, you can change $a^{-3}$ to $1/a^3$, and $1/a^{-3}$ to $a^3$.

**Example 20  Simplifying expressions containing indices**

Simplify the following expressions, ensuring that the simplified versions contain no negative indices.

(a) \( \frac{d^2}{d^{-4}} \)  
(b) \( \frac{b^{-3}}{b^2c^{-4}} \)  
(c) \( (2h^{-3}g)^2 \)

**Solution**

(a) Use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index, then use the law $a^ma^n = a^{m+n}$ to combine the powers.

\[
\frac{d^2}{d^{-4}} = d^2 \times \frac{1}{d^{-4}} = d^2 d^4 = d^6
\]

Alternatively, use the law $a^{-n}/a^n = a^{m-n}$.

\[
\frac{d^2}{d^{-4}} = d^{2-(-4)} = d^6
\]

(b) Use the law $a^{-n} = 1/a^n$ to change the negative indices into positive indices, then use the law $a^ma^n = a^{m+n}$ to combine the powers of $b$.

\[
\frac{b^{-3}}{b^2c^{-4}} = \frac{c^4}{b^4b^3} = \frac{c^4}{b^7}
\]

Alternatively, use the law $a^m/a^n = a^{m-n}$ to combine the powers of $b$, then use the law $a^{-n} = 1/a^n$ to change the negative indices into positive indices.

\[
\frac{b^{-3}}{b^2c^{-4}} = \frac{b^{-5}}{c^{-4}} = \frac{c^4}{b^5}
\]

(c) Remove the brackets by using the law $(ab)^n = a^n b^n$, then the law $(a^m)^n = a^{mn}$. Then use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index.

\[
(2h^{-3}g)^2 = 2^2(h^{-3})^2g^2 = 4h^{-6}g^2 = \frac{4g^2}{h^6}
\]

Alternatively, use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index, then use the law $(a/b)^n = a^n/b^n$, then the laws $(ab)^n = a^n b^n$ and $(a^m)^n = a^{mn}$.

\[
(2h^{-3}g)^2 = \left(\frac{2g}{h^3}\right)^2 = \frac{(2g)^2}{(h^3)^2} = \frac{2^2g^2}{h^6} = \frac{4g^2}{h^6}
\]
As mentioned above and illustrated in Example 20, when you’re simplifying an expression that contains indices, it’s often a good idea to aim for a final version that contains no negative indices. However, sometimes a final form that contains negative indices is simpler, or more useful. As with many algebraic expressions, there’s often no ‘right answer’ for the simplest form of an expression that contains indices. One form might be better for some purposes, and a different form might be better for other purposes.

**Activity 37  Simplifying expressions containing indices**

Simplify the following expressions, ensuring that the simplified versions contain no negative indices. (In parts (l), (n) and (o) you’re not expected to multiply out any brackets.)

(a) \(5g^{-1}\)  \(\frac{1}{y^{-1}}\)  \(\frac{2}{3x^{-5}}\)  \(\frac{a^{-3}}{b^{-4}}\)  \(\frac{P^2}{Q^{-5}}\)

(f) \((3h^2)^2\)  \((3h^2)^{-2}\)  \(\frac{(b^{-4})^3}{(3c)^2}\)  \(\frac{(A^{-1}B)^2}{(B^{-3})^3}\)

(j) \(\left(\frac{2y^{-1}}{x^2}\right)^5\)  \(x^{-5} \div x\)  \(\frac{(x - 1)^{-3}}{(x - 1)^2}\)  \(\left(\frac{3}{z^2}\right)^{-3}\)

(n) \(\frac{x}{(2x - 3)^3} \times x^{-2}\)  \(\left(\frac{(x + 2)^2}{x^5}\right) \div (x + 2)^{-4}\)

Now let’s consider indices that are fractions or any real numbers. To understand what’s meant by a fractional index, first consider the power \(2^{1/3}\). If the index law for raising a power to a power, \((a^m)^n = a^{mn}\), is to work for fractional indices, then, for example,

\[(2^{1/3})^3 = 2^{(1/3)\times3} = 2^1 = 2.\]

This calculation tells you that if you raise \(2^{1/3}\) to the power 3, then you get 2. So \(2^{1/3}\) must be the cube root of 2; that is, \(2^{1/3} = \sqrt[3]{2}\). In general, you can see that raising a positive number to the power \(1/n\), say, is the same as taking the \(n\)th root of the number. So \(5^{1/2} = \sqrt{5}\), and \(12^{1/4} = \sqrt[4]{12}\), and so on. This is the first rule in the following box.

To understand what’s meant by a fractional index when the numerator of the fraction isn’t 1, consider, for example, the power \(2^{5/3}\). If the index law for raising a power to a power is to work for fractional indices, then we must have

\[2^{5/3} = 2^{(1/3)\times5} = (2^{1/3})^5 = \left(\sqrt[3]{2}\right)^5.\]

So \(2^{5/3}\) must be the cube root of 2, raised to the power 5. However, by the same index law we must also have

\[2^{5/3} = 2^{5\times(1/3)} = (2^5)^{1/3} = \sqrt[3]{2^5}.\]
So $2^{5/3}$ must be the cube root of $2^5$. Luckily these two alternative definitions give the same answer, both in this case and for other fractional powers of a positive real number. This gives the second, more general rule in the following box.

**Converting between fractional indices and roots**

\[
a^{1/n} = \sqrt[n]{a} \\
a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}
\]

Notice that since the notation $\sqrt[n]{a}$ is defined only when $a$ is positive or zero, only positive numbers and zero can be raised to fractional powers. So the notation $a^n$ has no meaning if $a$ is negative and $n$ is not an integer.

**Activity 38  Converting fractional indices to root signs**

Rewrite the following expressions so that they contain root signs (such as $\sqrt{}$ and $\sqrt[3]{}$) instead of fractional indices.

Hint for parts (d), (f) and (g): first rewrite the expression to change the negative index into a positive index.

(a) $t^{1/2}$  
(b) $x^{1/3}$  
(c) $p^{2/3}$  
(d) $x^{-1/2}$  
(e) $(2x - 3)^{1/2}$  
(f) $(1 + x^2)^{-1/2}$  
(g) $(1 + x)x^{-1/2}$

Although you were asked to convert several different fractional indices to root signs in Activity 38, it’s often best to avoid using any root signs other than the square root sign $\sqrt{}$ in algebraic expressions. This is because the small numbers in root signs such as $\sqrt[3]{\cdot}$ and $\sqrt[4]{\cdot}$ can be easily misread, especially when they’re handwritten. For example,

\[p\sqrt[3]{q}\] could be misread as $p^3\sqrt[3]{q}$,

so it is better written as $pq^{1/3}$. 
Activity 39  Converting root signs to fractional indices

Rewrite the following expressions so that they contain fractional indices instead of root signs.

(a) \( \sqrt[4]{y} \)  \hspace{1cm} (b) \( \sqrt[5]{1-2x} \)  \hspace{1cm} (c) \( \frac{1}{\sqrt[3]{x}} \)  \hspace{1cm} (d) \( (\sqrt[5]{u})^2 \)  \hspace{1cm} (e) \( \frac{1}{\sqrt[3]{x^4}} \)  \hspace{1cm} (f) \( \sqrt[4]{(y+2)^3} \)

You should simplify an algebraic expression that contains fractional indices in the same ways as an expression that contains only integer indices – you should combine powers of the same base where possible, and so on. When you’ve done that, if the expression contains the index \( \frac{1}{2} \) or \( -\frac{1}{2} \), then you might consider rewriting it so that it contains the square root sign instead. For example, you might prefer to write

\[ x^{1/2} \text{ as } \sqrt{x}, \quad \text{and} \quad x^{-1/2} \text{ as } \frac{1}{\sqrt{x}}. \]

You should usually leave any other fractional indices as they are (except that it may be helpful to convert negative indices to positive ones). Here’s an example.

Example 21  Simplifying expressions containing fractional indices

Simplify the following expressions.

(a) \( \frac{c^{3/4}}{c^{5/4}} \)  \hspace{1cm} (b) \( (2h^{1/6})^2 \)

Solution

(a) \( \Box \). Use the index laws in the same ways as for integer indices. \( \Box \)

\[ \frac{c^{3/4}}{c^{5/4}} = c^{(3/4)-(5/4)} = c^{-2/4} = c^{-1/2} \]

\( \Box \). Usually, change the negative index to a positive one, and perhaps use a square root sign instead of the index \( \frac{1}{2} \). \( \Box \)

\[ = \frac{1}{c^{1/2}} = \frac{1}{\sqrt{c}} \]

(b) \( \Box \). Proceed as in part (a). Leave the final fractional index as it is, since it’s not \( \frac{1}{2} \) or \(-\frac{1}{2} \). \( \Box \)

\[ (2h^{1/6})^2 = 2^2(h^{1/6})^2 = 4h^{(1/6)\times2} = 4h^{1/3} \]
Activity 40  Simplifying expressions containing fractional indices

Simplify the following expressions. (In part (f) you’re not expected to multiply out any brackets.)

(a) $x^{1/3}x^{1/3}$  (b) $\frac{a}{a^{1/3}}$  (c) $\frac{a}{a^{1/2}}$  (d) $\frac{x}{x^{-1/2}}$  (e) $(2x^{1/5})^3$

(f) $\frac{(1 + x)^2}{\sqrt{1 + x}}$  (g) $\left(\frac{1}{u}\right)^{1/3}$  (h) $\frac{A^{5/2}}{A^3}$  (i) $\frac{x^{1/2}y^2}{x^{3}y^{1/2}}$  (j) $\sqrt[4]{x}$

It follows from the meaning of fractional indices that the two rules for square roots that you used in the last subsection,

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad \text{and} \quad \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}},$$

are just particular cases of the two index laws in the box on page 67. They’re these index laws with the index $n$ taken to be $\frac{1}{2}$.

So far in this subsection you’ve seen what’s meant by indices that are rational numbers. However, as mentioned earlier, an index can be any real number. For example, you can raise any positive number to the irrational index $\sqrt{2}$. A precise definition of the meaning of an irrational index is beyond the scope of this module, but the basic idea is that you can work out the value of any positive number raised to any irrational index as accurately as you like, by using as many decimal places of the decimal form of the irrational index as you like. For example, suppose that you’re interested in the value of $3^{\sqrt{2}}$. Since

$$\sqrt{2} \approx 1.41421356237309\ldots,$$

one approximation to $3^{\sqrt{2}}$ is

$$3^{1.414} = 4.72769503526853\ldots,$$

and a more accurate one is

$$3^{1.414213} = 4.72880146624114\ldots,$$

and so on. The indices here, 1.414 and 1.414213, and so on, are rational, as they are terminating decimals. It’s possible to show that the closer a rational index $r$ is to $\sqrt{2}$, the closer $3^r$ is to some fixed real number. The value of $3^{\sqrt{2}}$ is defined to be this number.

So an index can be any real number. All the index laws that you’ve met in this subsection hold for indices and base numbers that are any real numbers (except that the numbers must be appropriate for the operations – for example, you can’t divide by zero or apply $\sqrt{r}$ to a negative number). Here’s a summary of them all.
### Index laws

\[ a^m a^n = a^{m+n} \quad \frac{a^m}{a^n} = a^{m-n} \quad (a^m)^n = a^{mn} \]

\[ a^0 = 1 \quad a^{-n} = \frac{1}{a^n} \]

\[ (ab)^n = a^n b^n \quad \left(\frac{a}{b}\right)^n = a^n b^{-n} \]

\[ a^{m/n} = \sqrt[n]{a^m} \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m} \]

Here’s an activity involving indices that contain variables as well as numbers. You can deal with these using the index laws in the usual way.

### Activity 41  Simplifying indices that contain variables

Simplify the following expressions.

(a) \( a^{2p} a^{5p} \)  
(b) \( \frac{b^7}{b^4k} \)  
(c) \( (g^n)^k \)  
(d) \( (2y^{2t})^2y^{-t} \)  
(e) \( \frac{m^{3x}}{m^{-x}} \)

(f) \( \frac{c^{3y}}{c^{5y}} \)  
(g) \( \frac{(a^{-3t}b^{3t})^2}{a^{3t}b^t} \)  
(h) \( (d^{1/r})^2r \)  
(i) \( (3h)^{2p}(9h)^p \)

### Scientific notation

One use of indices is in **scientific notation** (also known as **standard form**). This is a way of writing numbers that’s particularly helpful when you’re dealing with very large or very small numbers. To express a number in scientific notation, you write it in the form

(a number between 1 and 10, but not including 10)  
× (an integer power of ten).

Here are some examples.

\[
\begin{align*}
427 &= 4.27 \times 100 &= 4.27 \times 10^2 \\
42.7 &= 4.27 \times 10 &= 4.27 \times 10^1 \\
4.27 &= 4.27 \times 1 &= 4.27 \times 10^0 \\
0.427 &= 4.27 \times 0.1 &= 4.27 \times 10^{-1} \\
0.0427 &= 4.27 \times 0.01 &= 4.27 \times 10^{-2}
\end{align*}
\]
Activity 42  Using scientific notation

(a) Express the following numbers in scientific notation.
   (i) 38,800,000  (ii) 4237  (iii) 0.0973  (iv) 1.303  
   (v) 0.000,000,028

(b) Express the following numbers in ordinary notation.
   (i) 2.8 \times 10^4  (ii) 5.975 \times 10^{-1}  (iii) 2.78 \times 10^{-7}
   (iv) 3.43 \times 10^7

In computer output, scientific notation is sometimes represented with the power of 10 indicated by the letter E (for exponent). For example, 4.27 \times 10^{-2} would be represented as 4.27E−2.

5 Equations

In this section you’ll revise how to rearrange and solve equations.

5.1 Terminology for equations

You saw earlier that to indicate that two expressions are equivalent, you put an equals sign between them. However, there’s another use of equals signs. You can place an equals sign between any two expressions, to form an equation. Here’s an example:

\[ 3(d + 1) = 7d - 5. \]  \hspace{1cm} (3)

Usually you form an equation like this when you’re interested in the values of the variables in the equation that make the equation true. These values are said to satisfy the equation, and are called solutions of the equation. For example, the value \( d = 2 \) satisfies equation (3), as shown in Example 22 below, but the equation isn’t satisfied by any other value of \( d \).

Notice that equation (3) contains the variable \( d \) and no other variable. This fact is expressed by saying that it’s an equation in \( d \). Similarly, an equation that contains the variables \( x \) and \( y \) and no other variables is an equation in \( x \) and \( y \), and so on.

The next example illustrates how you should set out your working when you’re checking whether an equation is satisfied. You should evaluate the left- and right-hand sides separately, and check whether each side gives the same answer. If you wish, you can use the abbreviations LHS and RHS for the left-hand side and right-hand side of the equation.
Example 22  Checking whether an equation is satisfied

Show that \( d = 2 \) satisfies the equation
\[ 3(d + 1) = 7d - 5. \]

Solution
If \( d = 2 \), then
LHS = \( 3(2 + 1) = 3 \times 3 = 9 \)
and
RHS = \( 7 \times 2 - 5 = 14 - 5 = 9 \).
Since LHS = RHS, \( d = 2 \) satisfies the equation.

In the next activity you’re asked to check whether another equation is satisfied for particular values of its variables. The right-hand side of this equation is just a number, so to check whether it’s satisfied you just need to evaluate the other side to see whether you get this number.

Activity 43  Checking whether an equation is satisfied

Show that the equation \( 2x + 3y = 5 \) is
(a) satisfied by \( x = 4 \) and \( y = -1 \);
(b) not satisfied by \( x = 3 \) and \( y = -2 \).

It’s traditional to use letters from near the end of the alphabet, such as \( x, y \) and \( z \), to represent unknowns, and letters from near the start of the alphabet, such as \( a, b \) and \( c \), to represent known constants. This tradition originates from René Descartes (see page 56). It’s thought that \( x \) was preferred to \( y \) and \( z \) because printers tended to have more type for the letter \( x \) than for the letters \( y \) and \( z \), due to the frequency of occurrence of these letters in the French and Latin languages.

An equation that is satisfied by all possible values of its variables is called an identity. For example, the equation
\[ x(x + y) = x^2 + xy \]
is an identity: it’s true no matter what the values of \( x \) and \( y \) are. However, in this section we’ll mostly be concerned with equations that are satisfied
by only some values of their variables. (In some texts, an identity is indicated using the symbol $\equiv$, rather than $=$.)

Sometimes the variables in an equation are restricted to numbers of a particular type. For example, if an equation contains a variable that represents the length of an object, then this variable would take only positive values.

This means that the solutions of an equation depend not only on the equation itself, but also on the possible values that its variables can take. For example, if $x$ can be any number, then the equation $x^2 = 4$ has two solutions, namely $x = 2$ and $x = -2$, but if you know that $x$ is a positive number, then the same equation has only one solution, $x = 2$.

It’s often clear from the context of an equation what type of values its variables can take. If it isn’t, then you should assume that the variables can be any real numbers.

The process of finding the solutions of an equation is called solving the equation. When we’re trying to solve an equation, the variables in the equation are often referred to as unknowns, since they represent particular numbers whose values aren’t yet known, rather than any numbers at all. The key to solving equations is the technique of rearranging them, which you’ll revise in the next subsection.

### 5.2 Rearranging equations

It’s often helpful to transform an equation into a different equation that contains the same variables, and is satisfied by the same values of these variables. This is called rearranging, manipulating or simply rewriting the equation. When you rearrange an equation, the original equation and the new one are said to be equivalent, or different forms of the same equation. Rewriting an equation as a simpler equation is called simplifying the equation.

There are three main ways to transform an equation into an equivalent equation – these are summarised below.

**Rearranging equations**

Carrying out any of the following operations on an equation gives an equivalent equation.

- Rearrange the expressions on one or both sides.
- Swap the sides.
- Do the same thing to both sides.
Here are some examples.

- The equations $y + y = 2$ and $2y = 2$ are equivalent, because the second equation is the same as the first but with the expression on the left-hand side rearranged.

- The equations $2x = 1$ and $1 = 2x$ are equivalent, because the second equation is the same as the first but with its sides swapped.

- The equations $y = 3x$ and $2y = 6x$ are equivalent, because the second equation is the same as the first but with both sides multiplied by 2.

Some of the things that you can do to both sides of an equation to obtain an equivalent equation are set out below. Note that a non-negative number is one that’s either positive or zero.

**Doing the same thing to both sides of an equation**

Doing any of the following things to both sides of an equation gives an equivalent equation.

- Add something.
- Subtract something.
- Multiply by something (provided that it is non-zero).
- Divide by something (provided that it is non-zero).
- Raise to a power (provided that the power is non-zero, and that the expressions on each side of the equation can take only non-negative values).

To understand why doing any of the things in the box above to both sides of an equation gives an equivalent equation, consider, for example, adding 2 to each side of the equation

$$2x - 3 = 9 - x.$$  

The value $x = 4$ satisfies the original equation, because with this value of $x$, each side of the equation is equal to 5. Adding 2 to both sides of the equation gives the new equation

$$2x - 3 + 2 = 9 - x + 2,$$

and the value $x = 4$ also satisfies this new equation, because with this value of $x$, each side of the new equation is equal to $5 + 2$, that is, 7. Similarly, the value $x = 3$ doesn’t satisfy the original equation, because it gives LHS = 3 and RHS = 6, and this value doesn’t satisfy the new equation either, because it gives LHS = 3 + 2 = 5 and RHS = 6 + 2 = 8. In general, doing any of the things in the box above to both sides of an equation doesn’t change the values of the variables that satisfy the equation.

Notice that there are some restrictions given in brackets in the box above. For example, the restriction on the third item in the box tells you that multiplying both sides of an equation by something is guaranteed to give an equivalent equation only if the something is non-zero. To see why this
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restriction is needed, notice that if you multiply both sides of any equation by zero, then you obtain the equation \( 0 = 0 \), which will usually not be equivalent to the original equation. There’s a more detailed discussion of the restrictions in the next subsection.

There are ‘shortcuts’ for adding or subtracting something on both sides of an equation, and for multiplying or dividing both sides, which can be useful in particular, common situations. These are described below. Note that some tutors recommend that it’s best *not* to use the first shortcut, ‘change the side, change the sign’, since it can easily lead to mistakes, as detailed below.

**Change the side, change the sign**

Suppose, for example, that you have the equation

\[
7x = 3 - 2y
\]  
(4)

and you want to rewrite it as an equation in which all the variables appear on the left-hand side only. You can achieve this by adding the term \( 2y \) to each side. This gives the equivalent equation

\[
7x + 2y = 3 - 2y + 2y,
\]

which can be simplified to

\[
7x + 2y = 3.
\]  
(5)

If you compare equations (4) and (5), then you can see that the overall effect on the original equation is that the term \( 2y \) has been moved to the other side, and its sign has been changed. In general, the technique of adding or subtracting on both sides of an equation leads to the following rule.

Moving a term of one side of an equation to the other side, and changing its sign, gives an equivalent equation.

This rule is sometimes summarised as ‘change the side, change the sign’. Some people find it useful, while others find it just as convenient to stick with thinking about adding or subtracting on both sides. If you do use the rule, then you need to do so carefully, as it can easily lead to mistakes of the kinds explained next.

When you use the rule, you must make sure that the term you’re moving is a term *of the whole expression* on one side of the equation, not just a term of a subexpression. For example, the equations

\[
3(a + b + c) = d \quad \text{and} \quad 3(a + b) = d - c
\]

are *not* equivalent, because the term that’s been moved, \( c \), is not a term of the whole expression on the left-hand side, but only of the subexpression \( a + b + c \). (The first equation *is* equivalent to \( 3(a + b) = d - 3c \), which is obtained by moving the term \( 3c \) that’s obtained when you multiply out the left-hand side of the original equation.)
You must also make sure that the term you’re moving becomes a term of the whole expression on the other side, not just a term of a subexpression. For example, the equations

\[ a + b = \frac{c + d}{2} \quad \text{and} \quad a = \frac{c + d - b}{2} \]

are not equivalent, because the term that’s been moved, \( b \), has not become a term of the whole expression on the right-hand side, but only of the subexpression \( c + d - b \). (The first equation is equivalent to \( a = \frac{c + d}{2} - b \)).

If you find that you tend to make these types of errors, then it might be better to avoid using the ‘change the side, change the sign’ rule, and instead think about adding or subtracting on both sides.

**Activity 44 Adding or subtracting on both sides correctly**

Which of the following are pairs of equivalent equations?

(a) \( y = \sqrt{x - 3} \) and \( y + 3 = \sqrt{x} \)
(b) \( u^2 = \frac{1}{2}v + 7 \) and \( u^2 - 7 = \frac{1}{2}v \)
(c) \( \frac{x^2 + x}{3} = \frac{x + 9}{2} \) and \( \frac{x^2}{3} = \frac{9}{2} \)

**Multiplying or dividing through**

Consider the equation

\[ a + b = c + d + e, \]

and suppose that you want to multiply both sides by 3. You know (by the box on page 33) that multiplying the left-hand side by 3 is the same as multiplying each individual term of the left-hand side by 3, and the same goes for the right-hand side. So multiplying both sides of the equation by 3 gives

\[ 3a + 3b = 3c + 3d + 3e. \]

You can see that the overall effect on the original equation is that each individual term on both sides has been multiplied by 3.

Similarly, if you divide both sides of the equation by 3, then (by the same box on page 33) the overall effect is that each individual term on both sides is divided by 3:

\[ \frac{a}{3} + \frac{b}{3} = \frac{c}{3} + \frac{d}{3} + \frac{e}{3}. \]

It’s useful to remember the following rules.
Multiplying each term on both sides of an equation by something (provided that it is non-zero) gives an equivalent equation.

Dividing each term on both sides of an equation by something (provided that it is non-zero) gives an equivalent equation.

Multiplying each term on both sides of an equation by something is known as **multiplying through** by the something, and similarly dividing each term on both sides of an equation by something is known as **dividing through** by the something. For example, the equation above was multiplied through by 3, and also divided through by 3.

When you use the rules in the box above, you must make sure that you’re multiplying or dividing terms of the whole expression on a side, not just terms of a subexpression. For example, multiplying the equation

\[(a + b)^2 = 5\]

through by 3 does **not** give

\[(3a + 3b)^2 = 15.\]

This is because \(a\) and \(b\) are not terms of the whole expression on the left-hand side of the first equation, but only of the subexpression \(a + b\). (Multiplying the first equation through by 3 gives \(3(a + b)^2 = 15\).)

One situation where it’s often useful to multiply through an equation is when it contains fractions. You can often obtain an equivalent equation with no fractions by multiplying through by an appropriate number or expression. For example, if you have the equation

\[\frac{x}{4} + 1 = x,\]

then you can multiply through by 4 to obtain

\[x + 4 = 4x.\]

This is known as **clearing**, or simply **removing**, the fractions in the equation. When you clear fractions in this way, you should keep in mind that multiplying an equation through by an expression is guaranteed to give an equivalent equation **only if the expression doesn’t take the value zero**. There’s more about this issue in Subsection 5.3.
Activity 45  Multiplying and dividing through correctly

Which of the following are pairs of equivalent equations?

(a) \( \frac{a}{3} = \frac{a^2}{6} - 3 \) and \( 2a = a^2 - 3 \)

(b) \( \frac{x^2 - 1}{2x} + x = \frac{1}{2x} \) and \( x^2 - 1 + x = 1 \) \( (x \neq 0) \)

(c) \( hy = x + 2 \) and \( y = \frac{x}{h} + \frac{2}{h} \) \( (h \neq 0) \)

(d) \( \sqrt{\frac{x}{2y}} = 1 \) and \( \sqrt{\frac{x}{y}} = 2 \) \( (y \neq 0) \)

Cross-multiplying

Suppose that you want to clear the fractions in the equation

\[ \frac{x}{2} = \frac{x + 1}{3}. \]

You can do this by multiplying through by 2, and then multiplying through by 3. This gives the equation

\[ 3x = 2(x + 1). \]

You can see that the overall effect on the original equation is that you have ‘multiplied diagonally across the equals sign’, like this:

\[ \frac{x}{2} \times \frac{x + 1}{3} \text{ gives } 3x = 2(x + 1). \]

This technique is called cross-multiplying. You can use it as a shortcut for multiplying through, whenever you have an equation of the form fraction = fraction. The general rule is summarised in the box below.

Cross-multiplying

If \( A, B, C \) and \( D \) are any expressions, then the equations

\[ \frac{A}{B} = \frac{C}{D} \text{ and } AD = BC \]

are equivalent (provided that \( B \) and \( D \) are never zero).

If only one side of an equation is a fraction, then you can still cross-multiply (the other side can be thought of as a fraction with denominator 1).
For example,
\[ \frac{x}{2} = x + 1 \] is equivalent to \[ x = 2(x + 1). \]

### Activity 46 Cross-multiplying correctly

Which of the following are pairs of equivalent equations?

(a) \( 3 = \frac{1}{x + 4} \) and \( 3(x + 4) = 1 \) \( (x \neq -4) \)

(b) \( b + \frac{a}{a - 3} = 6 \) and \( b + a = 6(a - 3) \) \( (a \neq 3) \)

(c) \( \frac{y}{2y - 1} = \frac{y + 1}{y} \) and \( y^2 = (y + 1)(2y - 1) \) \( (y \neq \frac{1}{2}, 0) \)

(d) \( 2 + x = \frac{9}{2x + 5} \) and \( 2 + x(2x + 5) = 9 \) \( (x \neq -\frac{5}{2}) \)

### 5.3 Solving linear equations in one unknown

In this subsection you’ll revise how to solve linear equations in one unknown. Equations of this type occur frequently, and are straightforward to solve.

A linear equation is one in which, after you’ve expanded any brackets and cleared any fractions, each term is either a constant term or a constant value times a variable. As you’d expect, a linear equation in one unknown is a linear equation that contains just one unknown.

For example,
\[ 2x - 5 = 8x \]
is a linear equation in the single unknown \( x \).

Usually, a linear equation in one unknown has exactly one solution, which can be found by using the following strategy.

#### Strategy:
**To solve a linear equation in one unknown**

Use the rules for rearranging equations to obtain successive equivalent equations. Aim to obtain an equation in which the unknown is alone on one side, with only a number on the other side. To achieve that, do the following, in order.
1. Clear any fractions and multiply out any brackets. To clear fractions, multiply through by a suitable expression.
2. Add or subtract terms on both sides to get all the terms in the unknown on one side, and all the other terms on the other side. Collect like terms.
3. Divide both sides by the coefficient of the unknown.

This strategy is illustrated in the next example.

**Example 23  Solving linear equations**

Solve the following equations.

(a) \( \frac{x}{5} - 4 = 3(4 - x) \)  
(b) \( \frac{1}{a} - 1 = \frac{1}{7a} \)

**Solution**

(a) \( \frac{x}{5} - 4 = 3(4 - x) \)

- There is a fraction with denominator 5, so multiply through by 5 to clear it.
  \( x - 20 = 15(4 - x) \)
- Multiply out the brackets.
  \( x - 20 = 60 - 15x \)
- Get all the terms in the unknown on one side, and all the other terms on the other side. Collect like terms.
  \( x + 15x = 60 + 20 \)
  \( 16x = 80 \)
- Divide both sides by 16, the coefficient of the unknown.
  \( x = 5 \)

The solution is \( x = 5 \).

- If you wish, check the answer by substituting into the original equation, as follows.
Check: if $x = 5$,

\[
LHS = \frac{5}{5} - 4 = 1 - 4 = -3,
\]

and

\[
RHS = 3(4 - 5) = 3 \times (-1) = -3.
\]

Since LHS = RHS, $x = 5$ satisfies the equation.

(b) \[
\frac{1}{a} - 1 = \frac{1}{7a}
\]

\(\therefore\) To clear the fractions, multiply through by a common multiple of the denominators, such as the lowest common multiple, $7a$. For this to be guaranteed to give an equivalent equation, you have to assume that $7a \neq 0$, that is, $a \neq 0$. \(\therefore\)

Assume that $a \neq 0$.

\[
\frac{7a}{a} - 7a = \frac{7a}{7a}
\]

\(\therefore\) Simplify, then proceed as in part (a). \(\therefore\)

\[
7 - 7a = 1
\]
\[
7 - 1 = 7a
\]
\[
6 = 7a
\]
\[
\frac{6}{7} = a
\]

The value $a = \frac{6}{7}$ satisfies the assumption $a \neq 0$, so it is the solution.

Notice that before you can multiply the equation in Example 23(b) through by $7a$ to clear the fractions, you have to make the assumption that $a$ never takes the value 0. In other words, you have to assume that the solutions of the equation are restricted to non-zero numbers. It’s fine to make this assumption, because it’s clear that $a = 0$ isn’t a solution of the equation anyway, because it makes the fractions undefined. But because the rearrangements of the equation are based on this assumption, you have to check the solution that you obtain: if you get $a = 0$, then this doesn’t count as a valid solution.

Notice also that in Example 23(b) the solution was left as $a = \frac{6}{7}$. It wasn’t converted to a rounded decimal, such as $a = 0.857$ to three decimal places. In general, in mathematics you should always try to use exact numbers, where it’s reasonably straightforward to do so. (However, it’s often helpful to give answers to practical problems as rounded decimals.)

You can practise solving linear equations in the next activity. When you’re solving a linear equation, you don’t need to rigidly follow the steps of the strategy above if you can see a better way to proceed. You can do
whatever you think will be helpful, as long as you’re following the rules for rearranging equations given in the boxes in Subsection 5.2.

**Activity 47  Solving linear equations**

Solve the following equations.

(a) \(12 - 5q = q + 3\)  
(b) \(3(x - 1) = 4(1 - x)\)  
(c) \(\frac{t - 3}{4} + 5t = 1\)  
(d) \(\frac{3}{b} = \frac{2}{3b} - \frac{1}{3}\)  
(e) \(2(A + 2) = \frac{A}{3} + 1\)  
(f) \(\frac{1}{z + 1} + \frac{1}{5(z + 1)} = 1\)

In the next activity, each of the equations is of a form that allows you to clear the fractions by cross-multiplying.

**Activity 48  Solving more linear equations**

Use cross-multiplication to help you solve the following equations.

(a) \(h + 1 = \frac{4h}{5}\)  
(b) \(\frac{2}{y} = \frac{3}{y + 2}\)  
(c) \(\frac{3}{2x - 1} = \frac{2}{3 - x}\)

Finally in this subsection, let’s have a more detailed look at why the restrictions in the box ‘Doing the same thing to both sides of an equation’ on page 77 are needed. Remember that this box sets out the things that you can do to both sides of an equation to obtain an equivalent equation, and the restrictions appear in brackets. Take a look back at the box before you read on.

The restrictions don’t normally cause problems when you’re solving linear equations in one unknown, but they can be an issue when you manipulate other types of equations.

For example, suppose that you want to solve the equation

\[x^2 = x.\]

You might think that you could simplify this equation by dividing both sides by \(x\), to give

\[x = 1.\]

However, something has gone wrong here, because the first equation has two solutions, namely \(x = 0\) and \(x = 1\), whereas the second equation has only one solution, \(x = 1\). So dividing the first equation by \(x\) has not given an equivalent equation. The problem here is that dividing by the variable \(x\) is guaranteed to give an equivalent equation only if you know that \(x\) can’t take the value 0 (this is the restriction on the fourth item in the box).
If you do know that \( x \) can’t take the value 0, then the first equation has only one solution, \( x = 1 \), and dividing the first equation by \( x \) does indeed give an equivalent second equation.

As another example, consider the simple equation
\[
x = 2.
\]
You might think that you could obtain an equivalent equation by raising both sides to the power 2 (that is, squaring both sides). This gives
\[
x^2 = 4.
\]
Again, something has gone wrong here, because when the variable \( x \) has the value \(-2\), it doesn’t satisfy the first equation, and yet it does satisfy the second equation. So raising both sides of the first equation to the power 2 has not given an equivalent equation. The problem this time is that raising both sides of an equation to a power is guaranteed to give an equivalent equation only if you know that the expression on each side of the equation can take only non-negative values (this is a restriction on the fifth item in the box).

By contrast, if you know that \( x \) is positive, then negative values of \( x \) such as \(-2\) cannot arise, and squaring both sides of the first equation does give an equivalent second equation (since \( x = -2 \) no longer counts as a solution).

Similar issues can arise when you’re solving an equation involving algebraic fractions. For example, consider the equation
\[
\frac{x^2}{x} = 0.
\]
This equation has no solutions. To see this, notice that the only way that a fraction can be equal to zero is for its numerator to be equal to zero. So the only possible solution of the equation is \( x = 0 \), but this value isn’t a solution, because it makes the fraction undefined. Now consider what happens when you cancel down the left-hand side of the equation. You obtain the equation
\[
x = 0,
\]
which has one solution, namely \( x = 0 \). So cancelling down the fraction on the left-hand side has not given an equivalent equation. This is because the fraction \( x^2/x \) is valid for all values of \( x \) except zero, whereas its cancelled-down version, \( x \), is valid for all values of \( x \), with no exceptions. The two expressions aren’t quite the same – this issue was mentioned on page 46. Essentially, the problem is that cancelling down an algebraic fraction in an equation is guaranteed to give an equivalent equation only if the factor that you cancel can’t take the value 0.
Activity 49  Thinking about the restrictions

What’s wrong with the following ‘proof’ that $1 = 2$?

Suppose that $a$ and $b$ are non-zero numbers such that $a = b$. Then we have:

\[
\begin{align*}
    a &= b \\
    a^2 &= ab & \text{(by multiplying through by $a$)} \\
    a^2 - b^2 &= ab - b^2 & \text{(by subtracting $b^2$ from both sides)} \\
    (a + b)(a - b) &= b(a - b) & \text{(by using the facts that $a^2 - b^2$ is a} \\
    & \quad \text{difference of two squares, and $ab - b^2$ has a common factor)} \\
    a + b &= b & \text{(by dividing through by $a - b$)} \\
    2b &= b & \text{(since $a = b$)} \\
    2 &= 1 & \text{(by dividing through by $b$)}
\end{align*}
\]

5.4 Making a variable the subject of an equation

If an equation contains more than one variable, and one side of the equation is just a single variable that doesn’t appear at all on the other side, then that variable is called the subject of the equation. For example, the equation

\[ a = c(c + b) \]  \hspace{1cm} (6)

has a subject, namely $a$. The subject of an equation is usually written on the left-hand side.

An equation with a subject is called a formula. We say that it’s a formula for whatever the subject is; for example, equation (6) is a formula for $a$.

The word formula is often used for the expression to which the subject is equal, as well as for the whole equation. For example, if $a$, $b$ and $c$ are related by equation (6), then we say that $c(c + b)$ is a formula for $a$.

The purpose of a formula is usually to allow you to find the value of the subject when you know the values of the other variables. For example, if $a$, $b$ and $c$ are related by equation (6), and you know that $b = 16$ and $c = 3$, then you can substitute these values of $b$ and $c$ into the equation to find the value of $a$:

\[ a = 3(3 + 16) = 3 \times 19 = 57. \]

If you have an equation relating two or more variables, then it’s often useful to rearrange it so that a particular variable becomes its subject. This isn’t always possible, but for many equations you can do it by using essentially the same method that you use to solve linear equations, treating the variable that you want to be the subject (which we’ll call the required subject) as the unknown. This method is summarised in the following strategy.
**Strategy:**

To make a variable the subject of an equation (this works for some equations but not all)

Use the rules for rearranging equations to obtain successive equivalent equations. Aim to obtain an equation in which the required subject is alone on one side. To achieve this, do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply through by a suitable expression.
2. Add or subtract terms on both sides to get all the terms containing the required subject on one side, and all the other terms on the other side. Collect like terms.
3. If more than one term contains the required subject, then take it out as a common factor.
4. Divide both sides by the expression that multiplies the required subject.

This strategy works provided that the equation is ‘linear in the required subject’. That is, its form must be such that if you replace every variable other than the required subject by a suitable number (one that doesn’t lead to multiplication or division by zero) then the result is a linear equation in the required subject.

As when you’re solving equations, you don’t need to follow the steps of the strategy rigidly if you can see a better way to proceed. You just have to make sure that you’re following the rules for rearranging equations given in the boxes in Subsection 5.2.

---

**Example 24  Making a variable the subject of an equation**

Rearrange the equation

\[ t(h - 1) = \frac{2}{t} + hr \]

to make \( h \) the subject.

**Solution**

\[ t(h - 1) = \frac{2}{t} + hr \]

Multiply through by \( t \) to clear the fraction (assume that \( t \neq 0 \)).

\[ t^2(h - 1) = 2 + thr \quad \text{(assuming } t \neq 0) \]
Multiply out the brackets.

\[ t^2h - t^2 = 2 + thr \]

Get all the terms in the required subject, \( h \), on one side, and all the other terms on the other side. Check for like terms – there are none.

\[ t^2h - thr = 2 + t^2 \]

Take the required subject, \( h \), out as a common factor.

\[ h(t^2 - tr) = 2 + t^2 \]

Divide both sides by the expression that multiplies the required subject \( h \) (assume that \( t \neq r \), to ensure that \( t^2 - tr \neq 0 \)).

\[ h = \frac{2 + t^2}{t^2 - tr} \quad \text{(assuming} \ t \neq r) \]

The working in the example above tells you that, provided the variable \( t \) does not take the value 0 or the same value as \( r \), then the initial and final equations are equivalent. It tells you nothing about what happens when \( t \) does take these values. In the next activity, you should note the assumptions that you make as you carry out the manipulations, as in the example above.

**Activity 50  Making variables the subjects of equations**

(a) Make \( m \) the subject of the equation \( am = 2a + 3m \).

(b) Make \( d \) the subject of the equation \( c = \frac{1}{3}(2 + 5d) \).

(c) Make \( X \) the subject of the equation \( \frac{Y}{X} = \frac{3Y + 2}{X + 1} \).

(d) Make \( c \) the subject of the equation \( a = \frac{b}{1 - 2c} \).

(e) Make \( B \) the subject of the equation \( A = \frac{2}{B} + AC \).

In the next example, the strategy given earlier for making a variable the subject of an equation doesn’t apply, but you can make the required variable the subject simply by raising both sides of the equation to an appropriate power, and simplifying.
Example 25   *Raising both sides of an equation to a power*

Rearrange the equation

\[ a^6 = 64bc^6 \]

to make \( a \) the subject. All the variables in this equation take only positive values.

**Solution**

\[ a^6 = 64bc^6 \]

Raise both sides to the power \( \frac{1}{6} \) (that is, take the sixth root of both sides). This is okay, because the expressions on both sides of the original equation are always positive – this follows from the fact that all the variables in the equation take only positive values.

\[ a = (64bc^6)^{\frac{1}{6}} \]

\[ a = 2b^{\frac{1}{6}}c \]

---

Activity 51   *Raising both sides of equations to powers*

In this activity, all the variables take only positive values.

(a) Make \( Q \) the subject of the equation \( Q^3 = PR^2 \).

(b) Make \( h \) the subject of the equation \( h^{\frac{1}{4}} = \frac{2k^{\frac{1}{2}}}{m} \).

(c) Make \( u \) the subject of the equation \( u^2 = v + w \).

You can often adapt the strategy given earlier for making a variable the subject of an equation to allow you to deal with equations for which the strategy doesn’t quite work. In the next example, the required subject, \( p \), is raised to the power 4 in the original equation, rather than just appearing as \( p \). The strategy doesn’t apply directly to situations like this, but you can make \( p \) the subject by first ‘making \( p^4 \) the subject’, then raising both sides to the power \( \frac{1}{4} \).
Example 26  Making a variable the subject of an equation, again

Rearrange the equation

\[ \frac{3p^4 - D}{c} = 1 \]

to make \( p \) the subject. All the variables in this equation take only positive values.

Solution

First ‘make \( p^4 \) the subject’. 

\[ \frac{3p^4 - D}{c} = 1 \]
\[ 3p^4 - D = c \]
\[ 3p^4 = c + D \]
\[ p^4 = \frac{c + D}{3} \]

Now raise both sides to the power \( \frac{1}{4} \). This is fine, because both sides are always positive – this follows from the fact that all the variables in the equation take only positive values.

\[ p = \left( \frac{c + D}{3} \right)^{1/4} \]

Activity 52  Making variables the subjects of equations, again

In this activity, all the variables take only positive values.

(a) Make \( a \) the subject of the equation \( a^2 - 2b^2 = 3c^2 \).

(b) Make \( t \) the subject of the equation \( a^2 = \frac{bt^2}{N} \).

(c) Make \( r \) the subject of the equation \( \left( \frac{r}{3s} \right)^5 = \sqrt{d} \).

(d) Make \( y \) the subject of the equation \( (4y)^{1/3} = x \).
6 Writing mathematics

An important part of studying mathematics at university level is learning how to communicate it effectively. In this section you’ll make a start on that, by learning how to write good solutions to TMA questions.

Usually you won’t be able to write down an answer to a TMA question, or part of a TMA question, immediately. You’ll need to carry out some working to find the answer, and it’s important that you include this working as part of the solution that you write out to send to your tutor. However, it’s not enough to simply write down your working. Instead, what you need to do is clearly explain how you reached your answer.

To illustrate the difference, let’s look at a sample TMA question. This particular question can be answered by using Pythagoras’ theorem, so before you see it, here’s a reminder about that.

You may remember that Pythagoras’ theorem allows you to work out the length of one of the sides of a right-angled triangle, if you know the lengths of the other two sides. The side opposite the right angle in a right-angled triangle is called the hypotenuse – this is always the longest side.

Pythagoras’ theorem

For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

For example, applying Pythagoras’ theorem to the right-angled triangle in Figure 9 gives

\[ c^2 = a^2 + b^2. \]

So if you know the side lengths \( a \) and \( b \), for example, then you can substitute them into this equation, and solve it to find the side length \( c \).

Now here’s the sample TMA question.

Question 1

A symmetrical, circular circus dais, shown below, has diameter 1.1 m at the top and 1.7 m at the bottom, and is 40 cm high. Find its slant height.

Here are two different solutions to the question.
6 Writing mathematics

Poorly-written solution

\[ 1.7 - 1.1 = 0.6 \times \frac{1}{2} \Rightarrow 0.3 \]
\[ 40 \Rightarrow 0.4 \]

Pythagoras’ theorem

\[ a^2 + b^2 = c^2 \]
\[ = 0.3^2 + 0.4^2 = c^2 \]
\[ = 0.25 = c^2 \]
\[ = c = \sqrt{0.25} = \pm 0.5 \]

Length ⇒ positive.
∴ \( c = 0.5 \)

Well-written solution

A cross-section of the dais is shown below.

The length \( l \) (in metres) shown in the diagram is given by

\[ l = \frac{1}{2}(1.7 - 1.1) = \frac{1}{2} \times 0.6 = 0.3. \]

So, by Pythagoras’ theorem, the slant height \( s \) (in metres) is given by

\[ s^2 = 0.4^2 + l^2 \]
\[ = 0.4^2 + 0.3^2 \]
\[ = 0.16 + 0.09 \]
\[ = 0.25. \]

Therefore, since \( s \) is positive,

\[ s = \sqrt{0.25} = 0.5. \]

That is, the slant height of the dais is 0.5 m.

Each of the two solutions above uses a correct method, obtains the correct answer and shows all the working. However, the author of the poorly-written solution has done little more than just jot down some working (and has also used notation incorrectly, as explained later), whereas the author of the well-written solution has written out a clear
explanation of how he or she has reached the answer. The result is that the well-written solution is much easier for a reader to understand.

Writing mathematics so that it can be easily understood by a reader is an important skill. It will be useful not only when you write TMA solutions for your tutor, but in other situations too. For example, it will be valuable if you need to write mathematical reports, or if you teach mathematics to others, or even when you make notes for yourself to be read at a later date. Also, explaining your thinking clearly can help you to deepen your understanding of the mathematics, identify any errors and remember the techniques.

Here are some things that you should try to do when you write mathematics, to make it easier for a reader to understand. The author of the well-written solution has done all of these things, whereas the author of the poorly-written solution has done few of them.

**Write in sentences.**

The well-written solution is written as a sequence of sentences. In contrast, the poorly-written solution is just a collection of fragments of English and mathematics, such as ‘Pythagoras’ theorem’ and ‘$40 \Rightarrow 0.4$’.

You should always aim to write your mathematics in sentences, though these sentences will often consist mostly of mathematical notation. However, if you’re answering a question that simply asks you to carry out a mathematical manipulation, such as rearranging an expression, then you don’t need to include any extra words.

**Explain your reasoning.**

In the well-written solution, the author has explained how each claim that he or she makes follows from something that was calculated earlier in the solution, or is given in the question, or is a known fact (such as Pythagoras’ theorem). Notice in particular how he or she has used the link words ‘so’, ‘therefore’ and ‘that is’ to indicate what follows from what. Other examples of useful link words and phrases are ‘hence’, ‘it follows that’ and ‘since’.

In your answers to TMA questions, you can use any of the facts that are stated in the *Handbook* or in the boxes in the units. You do not need to prove these facts, or state where you saw them.

**Use equals signs only between two numbers, quantities or algebraic expressions that are equal.**

In the first line of the poorly-written solution, the author has written ‘$1.7 - 1.1 = 0.6 \times \frac{1}{2}$’. This is incorrect! Whenever you write an equals sign, whatever is on the left of the sign must be equal to whatever is on the right.

There’s one situation in which it’s sometimes helpful to relax this rule, however. When you’re carrying out a calculation that includes units, it’s acceptable to omit the units until the end of the calculation.
For example, an alternative solution to the TMA question above might involve the calculation

\[ \sqrt{0.3^2 + 0.4^2} \text{m} = \sqrt{0.25} \text{m} = 0.5 \text{m}. \]

(You can see such a solution on page 98.) It’s acceptable to write this calculation as

\[ \sqrt{0.3^2 + 0.4^2} = \sqrt{0.25} = 0.5 \text{m}. \]

Although strictly it’s incorrect to write this, omitting units in this way can help to prevent calculations looking unnecessarily complicated.

**Don’t use equals signs to link equations.**

In the poorly-written solution, the author has solved the equation \(0.3^2 + 0.4^2 = c^2\) by writing down a sequence of equivalent equations, ending with \(c = \pm 0.5\). This is fine, but he or she has linked the equations in the sequence by writing equals signs between them. Don’t do this! Whenever you manipulate an equation in this way, the only equals signs should be the equals signs in the equations themselves.

You can link just two or three equivalent equations by using link words such as ‘that is’. When you want to link a longer sequence of equivalent equations, you can begin by making it clear that you’re manipulating an equation, and then just write the equivalent equations underneath each other, with no linking symbols, like this:

Solving the equation \(0.3^2 + 0.4^2 = c^2\) gives:

\[
\begin{align*}
0.3^2 + 0.4^2 &= c^2 \\
0.25 &= c^2 \\
c &= \pm \sqrt{0.25} \\
c &= \pm 0.5.
\end{align*}
\]

It’s important to appreciate that this point applies to *equations* (which contain equals signs), not *expressions* (which don’t). (The definitions of an expression and an equation are given on pages 24 and 74, respectively, and in the *Handbook.*) When you manipulate an *expression*, you *should* link the equivalent expressions with equals signs (often aligned vertically below each other). For example, you can see this done in the well-written solution, at the end of the sentence beginning ‘So, by Pythagoras’ theorem, …’.

**If you introduce new variables, explain what they are.**

The poorly-written solution includes the line \(a^2 + b^2 = c^2\), with no indication of what \(a, b\) and \(c\) represent. Don’t do this: if you introduce a variable that isn’t mentioned in the question, then explain what it represents.

If you introduce a variable that represents a physical quantity, then specify its units. For example, the author of the well-written solution has specified near the beginning that \(l\) and \(s\) are lengths *in metres*. 
Finish with a conclusion that clearly answers the question.

The poorly-written solution finishes with ‘\( c = 0.5 \)’. This is not a good conclusion, because the TMA question didn’t ask, for example, ‘What is the value of \( c \)?’. Also, it’s not clear whether the final answer, 0.5, is in metres, or centimetres, for example, as the author hasn’t specified units. The final line of the well-written solution is much more helpful: it’s a clear answer to the question that was asked, with units included.

Include enough detail, but no more than is needed.

You need to give enough detail to enable your reader to easily understand what you’ve done, but try to give no more than this. Excess detail can waste your time and your reader’s time, and it can actually make your solution harder to understand.

Sometimes you might be unsure about how much detail you need to include, and indeed as your mathematical experience grows it’s appropriate to include less detail of routine procedures, such as solving linear equations. As a general guideline, you should include enough detail to make your solution clear to a reader whose mathematical experience is about the same as yours. Another useful guideline is that the amount of detail should be similar to that in the worked examples and activity solutions in the module units. (Remember that any blue ‘thinks’ text is not part of the solution, but any other text is.) But, if in doubt, include the extra detail!

If there’s a worked example or an activity in the module that’s similar to a question that you’re answering, then it’s fine to use the format of its solution as a guide for the format of your solution. This isn’t plagiarism, as it’s the format that you’re copying, rather than the solution itself.

Make sure that the mathematical symbols you use are appropriate.

Often it’s better to use words instead. For example, the author of the poorly-written solution has written

\[
\text{Length } \Rightarrow \text{ positive.}
\]

It would be clearer, and more mathematically correct, to write something like

The variable \( c \) represents a length, so its value must be positive.

Similarly, rather than writing

\[
40 \Rightarrow 0.4,
\]

it would be better to write, for example,

The height of the dais is \( 40\,\text{cm} = 0.4\,\text{m} \).

The symbol \( \Rightarrow \), which means ‘implies’, is often used incorrectly, and it’s probably best not to use it at all in this module. It isn’t used in any of the materials provided (except here!). You can learn about its correct use in the module Essential mathematics 2 (MST125).
The symbol ∴ means ‘therefore’ and can be useful when you’re short of time, such as in an examination, or in rough or informal work. Usually, however, it’s better to use a word, such as ‘so’, ‘hence’ or ‘therefore’, as this makes your mathematical writing more pleasant to read. The symbol ∴ isn’t much used in university-level mathematics.

Display larger pieces of mathematical notation on separate lines.

Notice how the author of the well-written solution has done this. For example, the part of the solution immediately below the diagram would be less easy to read if it were written like this:

\[
l = \frac{1}{2}(1.7 - 1.1) = \frac{1}{2} \times 0.6 = 0.3.
\]

So, by Pythagoras’ theorem, ... 

Include the digit 0 in numbers such as 0.3.

In the poorly-written solution, the numbers 0.3 and 0.4 are written as .3 and .4 (on the diagram). Don’t do this. Whenever you write a decimal point, there should always be a digit on each side of it. This is because, for example, .3 could easily be read as 3 by mistake.

Make sure that diagrams are clear and helpful.

If you include a diagram – and it’s often useful to do so, even if the question doesn’t explicitly ask for one – then make sure that it’s neat, clearly labelled and not too cluttered, and that it’s clear how it relates to the question. It’s a good idea to use a ruler to draw straight lines. There’s usually no need to use graph paper for diagrams or graphs.

A geometric diagram, such as the one in the well-written solution, doesn’t need to be an exact scaled-down version of the situation that it represents – an approximate representation is fine.

Any well-written TMA solution will comply with the points above, but there’s no single ‘correct way’ to write a solution to a TMA question. For example, the following alternative solution to the TMA question on page 92 looks quite different from the well-written solution you saw on page 93, but it is just as acceptable.

Notice in particular that the final sentence of this solution consists of some explanation, some mathematical working and a clear final answer, including units, all within a single sentence. This can be appropriate for a straightforward calculation.
**Alternative well-written solution**

The radius of the top of the dais is
\[ \frac{1}{2} \times 1.1 \text{ m} = 0.55 \text{ m}, \]
and the radius of the bottom is
\[ \frac{1}{2} \times 1.7 \text{ m} = 0.85 \text{ m}. \]

The diagram on the right shows half of a cross-section of the dais. The base of the right-angled triangle shown is
\[ 0.85 \text{ m} - 0.55 \text{ m} = 0.3 \text{ m}. \]

Hence, by Pythagoras’ theorem, the slant height of the dais is
\[ \sqrt{0.3^2 + 0.4^2} \text{ m} = \sqrt{0.25} \text{ m} = 0.5 \text{ m}. \]

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**Activity 53  Improving a TMA solution**

Consider the following TMA question and poorly-written solution.

**Question 2**

A cold frame is constructed by vertically fixing a pane of glass that is 1.15 m high at a distance of 98 cm from a wall, and then fixing another pane of glass of the same height at an angle between the top of the first pane and the wall, as shown. (Two trapezium-shaped sides are also attached.) Find the height of the cold frame where it meets the wall, to the nearest centimetre.
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Poorly-written solution

\[ h^2 + 98^2 = 1.15^2 \]
\[ \Rightarrow h^2 + 98^2 = 1.15^2 \]
\[ \Rightarrow h^2 = 1.15^2 - 98^2 = 0.3621 \]
\[ \Rightarrow h = \sqrt{0.3621} = 0.6 \]
\[ \therefore h = 0.6 + 1.15 \Rightarrow 1.75 \]

(a) Criticise the solution: list the features that contribute to its being poorly-written and difficult to follow, and describe how it could be improved.

(b) Write out a better solution.

In many of the TMAs for this module, a few marks are allocated for ‘good mathematical communication’. The two well-written solutions that you’ve seen in this subsection (on pages 93 and 98) would merit full marks for their mathematical communication, as well as full marks for their mathematics. On the other hand, although the poorly-written solutions that you’ve seen (on page 93 and in Activity 53) would achieve some marks for their mathematics, they wouldn’t be awarded any marks for mathematical communication. They would also lose an additional mark or half-mark because their final answers don’t include units and hence are incomplete.

When you’re working on a TMA question, you might find it helpful to first write out a rough solution, and then work on a better version to send to your tutor. This may be especially helpful near the start of the module, while you’re learning the basics of mathematical communication. It’s also helpful to read through your TMA solutions after you’ve written them, to try to judge for yourself how easy they are to follow. Then you can improve them where this seems appropriate. If you have time, try to leave them aside for a few days, or even longer, between writing them and reading them through.

You’ll find that the feedback that your tutor provides on your TMA solutions will help you to improve your mathematical writing as you progress through the module.

You might be wondering whether you should handwrite your TMA solutions, or type them. The answer is that it’s your choice: either is just as acceptable as the other. Remember, though, that you’re expected to handwrite your answers in many mathematics exams, so you might find it useful to practise in preparation for that. Another thing to think about is that it can take time to learn to type mathematics, and it can take time to check typed mathematics for typing errors. If you’re short of time, then it would be better to concentrate on learning and practising the mathematics in the module rather than typing your TMA solutions.
If you do plan to type your TMA solutions, then you should use an equation editor to format most of the mathematics. The ordinary text features of a word processor aren’t adequate for the mathematics that you’ll learn in this module. The module *Essential mathematics 2* (MST125) teaches you how to typeset mathematics, using your choice from three different typesetting programs, so if your study programme includes this module, then you may wish to delay learning to type mathematics until you take it. Alternatively, you can access the MST125 teaching materials on this topic via the MST124 website.

Another thing that you might be wondering about is how you should write a solution to a mathematics question in a written examination. The short answer is that you should try to use many of the same writing skills that you use for TMA questions, but you need to adapt them so that you can get the questions done as quickly as possible. For example, you might use abbreviated forms of sentences, and the ‘therefore’ symbol.

## Learning outcomes

After studying this unit, you should be able to:

- work fluently and accurately with different types of numbers, including negative numbers
- understand how to avoid some common types of errors, such as rounding errors, errors arising from incorrect use of the BIDMAS rules and errors in algebraic manipulation
- manipulate and simplify algebraic expressions fluently and accurately, including those involving brackets, algebraic fractions and indices
- manipulate and simplify surds
- solve linear equations, including those involving algebraic fractions
- rearrange equations fluently and accurately
- appreciate some principles of writing mathematics, and begin to apply them.
Solutions to activities

Solution to Activity 1

(a) (i) \[23 - 2 \times 3 + (4 - 2)\] 
\[= 23 - 2 \times 3 + 2\] 
\[= 23 - 6 + 2\] 
\[= 19\]

(ii) \[2 - \frac{1}{2} \times 4 = 2 - 2 = 0\]

(iii) \[4 \times 3^2 = 4 \times 9 = 36\]

(iv) \[2 + 2^2 = 2 + 4 = 6\]

(v) \[\frac{1 + 2}{1 + 3^2} = \frac{3}{1 + 9} = \frac{3}{10}\]

(vi) \[1 - 2/3^2 = 1 - 2/9 = 7/9\]

(b) (i) \[3(b - a)^2 = 3(5 - 3)^2\] 
\[= 3 \times 2^2 = 3 \times 4 = 12\]

(ii) \[a + b(2a + b) = 3 + 5(2 \times 3 + 5)\] 
\[= 3 + 5(6 + 5)\] 
\[= 3 + 5 \times 11\] 
\[= 3 + 55\] 
\[= 58\]

(iii) \[a + 9\left(\frac{b}{a}\right) = 3 + 9 \times \frac{5}{3}\] 
\[= 3 + 15\] 
\[= 18\]

(iv) \[30/(ab) = 30/(3 \times 5)\] 
\[= 30/15\] 
\[= 2\]

Solution to Activity 2

(a) \[41.394 = 41.4\] (to 1 d.p.)

(b) \[22.325 = 22.3\] (to 3 s.f.)

(c) \[80.014 = 80.000\] (to 3 s.f.)

(d) \[0.05697 = 0.057\] (to 2 s.f.)

(e) \[0.006996 = 0.00700\] (to 3 s.f.)

(f) \[56.311 = 56.300\] (to the nearest 100)

(g) \[72.991 = 73.000\] (to the nearest 100)

Solution to Activity 3

(a) Let the radius of the circle be \(r\) (in cm). Then \[77.2 = 2\pi r,\] 
so \[r = \frac{77.2}{2\pi} = 12.28676160\ldots = 12.3\] (to 3 s.f.). That is, the radius of the circle is 12.3 cm (to 3 s.f.).

(b) Let the area of the circle be \(A\) (in cm\(^2\)). Then \[A = \pi r^2\] 
\[= \pi \times (12.28676160\ldots)^2\] 
\[= 474.268998\ldots\] 
\[= 470\] (to 2 s.f.). That is, the area of the circle is 470 cm\(^2\) (to 2 s.f.).

Solution to Activity 4

(a) \[-3 + (-4) = -3 - 4 = -7\]

(b) \[2 + (-3) = 2 - 3 = -1\]

(c) \[2 - (-3) = 2 + 3 = 5\]

(d) \[-1 - (-5) = -1 + 5 = 4\]

(e) \[5 \times (-4) = -20\]

(f) \[\frac{-15}{-3} = 5\]

(g) \[(-2) \times (-3) \times (-4) = 6 \times (-4) = -24\]

(h) \[6(-3 - (-1)) = 6(-3 + 1) = 6 \times (-2) = -12\]

(i) \[20 - (-5) \times (-2) = 20 - 10 = 10\]

(j) \[-5 + (-3) \times (-1) - 2 \times (-2) = -5 + 3 + 4 = 2\]

(k) \[\frac{-2 - (-1) \times (-2)}{-8} = \frac{-2 - 2}{-8} = \frac{-4}{-8} = \frac{1}{2}\]
Solution to Activity 5
(a) \(-5^2 = -25\)
(b) \((-5)^2 = 25\)
(c) \(-(-8) = 8\)
(d) \(-(-8)^2 = 64\)
(e) \(-2^2 + 7 = -4 + 7 = 3\)
(f) \(-(-5) - (-1) = 5 + 1 = 6\)
(g) \(-4^2 - (-4)^2 = -16 - 16 = -32\)
(h) \(-3 \times (-2^2) = -3 \times (-4) = 12\)

Solution to Activity 6
(a) \(-b = -(-3) = 3\)
(b) \(-a - b = -(-2) - (-3) = 2 + 3 = 5\)
(c) \(-b^2 = -(-3)^2 = -9\)
(d) \(a^2 + ab = (-2)^2 + (-2) \times (-3) = 4 + 6 = 10\)
(e) \(\frac{3 - a^2}{b} = \frac{3 - (-2)^2}{-3} = \frac{3 - 4}{-3} = \frac{-1}{-3} = \frac{1}{3}\)
(f) \(a^2 - 2a + 5 = (-2)^2 - 2 \times (-2) + 5 = 4 + 4 + 5 = 13\)
(g) \((6 - a)(2 + b) = (6 - (-2)) \times (2 + (-3)) = (6 + 2) \times (2 - 3) = 8 \times (-1) = -8\)
(h) \(a^3 = (-2)^3 = -8\)
(i) \(-b^3 = -(-3)^3 = -(-27) = 27\)

Solution to Activity 7
(a) (i) The positive factor pairs of 28 are 
1, 28; 2, 14; 4, 7.
The positive factors of 28 are 
1, 2, 4, 7, 14, 28.
(ii) The positive factor pairs of 25 are 
1, 25; 5, 5.
The positive factors of 25 are 
1, 5, 25.
(iii) The positive factor pairs of 36 are 
1, 36; 2, 18; 3, 12; 4, 9; 6, 6.
The positive factors of 36 are 
1, 2, 3, 4, 6, 9, 12, 18, 36.
(b) (i) The factor pairs of 28 are 
1, 28; 2, 14; 4, 7; 
-1, -28; -2, -14; -4, -7.
(ii) The factor pairs of -28 are 
1, -28; 2, -14; 4, -7; 
-1, 28; -2, 14; -4, 7.

Solution to Activity 8
(a) \(594 = 2 \times 3^3 \times 11\)
(b) \(525 = 3 \times 5^2 \times 7\)
(c) \(221 = 13 \times 17\)
(d) \(223 = 223\)
(The number 223 is prime.)

Solution to Activity 9
The prime factorisations are
\(9 = 3^2\)
\(18 = 2 \times 3^2\)
\(24 = 2^3 \times 3\)
(a) The LCM of 18 and 24 is \(2^3 \times 3^2 = 72\).
The HCF of 18 and 24 is \(2 \times 3 = 6\).
(b) The LCM of 9, 18 and 24 is \(2^3 \times 3^2 = 72\).
The HCF of 9, 18 and 24 is 3.
(c) The LCM and HCF of -18 and -24 are the same as the LCM and HCF of 18 and 24 found in part (a). That is, the LCM is 72 and the HCF is 6.

Solution to Activity 10
(a) \(-(-uv) = +uv = uv\)
(b) \(+(-9p) = -9p\)
(c) \(-(-4r^2) = +4r^2 = 4r^2\)
(d) \(-8z = -8z\)
(The expression in part (d) can be simplified by deleting the brackets, as shown. These brackets aren’t necessary, as the multiplication is done before taking the negative, anyway, by the BIDMAS rules.)
(e) \(2x^2y^2 \times 5xy^4 = 10x^3y^6\)
(f) \(-P(-PQ) = P^2Q\)
(g) $5m \times \left(-\frac{2}{3}n\right) = -2mn$

(h) $(-a^3)(-2b^3)(-2a^3) = -4a^6b^3$

(i) $(cd)^2 = cd \times cd = c^2d^2$

(j) $-(3x)^2 = -3x \times -3x = 9x^2$

(k) $-(3x)^2 = -3x \times -3x = -9x^2$

(l) $-(-3x)^2 = -3x \times -3x = -9x^2$

(m) $-(2x)^3 = -(2x) \times -(2x) \times -(2x) = -8x^3$

(n) $-(-2x)^3 = -(2x) \times -(2x) \times -(2x) = -8x^3$

**Solution to Activity 11**

(a) $3a \times 3b - 2b \times 3b = 9ab - 6b^2$

(b) $5x \times 8x - 3x(-3x)$

$= 40x^2 + 9x^2 = 49x^2$

(c) $3x^2 - (-3y^2) + (-x^2) + (2y^2)$

$= 3x^2 + 3y^2 - x^2 + 2y^2 = 2x^2 + 5y^2$

(d) $-3cd + (-5c \times 2d^2) - (-cd^2)$

$= -3cd - 10cd^2 + cd^2$

$= -3cd - 9cd^2$

(e) $-6p(-\frac{1}{8}p) + (-5p \times p) - 2(-\frac{1}{2}p^2)$

$= 2p^2 - 5p^2 + p^2$

$= -2p^2$

(f) $A(-B) + (-AB) - (-AB) + (-A)(-B)$

$= -AB - AB + AB + AB = 0$

**Solution to Activity 12**

(a) $a(a^4 + b) = a^5 + ab$

(b) $-x(6x - x^2) = -6x^2 + x^3 = x^3 - 6x^2$

(c) $3pq(2p + 3q - 1) = 6p^2q + 9pq^2 - 3pq$

(d) $(C^3 - C^2 - C)C^2 = C^5 - C^4 - C^3$

(e) $-\frac{1}{2}x \left(\frac{1}{3}x^2 + \frac{2}{3}x\right) = -\frac{1}{6}x^3 - \frac{1}{3}x^2$

**Solution to Activity 13**

(a) $-(-2x^2 + x - 1) = 2x^2 - x + 1$

(b) $+(2x - 3y + z) = 2x - 3y + z$

(c) $-(p - 2q) = -p + 2q$

**Solution to Activity 14**

(a) $\underline{x + x^2(1 + 3x)} = x + x^2 + 3x^3$

(b) $\underline{7ab - b(a + 2b)} = 7ab - ab - 2b^2$

$= 6ab - 2b^2$

(c) $\underline{-6(c + d) + 3(c - d)}$

$= -6c - 6d + 3c - 3d$

$= -3c - 9d$

(d) $\underline{2X - 5Y(-4X + 2Y)}$

$= 2X + 20XY - 10Y^2$

(e) $\underline{(1 - p^4)p + p^2 - p}$

$= p - p^5 + p^2 - p$

$= -p^5 + p^2$

**Solution to Activity 15**

(a) $(a + b)(c + d + e)$

$= a(c + d + e) + b(c + d + e)$

$= ac + ad + ae + bc + bd + be$

(b) $(x + 3)(x + 5)$

$= x(x + 5) + 3(x + 5)$

$= x^2 + 5x + 3x + 15$

$= x^2 + 8x + 15$

(c) $(x^2 - 2x + 3)(3x^2 - x - 1)$

$= x^2(3x^2 - x - 1) - 2x(3x^2 - x - 1)$

$+ 3(3x^2 - x - 1)$

$= 3x^4 - x^3 - x^2 - 6x^3 + 2x^2 + 2x$

$+ 9x^2 - 3x - 3$

$= 3x^4 - 7x^3 + 10x^2 - x - 3$

**Solution to Activity 16**

(a) $(x + 5)(x - 7)$

$= x^2 - 7x + 5x - 35$

$= x^2 - 2x - 35$

(b) $(x - 3)(x - 1)$

$= x^2 - x - 3x + 3$

$= x^2 - 4x + 3$

(c) $(2x - 1)(8x + 3)$

$= 16x^2 + 6x - 8x - 3$

$= 16x^2 - 2x - 3$

(d) $(2 - 5x)(x - 9)$

$= 2x - 18 - 5x^2 + 45x$

$= -5x^2 + 47x - 18$

(e) $(c - 2d)(1 + c)$

$= c + c^2 - 2d - 2cd$
Solution to Activity 17
(a) \((x + 1)^2 = (x + 1)(x + 1)
\= x^2 + x + x + 1
\= x^2 + 2x + 1\)
(b) \((3x - 2)^2 = (3x - 2)(3x - 2)
\= 9x^2 - 6x - 6x + 4
\= 9x^2 - 12x + 4\)
(c) \((2p + 3q)^2 = (2p + 3q)(2p + 3q)
\= 4p^2 + 6pq + 6pq + 9q^2
\= 4p^2 + 12pq + 9q^2\)

Solution to Activity 18
(a) \((x + 6)^2 = x^2 + 2 \times x \times 6 + 6^2
\= x^2 + 12x + 36\)
(b) \((x - 2)^2 = x^2 - 2 \times x \times 2 + 2^2
\= x^2 - 4x + 4\)
(c) \((1 + m)^2 = 1^2 + 2 \times 1 \times m + m^2
\= 1 + 2m + m^2\)
(d) \((1 - 2u)^2 = 1^2 - 2 \times 1 \times (2u) + (2u)^2
\= 1 - 4u + 4u^2\)
(e) \((2x - 3)^2 = (2x)^2 - 2 \times (2x) \times 3 + 3^2
\= 4x^2 - 12x + 9\)
(f) \((3c + d)^2 = (3c)^2 + 2 \times (3c) \times d + d^2
\= 9c^2 + 6cd + d^2\)

Solution to Activity 19
(a) (i) \(4a^2 = 2a \times 2a\) and \(2ab = 2a \times b\)
(ii) \(x^3 = x^2 \times x\) and \(x^5 = x^2 \times x^3\)
(iii) \(18z^2 = 6z \times 3z\), \(6z^2 = 6z \times z\) and \(6z = 6z \times 1\)
(b) (i) \(10cd^2 = 5c \times 2d^2\) and \(10cd^2 = 2cd \times 5d\)
(ii) \(p^7 = p^3 \times p^4\) and \(p^7 = p^3 \times p^4\)
(iii) \(9y^2 = 3 \times 3y^2\), \(9y^2 = 9y^2 \times 1\) and \(9y^2 = 3y \times 3y\)

Solution to Activity 20
(a) (i) The highest common factor is \(3x\).
\(3x^2 = 3x \times x\) and \(9xy = 3x \times 3y\)
(ii) The lowest common multiple is \(9x^2y\).
\(9x^2y = 3x^2 \times 3y\) and \(9x^2y = 9xy \times x\)
(b) (i) The highest common factor is \(2pq\).
\(6p^2q^3 = 2pq \times 3pq^2\), \(4pq^2 = 2pq \times 2q\) and \(2pq = 2pq \times 1\)
(ii) The lowest common multiple is \(12p^2q^3\).
\(12p^2q^3 = 6p^2q^3 \times 2\), \(12p^2q^3 = 4pq^2 \times 3pq\)
and \(12p^2q^3 = 2pq \times 6pq^2\)

Solution to Activity 21
(a) \(pq + 12qr = q(p + 12r)\)
(b) \(14cd - 7cd^2 = 7cd(2 - d)\)
(c) \(m^3 - 7m - 8m^2 = m^2(m - m^2 - 8)\)
(d) \(-6AB + 3A^2B - 12A^3B^2\)
\= 3AB(-2 + A - 4A^2B)
\= 3AB(A - 4A^2B - 2)\)
(e) \(\sqrt{T} - s\sqrt{T} = \sqrt{T}(1 - s)\)
(f) \(5x^2 - 10x = 5x(x - 2)\)
(g) \(18y^2 + 6 = 6(3y^2 + 1)\)

Solution to Activity 22
(a) \(-2x^3 + 3x^2 - x - 5 = -(2x^3 - 3x^2 + x + 5)\)
(b) \(-ab - a - b = -(ab + a + b)\)
(c) \(5cd^2 - 10c^2d - 5cd\)
\= -5cd(-d + 2c + 1)\)
\= -5cd(2c - d + 1)\)
Solution to Activity 23
(a) \( \frac{1}{2}a^2 + \frac{3}{2}a = \frac{1}{2}(a^2 + 3a) = \frac{1}{2}a(a + 3) \)
(b) \( \frac{1}{3}x - \frac{1}{6} = \frac{1}{6}(2x - 1) \)
(c) \( 2x^2 - \frac{1}{2}x + \frac{1}{3} = \frac{1}{4}(8x^2 - 2x + 1) \)
(d) \( \frac{2}{3}u^2v^2 + \frac{4}{3}u^3v = \frac{1}{6}(4u^2v^2 + 3u^3v) \)
= \( \frac{1}{6}u^2v(4v + 3u) \)

Solution to Activity 24
(a) \( \frac{8xy^3}{6x^2y^2} = \frac{4y}{3x} \)
(b) \( \frac{2(3x - 1)}{10(3x - 1)^2} = \frac{1}{5(3x - 1)^2} \)
(c) \( \frac{(x - 1)^2(x - 2)}{(x - 1)(x - 2)^2} = \frac{x - 1}{x - 2} \)
(The fraction in part (c) can’t be cancelled down any further. There’s an \( x \) in both the numerator and the denominator, but \( x \) isn’t a factor of either the numerator or the denominator.)

Solution to Activity 25
(a) \( \frac{ab + a}{a^2 + a} = \frac{b + 1}{a + 1} \)
(b) \( \frac{9 + 6y^2}{u^2 - u^3} = \frac{3 + 2y^2}{1 - u} \)
(c) \( \frac{2x^2 - 4x^3}{6x^4 - 2x^2} = \frac{x^2 - 2x^3}{3x^4 - x^2} = \frac{1 - 2x}{3x^2 - 1} \)
(d) \( \frac{x^3 + x^2}{2x + 2} = \frac{x^2(x + 1)}{2(x + 1)} = \frac{x^2}{2} \)
(e) \( \frac{1 - n}{n^2 - n} = \frac{1 - n}{n(n - 1)} = -\frac{(n - 1)}{n(n - 1)} = -\frac{1}{n} = -\frac{1}{n} \)

Solution to Activity 26
(a) \( \frac{1 + y}{x} = \frac{1 + y}{x} \)
(b) \( \frac{c + 2}{c^2 + c} - \frac{1}{c^2 + c} = \frac{c + 2 - 1}{c^2 + c} = \frac{c + 1}{c(c + 1)} = \frac{1}{c} \)
(c) \( \frac{1 - \frac{1}{ab}}{bc} = \frac{c}{abc} - \frac{a}{abc} = \frac{c - a}{abc} \)
(d) \( \frac{2}{3a} + \frac{1}{2a} = \frac{4}{6a} + \frac{3}{6a} = \frac{7}{6a} \)
(e) \( \frac{1}{x^2} - \frac{2}{x} + 3 = \frac{1 - 2x + 3x^2}{x^2} \)
(f) \( 5 - \frac{1}{x} + \frac{2}{y} = \frac{5 - \frac{y}{xy} + \frac{2x}{xy}}{xy} \)
(g) \( A - \frac{A^2 - 1}{2A} = \frac{2A^2 - A^2 + 1}{2A} \)
= \( \frac{A^2 + 1}{2A} \)
(h) \( \frac{3}{2u - 1} + u = \frac{3 + u(2u - 1)}{2u - 1} \)
= \( \frac{2u^2 - u + 3}{2u - 1} \)
(i) \( \frac{x + 2}{x + 1} + \frac{1}{x} = \frac{x(x + 2)}{x(x + 1)} + \frac{1}{x(x + 1)} \)
= \( \frac{x(x + 2) + x + 1}{x(x + 1)} \)
= \( \frac{x^2 + 3x + 1}{x(x + 1)} \)

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Solution to Activity 27
(a) \[ \frac{a^2 + a^5 - 1}{a^2} = \frac{a^5}{a^2} + \frac{a^5}{a^2} - \frac{1}{a^2} = 1 + a^3 - \frac{1}{a^2} \]

(b) \[ \frac{2 - 5cd}{c} = \frac{2}{c} - \frac{5cd}{c} = \frac{2}{c} - 5d \]

(c) \[ \frac{2a + 3a^2}{6} = \frac{2a}{6} + \frac{3a^2}{6} = \frac{a}{3} + \frac{a^2}{2} \]

Solution to Activity 28
(a) \[ \frac{40A}{B} \times \frac{16A^4}{5C} = \frac{2A^3}{2} \]

(b) \[ \frac{b}{c^2} \div c^3 = \frac{b}{c^2} \times \frac{1}{c^3} = \frac{b}{c^5} \]

(c) \[ \left( \frac{6y}{x^7} \right) \div \left( \frac{15y^{10}}{x^4} \right) = \frac{6y}{x^7} \times \frac{x^4}{15y^{10}} = \frac{2}{5xy^9} \]

(d) \[ \frac{a/(a + 1)}{a^6/(a + 1)^2} = \frac{a}{a + 1} \div \frac{a^6}{(a + 1)^2} = \frac{a}{a + 1} \times \frac{(a + 1)^2}{a^6} = \frac{a + 1}{a} \]

(e) \[ \frac{3x}{y} \div \frac{6}{y^2} = \frac{3x}{y} \times \frac{y^2}{6} = \frac{xy}{2} \]

(f) \[ g \times \frac{5}{k} = \frac{g}{1} \times \frac{5}{k} = \frac{5g}{k} \]

Solution to Activity 29
(a) \[ \sqrt{5} \sqrt{6} = \sqrt{5 \times 6} = \sqrt{30} \]

(b) \[ \sqrt{\frac{75}{15}} = \sqrt{\frac{75}{15}} = \sqrt{5} \]

(c) \[ 3 + \sqrt{10} \sqrt{10} = 3 + 10 = 13 \]

(d) \[ \sqrt{8} \sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = 4 \]

(e) \[ \sqrt[3]{\frac{5}{15}} = \sqrt[3]{\frac{5}{15}} = \sqrt[3]{\frac{1}{3}} = \frac{\sqrt[3]{1}}{\sqrt[3]{3}} = \frac{1}{\sqrt[3]{3}} \]

Solution to Activity 30
(a) \[ \sqrt[3]{2(\sqrt{2} + \sqrt{3})} = 2\sqrt{\sqrt{2} \sqrt{3} + \sqrt{3} \sqrt{3}} = 2\sqrt{6} + 3 = 3 + 2\sqrt{6} \]

(It’s slightly tidier to write the term containing the irrational root as the last term.)

(b) \[ \sqrt{2(1 + \sqrt{3})} + 9\sqrt{2} = \sqrt{2 + \sqrt{2} \sqrt{3} + 9\sqrt{2}} = \sqrt{6 + 10\sqrt{2}} \]

(c) \[ (1 - \sqrt{5})(6 - 2\sqrt{5}) = 6 - 2\sqrt{5} - 6\sqrt{5} + 2\sqrt{5} \sqrt{5} = 6 - 8\sqrt{5} + 10 = 16 - 8\sqrt{5} \]

(d) \[ (6 - \sqrt{7})(6 + \sqrt{7}) = 36 + 6\sqrt{7} - 6\sqrt{7} - \sqrt{7} \sqrt{7} = 36 - 7 = 29 \]

(You can shorten your working in part (d) by noticing that the given expression has the form that
Solution to Activity 31

(a) (i) \( \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2} \)
(ii) \( \sqrt{150} = \sqrt{25 \times 6} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6} \)
(iii) \( \sqrt{22} \) can’t be simplified.
(iv) \( \sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2} \)
(v) \( 5 + \sqrt{108} = 5 + \sqrt{36 \times 3} \)
= \( 5 + 6\sqrt{3} \)
= \( 5 + 6\sqrt{3} \)
(vi) \( \sqrt{12} + \sqrt{4} = \sqrt{4 \times 3} + 2 \)
= \( 2\sqrt{3} + 2 \)
= \( 2 + 2\sqrt{3} \)
= \( 2(1 + \sqrt{3}) \)
(It’s slightly tidier to write the term containing the irrational root as the last term. It’s not essential to take out the common factor 2.)
(vii) \( \sqrt{27} - \sqrt{3} = \sqrt{9 \times 3} - \sqrt{3} \)
= \( \sqrt{9} \sqrt{3} - \sqrt{3} \)
= \( 3\sqrt{3} - \sqrt{3} \)
= \( 2\sqrt{3} \)
(b) (i) \( \sqrt{6} (\sqrt{3} + \sqrt{2}) = \sqrt{3 \sqrt{6}} + \sqrt{2 \sqrt{6}} \)
= \( \sqrt{3} \sqrt{3} \times 2 + \sqrt{2} \sqrt{2} \times 3 \)
= \( 3\sqrt{3} \sqrt{2} + 2\sqrt{2} \sqrt{3} \)
= \( 3\sqrt{2} + 2\sqrt{3} \)

Solution to Activity 32

(a) \( \frac{3}{\sqrt{7}} = \frac{3 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{3\sqrt{7}}{7} \)
(b) \( \frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{2} \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}} = \frac{\sqrt{12}}{6} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \)
(c) \( \frac{5}{\sqrt{5}} = \frac{5 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5} \)
(d) \( \frac{2}{1 + \sqrt{17}} = \frac{2 \times (1 - \sqrt{17})}{1 - \sqrt{17} \times (1 - \sqrt{17})} \)
= \( \frac{2 - 2\sqrt{17}}{1 - (\sqrt{17})^2} \)
= \( \frac{2 - 2\sqrt{17}}{1 - 17} \)
= \( \frac{2 - 2\sqrt{17}}{-16} \)
= \( \frac{1 - \sqrt{17}}{-8} \)
= \( \frac{\sqrt{17} - 1}{8} \)
(Remember that \( 1 - \sqrt{17} \) is the negative of \( 1 + \sqrt{17} \); see page 32.)
### Solution to Activity 33
(a) $2^4 = 2 \times 2 \times 2 \times 2 = 16$
(b) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$
(c) $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$
(d) $(-3)^3 = (-3) \times (-3) \times (-3) = -27$
(e) $\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
(f) $\left(\frac{1}{3}\right)^3 = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

### Solution to Activity 34
(a) $\frac{a^{20}}{a^5} = a^{15}$
(b) $(y^4)^5 = y^{20}$
(c) $\frac{b^4b^7}{b^3} = \frac{b^{11}}{b^3} = b^8$
(d) $p^3p^5p^3p = p^{12}$
(e) $(x^3)^3x^2 = x^9x^2 = x^{11}$
(f) $(m^3m^2)^5 = (m^5)^5 = m^{25}$
(g) $\left(\frac{c^5}{c^3}\right)^2 = (c^2)^2 = c^6$

### Solution to Activity 35
(a) $(x^3 - 1)(2x^3 + 5) = 2x^6 + 5x^3 - 2x^3 - 5 = 2x^6 + 3x^3 - 5$
(b) $(a^4 + b^4)^2 = (a^4 + b^4)(a^4 + b^4) = a^8 + 2a^4b^4 + b^8$
(c) $\frac{p^8 + p^2}{p^2} = p^6 + 1$

### Solution to Activity 36
(a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$
(b) $3^{-1} = \frac{1}{3}$
(c) $5^0 = 1$
(d) $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$
(e) $\left(\frac{1}{2}\right)^{-1} = 3$
(f) $\left(\frac{1}{3}\right)^{-2} = \left(\left(\frac{1}{3}\right)^{-1}\right)^2 = 3^2 = 9$

### Solution to Activity 37
(a) $5g^{-1} = \frac{5}{g}$
(b) $\frac{1}{y^{-1}} = y$
(c) $\frac{2}{3x^{-5}} = \frac{2x^5}{3}$
(d) $\frac{a^{-3}}{b^{-4}} = \frac{b^4}{a^3}$
(e) $\frac{p^2}{Q^{-5}} = P^2Q^5$
(f) $(3h^2)^2 = 3^2(h^2)^2 = 9h^4$
(g) $(3h^2)^{-2} = \frac{1}{(3h^2)^2} = \frac{1}{3^2(h^2)^2} = \frac{1}{9h^4}$
(h) $\frac{(b^{-4})^3}{(3c)^2} = \frac{b^{-12}}{3^2c^2} = \frac{1}{9b^{12}c^2}$
(i) $\frac{(A^{-1}B)^2}{(B^{-3})^3} = \frac{(A^{-1})^2B^2}{(B^{-3})^3} = \frac{A^{-2}B^2}{B^{-9}} = \frac{B^2B^9}{A^2} = \frac{B^{11}}{A^2}$
(j) $\left(\frac{2y^{-1}}{z^2}\right)^5 = \left(\frac{2y^{-1}}{z^2}\right)^5 = \frac{2^5y^{-5}}{z^{10}} = \frac{32}{y^5z^{10}}$
(k) $\frac{x^{-5}}{x} = \frac{1}{x^5x} = \frac{1}{x^6}$
(l) $\frac{(x-1)^{-3}}{(x-1)^2} = \frac{1}{(x-1)^3(x-1)^2} = \frac{1}{(x-1)^5}$
Solutions to activities

Solution to Activity 38
(a) \( t^{1/2} = \sqrt{t} \)
(b) \( x^{1/3} = \sqrt[3]{x} \)
(c) \( p^{2/3} = \sqrt[3]{p^2} \quad \text{(or} \quad \sqrt[3]{(\sqrt{p})^2}) \)
(d) \( x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}} \)
(e) \( (2x - 3)^{1/2} = \sqrt{2x - 3} \)

(f) \( (1 + x^2)^{-1/2} = \frac{1}{(1 + x^2)^{1/2}} = \frac{1}{\sqrt{1 + x^2}} \)
(g) \( (1 + x)x^{-1/2} = \frac{1 + x}{x^{1/2}} = \frac{1 + x}{\sqrt{x}} \)

Solution to Activity 39
(a) \( \sqrt[4]{y} = y^{1/4} \)
(b) \( \sqrt[5]{1 - 2x} = (1 - 2x)^{1/5} \)
(c) \( \frac{\sqrt[3]{x}}{1} = \frac{1}{x^{1/3}} \)
(d) \( (\sqrt[5]{u})^2 = u^{2/5} \)
(e) \( \frac{\sqrt[4]{x^4}}{1} = \frac{1}{x^{1/4}} \)
(f) \( \sqrt[3]{(y + 2)^3} = (y + 2)^{3/4} \)

Solution to Activity 40
(a) \( x^{1/3} x^{1/3} = x^{2/3} \)
(b) \( \frac{a}{a^{1/3}} = a^{2/3} \)
(c) \( \frac{a}{a^{1/2}} = a^{1/2} = \sqrt{a} \quad \text{(It’s fine to leave the answer as} \ a^{1/2}. \text{)} \)
(d) \( \frac{x}{x^{1/2}} = x^{3/2} \)
(e) \( (2x^{1/5})^3 = 8x^{3/5} \)
(f) \( \frac{(1 + x)^2}{\sqrt{1 + x}} = \frac{(1 + x)^2}{(1 + x)^{1/2}} = (1 + x)^{3/2} \)

(g) \( \left( \frac{1}{u} \right)^{1/3} = \frac{1}{u^{1/3}} \quad \text{(It’s fine to leave the answer as} \ u^{-1/3}. \text{)} \)
(h) \( \frac{A^{5/2}}{A^3} = A^{(5/2) - 3} = A^{-1/2} = \frac{1}{A^{1/2}} = \frac{1}{\sqrt{A}} \quad \text{(It’s fine to write the answer as} \ \frac{1}{A^{1/2}}, \text{or as} \ A^{-1/2}. \text{)} \)

(i) \( \frac{x^{1/2} y^2}{x^3 y^{1/2}} = \frac{y^{3/2}}{x^{5/2}} \quad \text{(or} \quad \sqrt[5]{\frac{y^3}{x^5}}) \)

(j) \( \sqrt{4x} = \sqrt{4} \sqrt{x} = 2\sqrt{x} \)

Solution to Activity 41
(a) \( a^{2p} a^{5p} = a^{7p} \)
(b) \( \frac{b^{7k}}{b^{4k}} = b^{3k} \)
(c) \( (g^n)^k = g^{nk} \)
(d) \((2y^{2t})^2y^{-t} = 2^2(y^{2t})^2y^{-t}\)
\[= 4y^{4t}y^{-t}\]
\[= 4y^{3t}\]

(e) \(\frac{m^{3x}}{m^{-x}} = m^{3x}m^x\)
\[= m^{4x}\]

(f) \(\frac{c^{3y}}{c^{5y}} = \frac{1}{c^{2y}}\)

(g) \(\frac{(a^{-3t}b^{3t})^2}{a^{3t}b^t} = \frac{(a^{-3t})^2(b^{3t})^2}{a^{3t}b^t}\)
\[= \frac{a^{-6t}b^{6t}}{a^{3t}b^t}\]
\[= \frac{b^{5t}}{a^{3t}}\]

(h) \((d^{1/r})^{2r} = d^2\)

(i) \((3h)^2(9h)^p = 3^{2p}h^{2p}9^p h^p\)
\[= 3^{2p}h^{3p}(3^2)^p\]
\[= 3^{2p}h^{3p}3^{2p}\]
\[= 3^{4p}h^{3p}\]

Solution to Activity 44

(a) These equations are not equivalent. The term in the first equation that’s been moved, \(-3\), isn’t a term of the whole expression on the right-hand side, but only of the subexpression \(x - 3\).

(b) These equations are equivalent.

(c) These equations are not equivalent. The second equation is obtained from the first by subtracting \(x/3\) from the left-hand side and \(x/2\) from the right-hand side, but this does not give an equivalent equation.

Solution to Activity 45

(a) These equations are not equivalent. Multiplying the first equation through by 6 gives
\[2a = a^2 - 18.\]

(b) These equations are not equivalent. Multiplying the first equation through by \(2x\) gives
\[x^2 - 1 + 2x^2 = 1.\]

(c) These equations are equivalent.

(d) These equations are not equivalent. Multiplying the first equation through by 2 gives \(\sqrt{\frac{2x}{y}} = 2.\)

Solution to Activity 46

(a) These equations are equivalent.

(b) These equations are not equivalent. Multiplying both sides of the first equation by \(a - 3\) gives
\[b(a - 3) + a = 6(a - 3).\]
Alternatively, you can rearrange the first equation to give
\[\frac{a}{a - 3} = 6 - b,\]
and cross-multiply to give
\[a = (6 - b)(a - 3).\]

(c) These equations are equivalent.

(d) These equations are not equivalent.
Cross-multiplying in the first equation gives
\[(2 + x)(2x + 5) = 9.\]
Solution to Activity 47

(a) \( 12 - 5q = q + 3 \)
\[ 12 - 3 = q + 5q \]
\[ 9 = 6q \]
\[ \frac{3}{2} = q \]
The solution is \( q = \frac{3}{2} \).

(b) \( 3(x - 1) = 4(1 - x) \)
\[ 3x - 3 = 4 - 4x \]
\[ 3x + 4x = 4 + 3 \]
\[ 7x = 7 \]
\[ x = 1 \]
The solution is \( x = 1 \).

(c) \[ \frac{t - 3}{4} + 5t = 1 \]
\[ \frac{4(t - 3)}{4} + 4 \times 5t = 4 \]
\[ t - 3 + 20t = 4 \]
\[ t + 20t = 4 + 3 \]
\[ 21t = 7 \]
\[ t = \frac{1}{3} \]
The solution is \( t = \frac{1}{3} \).

(d) \[ \frac{3}{b} = \frac{2}{3b} - \frac{1}{3} \]
\[ \frac{3 \times 3b}{b} = \frac{2 \times 3b}{3b} - \frac{3b}{3} \]
\[ (\text{assuming } b \neq 0) \]
\[ 9 = 2 - b \]
\[ 7 = -b \]
\[ -7 = b \]
The solution is \( b = -7 \).

(e) \[ 2(A + 2) = \frac{A}{3} + 1 \]
\[ 6(A + 2) = \frac{3A}{3} + 3 \]
\[ 6A + 12 = A + 3 \]
\[ 6A - A = 3 - 12 \]
\[ 5A = -9 \]
\[ A = -\frac{9}{5} \]
The solution is \( A = -\frac{9}{5} \).

(f) \[ \frac{1}{z + 1} + \frac{1}{5(z + 1)} = 1, \]
so assuming that \( z \neq -1 \),
\[ \frac{5(z + 1)}{z + 1} + \frac{5(z + 1)}{5(z + 1)} = 5(z + 1) \]
\[ 5 + 1 = 5z + 5 \]
\[ 1 = 5z \]
\[ \frac{1}{5} = z \]
The solution is \( z = \frac{1}{5} \).

(The fourth equation above is obtained by subtracting 5 from each side of the third equation.)

Solution to Activity 48

(a) \[ h + 1 = \frac{4h}{5} \]
\[ 5(h + 1) = 4h \]
\[ 5h + 5 = 4h \]
\[ 5h - 4h = -5 \]
\[ h = -5 \]
The solution is \( h = -5 \).

(b) \[ \frac{2}{y} = \frac{3}{y + 2} \]
\[ 2(y + 2) = 3y \] (assuming \( y \neq -2, 0 \))
\[ 2y + 4 = 3y \]
\[ 4 = 3y - 2y \]
\[ 4 = y \]
The solution is \( y = 4 \).

(c) \[ \frac{3}{2x - 1} = \frac{2}{3 - x} \]
\[ 3(3 - x) = 2(2x - 1) \] (assuming \( x \neq \frac{1}{2}, 3 \))
\[ 9 - 3x = 4x - 2 \]
\[ 9 + 2 = 4x + 3x \]
\[ 11 = 7x \]
\[ \frac{11}{7} = x \]
The solution is \( x = \frac{11}{7} \).

Solution to Activity 49

The problem with the ‘proof’ is that it involves dividing by \( a - b \), which is equal to zero, since \( a = b \).
Solution to Activity 50
(a) \(am = 2a + 3m\)
\(am - 3m = 2a\)
\(m(a - 3) = 2a\)
\(m = \frac{2a}{a - 3}\) (assuming \(a \neq 3\))

(b) \(c = \frac{1}{3}(2 + 5d)\)
\(3c = 2 + 5d\)
\(3c - 2 = 5d\)
\(5d = 3c - 2\)
\(d = \frac{1}{5}(3c - 2)\)

(The alternative form \(d = \frac{3c - 2}{5}\) is fine.)

(c) \(Y = \frac{3Y + 2}{X + 1}\)
\(Y(X + 1) = X(3Y + 2)\) (assuming \(X \neq 0\), \(-1\))
\(XY + Y = 3XY + 2X\)
\(Y = 3XY - XY + 2X\)
\(Y = 2XY + 2X\)
\(Y = 2X(Y + 1)\)
\(\frac{Y}{2(Y + 1)} = X\) (assuming \(Y \neq -1\))
\(X = \frac{Y}{2(Y + 1)}\)

(d) \(a = \frac{b}{1 - 2c}\)
\(a(1 - 2c) = b\) (assuming \(c \neq \frac{1}{2}\))
\(a - 2ac = b\)
\(a - b = 2ac\)
\(2ac = a - b\)
\(c = \frac{a - b}{2a}\) (assuming \(a \neq 0\))

(e) \(A = \frac{2}{B} + AC\)
\(A - AC = \frac{2}{B}\)
\(B(A - AC) = 2\) (assuming \(B \neq 0\))
\(AB(1 - C) = 2\)
\(B = \frac{2}{A(1 - C)}\) (assuming \(A \neq 0\), \(C \neq 1\))

Solution to Activity 51
(a) \(Q^3 = PR^2\)
\(Q = (PR^2)^{1/3}\)
\(Q = P^{1/3}R^{2/3}\)

(The forms \(Q = (PR^2)^{1/3}\) and \(Q = \frac{3}{2}PR^{2/3}\) are also suitable answers.)

(b) \(h^{1/4} = \frac{2k^{1/2}}{m}\)
\(h = \left(\frac{2k^{1/2}}{m}\right)^4\)
\(h = \frac{16k^2}{m^4}\)

(c) \(u^2 = v + w\)
\(u = \sqrt{v + w}\)

Solution to Activity 52
(a) \(a^2 - 2b^2 = 3c^2\)
\(a^2 = 2b^2 + 3c^2\)
\(a = \sqrt{2b^2 + 3c^2}\)

(b) \(a^2 = \frac{bt^2}{N}\)
\(a^2N = bt^2\)
\(\frac{a^2N}{b} = t^2\)
\(t^2 = \frac{a^2N}{b}\)
\(t = \sqrt{\frac{a^2N}{b}}\)
\(t = a\sqrt{\frac{N}{b}}\)

(c) \(\left(\frac{r}{3s}\right)^5 = \sqrt{d}\)
\(\left(\frac{r}{3s}\right)^5 = d^{1/2}\)
\(\frac{r}{3s} = (d^{1/2})^{1/5}\)
\(r = 3sd^{1/10}\)

(d) \((4y)^{1/3} = x\)
\(4y = x^3\)
\(y = \frac{1}{4}x^3\)

(The alternative form \(y = \frac{x^3}{4}\) is fine.)
Solution to Activity 53

(a) Here are some ways in which the solution could have been improved.

- It should be written in sentences.
- It needs some words of explanation.
- The three ⇒ symbols at the left should be omitted.
- The fourth ⇒ symbol (the one just before the number 1.75) should be replaced by an equals sign. Here, the author of the solution has worked out that 0.6 + 1.15 is equal to 1.75, so an equals sign is the correct symbol.
- The equation $\sqrt{0.3621} = 0.6$ should be changed to $\sqrt{0.3621} = 0.60$ (to 2 d.p.)
or $\sqrt{0.3621} \approx 0.60$.

The reason why either ‘(to 2 d.p.)’ should be appended or the equals sign changed to ‘≈’ is that, without such a change, the equals sign is used incorrectly. This is because $\sqrt{0.3621}$ is not equal to 0.6. These numbers are only approximately equal.

The reason why the rounded answer 0.6 should be written as 0.60 is to indicate that it has been rounded to two decimal places.
- The solution should explain what the variable $h$ represents.
- The first equation, $h^2 + 98^2 = 1.15^2$, should not have been written at all. It is incorrect, because it has been obtained from Pythagoras’ theorem by substituting in one length expressed in centimetres and another expressed in metres. Any lengths substituted into Pythagoras’ theorem must be in the same units.
- Ideally the letter $h$ should not be used to represent two different quantities, and it should certainly not be used in this way without explanation. (In most of the solution the letter $h$ seems to represent the vertical height of the sloping pane of glass, but in the final line it seems to represent the total height of the cold frame. This makes the solution difficult to understand.)
- The solution should finish with a clear answer to the question, including units.
- The lengths in the diagram should include units (or they should be expressed in the same units and it should be made clear what the units are).
- The diagram should relate more clearly to the question.
- Ideally, the ∴ symbol should be replaced by a word, such as ‘So’.

(b) Here is a better solution.

A side view of the cold frame is shown below.

By Pythagoras’ theorem, the vertical height $h$ (in metres) of the slant pane of glass is given by $h^2 + 0.98^2 = 1.15^2$.

Solving this equation (and using the fact that $h$ is positive) gives

$h^2 = 1.15^2 - 0.98^2$
$h^2 = 0.3621$
$h = \sqrt{0.3621}$
$h = 0.601747\ldots$

Hence the height of the cold frame where it meets the wall is

$(1.15 + 0.601747\ldots)\ m$
$= 1.751747\ldots\ m$
$= 1.75\ m$ (to the nearest cm).
Acknowledgements

Grateful acknowledgement is made to the following sources:

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Unit 2

Graphs and equations
Introduction

Mathematics is often used to represent and analyse the relationship between two quantities. For example, consider a car travelling along a road from some starting point. There is a relationship between the time that has elapsed since the start of the journey and the distance the car has travelled. This relationship can be investigated mathematically.

The use of mathematics to represent and study real-life situations is known as **mathematical modelling**. When creating a **mathematical model**, we usually simplify the real-life situation in order to concentrate on the aspects that we think are the most important. This often allows the relationship between the quantities of interest to be expressed as an equation.

In the example of the travelling car, if the variable $t$ is used to represent the time in hours since the car left its starting point, and the variable $s$ is used to represent the distance in kilometres that the car has travelled since the start, then there is a relationship between the variables $t$ and $s$. In this unit you will consider some cases where the relationship between the variables $s$ and $t$ is modelled by using a relatively simple equation. Note that mathematicians often use $s$ to represent distance.

Every equation in two variables, whether it models a practical situation or not, represents a relationship between those variables. For example, the equation

$$ y = 2x - 1 $$

represents a relationship between the variables $x$ and $y$. Each particular value of $x$ corresponds to a particular value of $y$.

A useful way to visualise a relationship between two variables is to draw its **graph**. If the relationship is specified by an equation, then this graph is called the graph of the equation. Figure 1 shows the graph of equation (1).

![Figure 1](Image)
Each point on the graph corresponds to a pair of values of $x$ and $y$ that are related by the equation. The graph of an equation in two variables can give new insights into the relationship between the variables. These complement those that you can obtain by using an algebraic approach.

In this unit you will revise two particular types of equation and their graphs. You have probably met these in your previous study of mathematics. They are covered here because they occur frequently in the later units of this module and you need to be confident in working with them, both graphically and algebraically. If you are familiar with them, then you may not wish to spend long reading the sections about them, but you should try the activities and ensure that you are not missing anything new to you, particularly in Subsection 2.4, which is important for the calculus that you’ll study later. You will also begin to learn how to use the module computer algebra system to help you investigate mathematical problems.

In Section 1 you will revise the concepts of coordinates and graphs, and then in Sections 2 and 3 you will consider equations of the form

$$y = mx + c,$$

where $x$ and $y$ are variables, and $m$ and $c$ are constants. Equation (1) is of this form, with $m = 2$ and $c = −1$. The graph of every equation of this form is a straight line. You will see how such equations can be used to model some real-life situations, and revise how to solve pairs of such equations simultaneously.

In Section 4, you will consider equations of the form

$$y = ax^2 + bx + c,$$

where again $x$ and $y$ are variables, and $a$, $b$ and $c$ are constants, with $a \neq 0$. For example, the equation

$$y = x^2 − 2$$

is of this form, with $a = 1$, $b = 0$ and $c = −2$. The graph of every equation of this form has a particular curved shape called a parabola. Figure 2 shows the parabola that is the graph of equation (2).

![Figure 2](image_url)  

**Figure 2** The graph of $y = x^2 − 2$
Equations of this form can be used to model some types of real-life situations, such as the motion of an object falling under the influence of gravity. To find the values of $x$ where a graph such as that in Figure 2 crosses the $x$-axis you have to solve a *quadratic equation*, and you will revise various techniques for solving equations of this type.

Finally, in Section 5 you will learn how to use the module computer algebra system to plot the graphs of various equations in two variables, manipulate expressions and solve equations.

## 1 Plotting graphs

In this section you’ll revise the idea of the graph of an equation, and practise plotting graphs using tables of values. We begin with a brief reminder about *coordinates*.

### 1.1 Coordinates

The location of each point in a plane (that is, on a flat surface) can be specified using a pair of *coordinates* that give the position of the point relative to two *axes* at right angles to each other, as illustrated in Figure 3.

The horizontal and vertical axes are usually labelled $x$ and $y$, respectively, and are referred to as the *$x$-axis* and the *$y$-axis*. The point where the axes intersect is called the *origin*, and is sometimes labelled $O$. Each axis is drawn with an arrowhead to indicate the positive direction (the direction in which the numbers increase), and is usually marked with numbers to show the scale.

**Figure 3** Coordinates

The coordinates of a point are written in brackets, separated by a comma, like this: $(1, 2)$. The first number is the number on the $x$-axis directly
below or above the point, and is called the \textit{x-coordinate}. The second
number is the number on the \textit{y-axis} directly to the left or right of the
point, and is called the \textit{y-coordinate}. For example, Figure 3 shows two
points, A and B, with coordinates (1.5, 2.5) and (−3, −2), respectively.

This method of specifying the position of points is known as a \textit{rectangular}
or \textit{Cartesian} coordinate system. The plane in which the axes and the
points lie is sometimes called the \textbf{Cartesian plane}, the \textit{x,y-plane}, the
\textbf{coordinate plane} or, when there is no ambiguity, simply the \textit{plane}. The
axes are sometimes referred to as the \textbf{coordinate axes}.

The adjective \textit{Cartesian} comes from the surname of the French
mathematician and philosopher René Descartes (1596–1650). He is
credited with being the first person to realise that the shapes of
curves and surfaces can be studied using algebra. Poor health at
school led to his life-long habit of never rising from bed until 11
o’clock in the morning. He later suggested this was an essential
requirement for doing good mathematics!

Although \(x\) and \(y\) are the standard labels for graph axes, other labels can
be used. For example, if a graph represents the relationship between the
variables \(s\) and \(t\), then these letters are used to label the axes. You can
also refer to the horizontal and vertical axes, and the horizontal and
vertical coordinates, and adjust the way you describe other quantities
accordingly. Whatever the axis labels are, the first number in a pair of
coordinates is always the position along the horizontal axis, and the second
number is always the position along the vertical axis.

The position of a point in a plane can be marked with a dot, as in
Figure 3, or with a small cross. It can be left unlabelled, or it can be
labelled in any of various ways: with its coordinates, or with a letter, or
with both, as in Figure 3.

\section*{1.2 Graphs of equations}

Suppose that you have an equation in \(x\) and \(y\), such as
\[
    y = 2x - 1, \quad y = x^2 + x + 3 \quad \text{or} \quad x^2 + y^2 = 1.
\]
A point \((x, y)\) is said to \textbf{satisfy} the equation if the equation is true for the
point’s values of \(x\) and \(y\).
Example 1  Checking whether a point satisfies an equation
Show that the point (3, 2) satisfies the equation \( y = x^2 - 2x - 1 \).

Solution

\( \checkmark \) Substitute the \( x \)- and \( y \)-coordinates of the point into the left-hand side and right-hand side of the equation, and check that the two sides are equal. \( \checkmark \)

When \( x = 3 \) and \( y = 2 \),

\[
\text{LHS} = y = 2
\]

and

\[
\text{RHS} = x^2 - 2x - 1 = 3^2 - 2 \times 3 - 1 = 9 - 6 - 1 = 2.
\]

The LHS and RHS are equal, so the point (3, 2) satisfies the equation.

The method in Example 1 is the same as that in Example 22 in Unit 1.

Activity 1  Checking whether points satisfy an equation

Determine whether the following points satisfy the equation \( y - 2 = 3x \).

(a) (6, 20)  (b) (−2, 8)

You know from Unit 1 that if a pair of values of \( x \) and \( y \) satisfy an equation in \( x \) and \( y \), then they also satisfy any rearranged version of the equation. This means that the collection of points that satisfy an equation does not change when the equation is rearranged. For example, the points that satisfy the equation \( y - 2x = -1 \) are the same as the points that satisfy the equation \( y = 2x - 1 \).

Usually, for any particular equation in \( x \) and \( y \), the points that satisfy the equation are the points that lie on a particular line or curve. If the equation has the property that it can be rearranged to express \( y \) as a formula in terms of \( x \) (so each value of \( x \) determines just one value of \( y \)), then the line or curve is called the graph of the equation. The word ‘graph’ is also used to refer to the whole diagram, including the line or curve and the coordinate axes.

A simple way to get an idea of the shape of the graph of an equation is to choose a few values for \( x \), substitute them into the equation to find the corresponding values of \( y \), plot the resulting points, and draw a smooth line or curve through them. This is illustrated in the next example. It is convenient to use a table of values to record the values of \( x \) and \( y \).
**Example 2  Plotting the graph of an equation**

Plot the graph of the equation \( y = \frac{x}{2} + 2 \).

**Solution**

1. Construct a table of values. Choose some equally-spaced values of \( x \), and work out the corresponding values of \( y \) by substituting into the equation. For example, substituting \( x = -2 \) into the equation gives \( y = \frac{(-2)}{2} + 2 = -1 + 2 = 1 \).

A table of values for the equation \( y = \frac{x}{2} + 2 \) is as follows.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
</tbody>
</table>

2. Draw the axes and plot the points. They seem to lie in a straight line, so draw the straight line through them. Label the line with its equation, either on the graph or in a title.

We will consider why the points in Example 2 lie on a straight line in Subsection 2.2.

Notice that the graph drawn in Example 2 has been extended beyond the points that were calculated. This is because every straight line continues infinitely far in each direction. A graph can show only a small, finite part of a line. A finite part of a line, such as the part between two particular points, is called a line segment. Also notice that the graph was plotted with \( x \) on the horizontal axis and \( y \) on the vertical axis. When you plot the graph of an equation in which one variable \( y \) is expressed in terms of another variable \( x \), the variable \( x \) is always on the horizontal axis and the variable \( y \) on the vertical axis.
You can practise using a table of values to plot a graph in the next activity. Remember that whenever you draw a graph, you should label it with its equation, either on the graph or in a title.

**Activity 2  Plotting the graph of an equation**

Complete the table of values below for the equation

\[ y = x^2 + 2x + 2, \]

and hence plot the graph of this equation.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>|  |  |  |</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When you plot the graph of an equation using a table of values alone, you cannot be absolutely sure that the graph is correct. This is because you do not know what happens between the points that you plotted and to each side of them. For example, consider the equation \( y = 5x^3 - x^5 \). A table of values for this equation, using integer values of \( x \) from -2 to 2, is given in Table 1.

**Table 1  A table of values for \( y = 5x^3 - x^5 \)**

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-8</td>
<td>-4</td>
<td>0</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

These points lie on a straight line, as shown in Figure 4(a). However this line is *not* the graph of the equation \( y = 5x^3 - x^5 \). A correct graph is shown in Figure 4(b).

![Graphs](image.png)

**Figure 4  (a) The straight line through the points in Table 1 (b) the graph of \( y = 5x^3 - x^5 \), which goes through the same points**
Because you can get the wrong idea about the shape of a graph from an unfortunate choice of plotted points, it’s helpful to get to know the general shapes of the graphs of some standard types of equation, and learn how to sketch such graphs without using tables of values. You’ll do this for various types of equation throughout this module, starting in Subsection 2.2 with equations whose graphs are straight lines.

2 Straight-line graphs

In this section you’ll revise how to recognise equations that have straight-line graphs, and how to sketch such graphs from their equations. First, it’s helpful to revise some basic properties of lines. Note that, in mathematics, the word ‘line’ is generally used to mean a straight line.

2.1 Gradients and intercepts of straight lines

Gradients

The gradient of a straight line is a measure of how steep it is. To understand what gradient means, imagine tracing your pen tip along a straight line. The gradient (or slope) of the line is the number of units that your pen tip moves up for every one unit that it moves to the right. For example, you can see that the line in Figure 5(a) has gradient 2.

![Figure 5](image)

(a) (b)

**Figure 5** Straight lines with gradients (a) 2 (b) −3

If you imagine tracing your pen tip along the line in Figure 5(b), you can see that it will move down, rather than up, as it moves to the right. It will move down by 3 units for every one unit that it moves to the right. A movement of 3 units down can be thought of as a movement of −3 units.
up. So your pen tip moves up by \(-3\) units for every one unit that it moves to the right, and hence the gradient of this line is \(-3\).

It’s helpful to remember the following facts.

A line that slopes up from left to right has a positive gradient.
A line that slopes down from left to right has a negative gradient.

**Activity 3  Thinking about gradients**

By thinking about moving your pen tip along each of the lines below, and counting how many units it would move up or down for each unit it moves to the right, write down the gradients of the following lines.
Lines with large positive or negative gradients, such as 10, 50, −10 or −50, are steeper than those with smaller positive or negative gradients, such as 1, 0.5, −0.5 or −1. This fact can be expressed more neatly by using the idea of the magnitude of a number, which is its value without its minus sign, if it has one. For example, the magnitudes of 5 and −5 are both 5. The magnitude of a number is also called its modulus or its absolute value. The greater the magnitude of the gradient of a line, the steeper the line.

When the coordinate axes have equal scales (that is, when the distance representing one unit is the same for both the horizontal and vertical axes), a line with gradient 1 or −1 makes an angle of 45° with the horizontal axis, as shown in Figure 6. So a line whose gradient has magnitude greater than 1 makes an angle of more than 45° with the horizontal axis, and a line whose gradient has magnitude less than 1 makes an angle of less than 45° with the horizontal axis. Remember, though, that these facts are true only if the coordinate axes have equal scales.

**Figure 6**  Lines with gradient (a) 1 (b) −1

**Calculating gradients**

You can calculate the gradient of a straight line by choosing any two points on the line, and proceeding as follows.

First you choose one of the two points (it doesn’t matter which) to be the ‘first point’, and the other point to be the ‘second point’. Then you find the number of units by which $x$ increases as you trace your pen tip from the first point to the second point. This is known as the run from the first point to the second point. You also find the number of units by which $y$ increases as you trace your pen tip from the first point to the second point. This is known as the rise from the first point to the second point. If $x$ or $y$ (or both) actually decreases as you trace your pen tip from the first point to the second point, then the run or rise (or both) is negative.
For example, in Figure 7(a) the run is 4 and the rise is 2, whereas in Figure 7(b) the run is −1 and the rise is 3.

![Diagram](https://via.placeholder.com/150)

(a) **run = 4**  \( \text{rise} = 2 \)  
(b) **run = −1**  \( \text{rise} = 3 \)

**Figure 7**  The run and rise from one point to another

Once you have found the run and rise from the first point to the second point, you can calculate the gradient as follows:

\[
\text{gradient} = \frac{\text{rise}}{\text{run}}.
\]

For example, the gradient of the line in Figure 7(a) is

\[
\frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2}.
\]

and the gradient of the line in Figure 7(b) is

\[
\frac{\text{rise}}{\text{run}} = \frac{3}{-1} = -3.
\]

This method for calculating the gradient of a line can be expressed as a formula in terms of the coordinates of the two points on the line. Let’s denote the first and second points by \((x_1, y_1)\) and \((x_2, y_2)\), respectively.

Here \(x_1\) and \(x_2\) are particular values of \(x\), and \(y_1\) and \(y_2\) are particular values of \(y\). Mathematicians often use subscripts in this way to indicate particular values of variables. When you work with subscripts, be careful not to confuse \(x_2\) with \(x^2\), for example.

With this notation,

\[
\text{run} = x_2 - x_1 \quad \text{and} \quad \text{rise} = y_2 - y_1.
\]

So

\[
\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.
\]

**The gradient of a straight line**

The gradient of the straight line through the points \((x_1, y_1)\) and \((x_2, y_2)\), where \(x_1 \neq x_2\), is given by

\[
\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.
\]
Although it doesn’t matter which point on the line you choose to be \((x_1, y_1)\) and which to be \((x_2, y_2)\) when you use the formula above, it is important to take them the same way round in both the numerator and the denominator.

**Activity 4  Using the formula for gradient**

Consider the following diagram.

(a) Write down the coordinates of the points \(A, B, C\) and \(D\).

(b) Use the formula for gradient to calculate the gradients of the lines that pass through the following pairs of points.

(i) \(A\) and \(B\)  
(ii) \(A\) and \(D\)  
(iii) \(B\) and \(C\)

**Gradients of horizontal and vertical lines**

The gradient of a horizontal line is zero. This is because the gradient is the rise divided by the run, and the rise between any two points on a horizontal line is zero, as illustrated in Figure 8(a). On the other hand, the gradient of a vertical line is undefined. This is because, again, the gradient is the rise divided by the run, but the run between any two points on a vertical line is zero, as illustrated in Figure 8(b). Since it is not possible to divide by zero, the gradient of a vertical line does not exist.
2 Straight-line graphs

Figure 8  (a) The rise between two points on a horizontal line is zero
(b) the run between two points on a vertical line is zero

Intercepts

The value of $x$ where a line crosses the $x$-axis is called its $x$-intercept, and the value of $y$ where it crosses the $y$-axis is called its $y$-intercept. For example, in Figure 9 the $x$-intercept is $-3$ and the $y$-intercept is $2$.

Some mathematicians use the word ‘intercept’ to describe the point at which a line crosses an axis, rather than the value of the $x$- or $y$-coordinate there. They would say that the $x$- and $y$-intercepts in Figure 9 are $(-3,0)$ and $(0,2)$.

Figure 9  The $x$- and $y$-intercepts of a line
2.2 Straight lines and their equations

Let’s now consider which equations in \( x \) and \( y \) are the equations of straight lines. First consider the simple equation \( y = 2x \). The points that satisfy this equation are those whose \( y \)-coordinate is twice their \( x \)-coordinate. These are the points that lie on the straight line through the origin with gradient 2, as illustrated in Figure 10(a).

![Figure 10](image)

**Figure 10** Lines through the origin with gradient (a) 2 (b) 3 (c) -2

Similarly, the points \((x, y)\) that satisfy the equations \( y = 3x \) and \( y = -2x \) are the points that lie on the straight lines through the origin with gradients 3 and -2, respectively, as you can see in Figure 10(b) and (c).

In general, for any value of \( m \), the points \((x, y)\) that satisfy the equation \( y = mx \) are the points that lie on the straight line through the origin with gradient \( m \).

It is traditional in the UK to use the letter \( m \) to represent gradient, though the reason is no longer known! Some countries traditionally use other letters, such as \( s \) or \( k \). The earliest known use of the letter \( m \) for gradient is by the Italian mathematician Vincenzo Riccati in 1757. In addition to making important contributions to several areas of mathematics, Riccati taught Italian literature and Latin, and was responsible for developing flood control measures around Venice and Bologna.

Vincenzo Riccati (1707–75)

Now consider the equation \( y = 2x + 1 \). Note that if a point satisfies \( y = 2x \), then adding 1 to its \( y \)-coordinate gives a point that satisfies \( y = 2x + 1 \).

So the points that satisfy the equation \( y = 2x + 1 \) are all the points on the line that is obtained by moving the line in Figure 10(a) vertically up by
1 unit, as shown in Figure 11. Moving the line up by this amount changes its $y$-intercept from 0 to 1.

![Graph of a straight line](image)

**Figure 11** The line with equation $y = 2x + 1$

In general, for any constant $c$ the graph of the equation $y = mx + c$ is obtained by moving the graph of the equation $y = mx$ vertically by $c$ units. So we have the following fact.

**Graphs of equations of the form $y = mx + c$**

The graph of the equation $y = mx + c$ is the straight line with gradient $m$ and $y$-intercept $c$.

It follows that the graph of any equation that can be rearranged into the form $y = mx + c$ is a straight line. For example, the graph of the equation $3x + 2y - 4 = 0$ is a straight line, since this equation can be rearranged as $y = -\frac{3}{2}x + 2$. Any equation that’s of the form $y = mx + c$, or that can be rearranged into this form, is called a linear equation in the variables $x$ and $y$. (You met the general definition of linear equation in Unit 1, where you revised how to solve linear equations in one unknown.)

When the equation of a line is written in the form $y = mx + c$, it’s straightforward to ‘read off’ the gradient and the $y$-intercept. The gradient is the coefficient of $x$, and the $y$-intercept is the constant term. For example, the line $y = -\frac{3}{2}x + 2$ has gradient $-\frac{3}{2}$ and $y$-intercept 2.

To find the $x$-intercept of a line, you need to find the value of $x$ for which $y = 0$. Here is an example.
Example 3  Finding the $x$-intercept of a line from its equation

Find the $x$-intercept of the line with equation $y = 4x - 3$.

Solution

- The $x$-intercept is the value of $x$ when $y = 0$.

Putting $y = 0$ gives

$$4x - 3 = 0.$$ 

Solving this equation gives

$$4x = 3$$

$$x = \frac{3}{4}.$$ 

Hence the $x$-intercept is $\frac{3}{4}$.

- Leave the answer as a fraction.

(Check: substituting $x = \frac{3}{4}$ into $y = 4x - 3$ gives $y = 4 \times \frac{3}{4} - 3 = 0$, as expected.)

As illustrated in Example 3, if the $x$-intercept, $y$-intercept or gradient of a straight line is a fraction, then there is no need to express it as a decimal.

Activity 5  Finding the gradients and intercepts from equations of lines

Find the gradient, and $x$- and $y$-intercepts, of each of the following lines.

(a)  $y = -4x + 3$  
(b)  $3y - x + 2 = 0$

Equations of horizontal and vertical lines

You saw earlier that the gradient of a horizontal line is zero. So the horizontal line with $x$-intercept $c$ has equation $y = 0x + c$; that is, $y = c$. For example, the horizontal line in Figure 12(a) has $y$-intercept 3, so its equation is $y = 3$.

An alternative way to think of this fact is to notice that every point on the line in Figure 12(a) has $y$-coordinate 3, so the equation $y = 3$ describes each point on the line. It is therefore the equation of the line.
What about vertical lines? A vertical line has no gradient, so it doesn’t have an equation of the form \( y = mx + c \). Vertical lines are the only lines that do not have equations of this form.

However, every point on a vertical line has the same \( x \)-coordinate, so the line has an equation of the form \( x = d \). The constant \( d \) is the \( x \)-intercept. For example, consider the vertical line shown in Figure 12(b). Every point on this line has \( x \)-coordinate 4, so the equation \( x = 4 \) describes each point on the line and is therefore the equation of the line.

**Equations of horizontal and vertical lines**

The horizontal line with \( y \)-intercept \( c \) has equation \( y = c \).

The vertical line with \( x \)-intercept \( d \) has equation \( x = d \).

In particular, the equation of the \( x \)-axis is \( y = 0 \), and the equation of the \( y \)-axis is \( x = 0 \).

**Drawing a line from its equation**

In Section 1 you saw how to plot the graph of an equation by finding several points that satisfy the equation, and drawing a smooth line or curve through them. If you can recognise an equation as the equation of a straight line, then to draw this line you just need to find two points that satisfy the equation, and draw the straight line through them.

One way to find two suitable points is to choose two values of \( x \), and use the equation to find the corresponding values of \( y \). You should try to choose values of \( x \) that are reasonably far apart and that lead to simple
calculations. For example, if the equation is $y = \frac{1}{3}x + 2$ then you might choose $x = 0$ and $x = 3$, to avoid fractions. An alternative way to find two suitable points is to find the $x$- and $y$-intercepts.

Drawing a horizontal or vertical line from its equation is even more straightforward. To draw the line with equation $y = c$, you just mark the $y$-intercept $c$ and draw the horizontal line through it. Similarly, to draw the line with equation $x = d$, you just mark the $x$-intercept $d$ and draw the vertical line through it.

**Activity 6  Drawing lines from their equations**

Draw the straight lines with the following equations.

(a) $y = \frac{1}{3}x + 2$  (b) $y = -2x + 4$  (c) $y = \frac{7}{2}$  (d) $x = -3$

**Finding the equation of a straight line**

Every straight line that you can draw in a plane, with the exception of any vertical line, has a gradient and a $y$-intercept, and hence has an equation of the form $y = mx + c$.

If you know the gradient and the $y$-intercept of the line, then you can immediately write down the equation of the line. For example, the line with gradient 3 and $y$-intercept $-5$ has equation $y = 3x - 5$.

Sometimes, however, you might know different information about a line. The next example demonstrates a method for finding the equation of a line when you know its gradient and a point on it.

---

**Example 4  Finding the equation of a line from its gradient and a point on it: Method 1**

Find the equation of the line that has gradient $-6$ and passes through the point $(-1, 4)$.

**Solution**

- A straight line has an equation of the form $y = mx + c$, where $m$ is the gradient.

The equation is of the form $y = -6x + c$.

- The point $(-1, 4)$ lies on the line, so this point must satisfy the equation.

Substituting $x = -1$ and $y = 4$ into the equation gives

$$4 = -6 \times (-1) + c$$

$$4 = 6 + c.$$
Solve this equation to find c.

\[-2 = c\]

So the equation of the line is \( y = -6x - 2 \).

(Check: substituting \( x = -1 \) into the equation \( y = -6x - 2 \) gives \( y = -6 \times (-1) - 2 = 4 \), so the point \((-1, 4)\) lies on the line, as expected.)

Here’s an alternative way to find the equation of a straight line from its gradient and a point on it. Suppose that the gradient is \( m \) and the point is \((x_1, y_1)\). If \((x, y)\) is any other point on the line, then, by the formula for the gradient on page 127,

\[
m = \frac{y - y_1}{x - x_1}.
\]  

(3)

Rearranging this equation gives

\[
y - y_1 = m(x - x_1).
\]  

(4)

Equation (3) does not hold when \((x, y) = (x_1, y_1)\), since that would require division by zero, but it does hold for all other points \((x, y)\) on the line. However, the rearranged equation, equation (4), holds for all points \((x, y)\) on the line, including \((x, y) = (x_1, y_1)\), since for this point both sides are equal to zero. So equation (4) is the equation of the line. This fact is summarised below.

The equation of the straight line with gradient \( m \) that passes through the point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1).
\]

Example 5Finding the equation of a line from its gradient and a point on it: Method 2.

Find the equation of the line that has gradient \(-2\) and passes through the point \((1, 4)\).

Solution

Substitute \( m = -2 \), \( x_1 = 1 \) and \( y_1 = 4 \) into the equation in the box above, and simplify it.

The equation of the line is

\[
y - 4 = -2(x - 1).
\]
It can be simplified as follows:
\[ y - 4 = -2x + 2 \]
\[ y = -2x + 6. \]
So the equation of the line is \( y = -2x + 6 \).
(Check: substituting \( x = 1 \) into \( y = -2x + 6 \) gives \( y = -2 \times 1 + 6 = 4 \), so the point \((1, 4)\) lies on the line, as expected.)

Sometimes you might want to find the equation of a line from the coordinates of two points on it. If the two points have the same \( x \)-coordinate or the same \( y \)-coordinate, then you can immediately write down the equation of the horizontal or vertical line that they lie on. Otherwise, you can use the coordinates of the two points to calculate the gradient of the line, and then apply the method of Example 4 or Example 5.

**Activity 7  Finding the equations of lines**

Find the equations of the following lines.
(a) The line through the point \((2, 1)\) with gradient 3
(b) The line through the points \((2, 3)\) and \((4, 5)\)
(c) The line with \(y\)-intercept 3 and gradient 2
(d) The line with \(x\)-intercept 2 and gradient \(-3\)
(e) The vertical line that passes through the point \((1, 0)\)
(f) The line through the points \((-2, 3)\) and \((4, 3)\)

### 2.3 Parallel and perpendicular lines

Two straight lines are **parallel** if they never cross, even when extended infinitely far in each direction, as illustrated in Figure 13.
In Activity 4 on page 128, the line through the points $A$ and $D$ is parallel to the line through the points $B$ and $C$. You may have noticed from your solution to this activity that these two lines have the same gradient. In general, saying that two non-vertical lines are parallel means the same as saying that they have the same gradient. Any two vertical lines are also parallel.

Two lines are **perpendicular** if they are at right angles to each other. You’d expect the gradients of two perpendicular lines to be related in some way – but how? To work this out, consider any two perpendicular lines that are not parallel to the axes, as illustrated in Figure 14.

Let $A$ be the point where the lines cross, let $B$ be any point that lies on one of the lines and is above and to the right of $A$, and let $C$ be the point on the other line that is obtained by rotating $B$ anticlockwise through a quarter turn about $A$, as shown in Figure 14.

Let’s denote the run from $A$ to $B$ by $p$, and the rise from $A$ to $B$ by $q$. Both $p$ and $q$ are positive numbers. If you rotate the right-angled triangle with hypotenuse $AB$ in Figure 14 anticlockwise through a quarter turn about $A$, then it will lie exactly on top of the right-angled triangle with hypotenuse $AC$. It follows that the run from $A$ to $C$ is $-q$ (it’s negative because $C$ lies to the left of $A$), and the rise from $A$ to $C$ is $p$. 

---

**Figure 13** Two parallel lines

**Figure 14** Two perpendicular lines
Hence

gradient of the line through \( A \) and \( B = \frac{q}{p} \)

and

gradient of the line through \( A \) and \( C = \frac{p}{-q} = -\frac{p}{q} \).

Multiplying these two gradients together gives

\[
\frac{q}{p} \times \left( -\frac{p}{q} \right) = -1.
\]

So we have the following fact.

**Gradients of perpendicular lines**

The gradients of any two perpendicular lines (not parallel to the axes) have product \(-1\).

Note that if two perpendicular lines are parallel to the axes, then one of them is parallel to the \( y \)-axis and hence has undefined gradient.

**Activity 8 Finding the equation of a line perpendicular to another line**

(a) Find the gradient of a line perpendicular to the line \( y = 3x + 5 \).

(b) Hence find the equation of the line that is perpendicular to \( y = 3x + 5 \) and passes through the point \((2,1)\).

**2.4 Applications of straight-line graphs**

As mentioned in the introduction to this unit, the relationship between two real-life quantities can often be modelled by an equation in two variables. If the equation representing this relationship is of the form \( y = mx + c \), and hence has a straight-line graph, then we say that the model is **linear**.

In this subsection, you’ll look at some examples of linear models, and practise working with them. It’s particularly important that you work carefully through this subsection, as it underpins **calculus**, which you’ll start studying in Unit 6.
First, consider the graph in Figure 15, which represents the journey of a car along a road. It shows the relationship between the time that has elapsed since the car began its journey, and the distance that it has travelled since the start of the journey. The axes of the graph are labelled with ‘time’ and ‘distance’, as well as with \( t \) and \( s \), the letters chosen to represent these quantities. The units in which each quantity is measured, kilometres (abbreviated to km) and hours (abbreviated to h), are also included. A graph like this, in which distance is plotted against time, is known as a distance–time graph.

The variables \( s \) and \( t \) are often used for distance and time, respectively, so you need to be careful to avoid possible confusion if the unit \( s \) (seconds) is used for time.

\[
\begin{align*}
\text{distance, } s & \ (\text{km}) \\
\text{time, } t & \ (\text{h}) \\
\end{align*}
\]

**Figure 15** A distance–time graph for the journey of a car

This graph includes only non-negative values of \( t \) and \( s \), as the times elapsed and the distances travelled since the start of the journey are all non-negative.

You can calculate the gradient of the graph in Figure 15 by choosing two points on it in the usual way. The two points marked on the graph have coordinates \((0, 0)\) and \((3, 240)\), so

\[
\text{gradient} = \frac{(240 - 0) \text{ km}}{(3 - 0) \text{ h}} = 80 \text{ km/h}.
\]

Notice that, because the numbers on the axes of the graph have units, the gradient also has units. Since the rise is measured in kilometres (km) and the run is measured in hours (h), the units of the gradient are kilometres divided by hours, that is, kilometres per hour (km/h). In general, the units of the gradient of a graph are the units on the vertical axis divided by the units on the horizontal axis.

The units are shown in the calculation above to demonstrate this fact, but in general it’s not necessary to include units in the calculation of a gradient. You just need to state the units in the final answer.

Units such as km/h, which are obtained by combining simpler units (in this case km and h), are called **derived units**.

The fact that the gradient of the graph in Figure 15 is 80 km/h tells you that the distance travelled by the car changes by 80 kilometres for each hour of the journey. That is, it tells you that the car travels 80 kilometres
in each hour, or, in other words, that the speed of the car is 80 kilometres per hour. The fact that the graph is a straight line tells you that the car is travelling at a constant speed.

In general, if a graph is a straight line, then it means that the quantity on the vertical axis is changing at a constant rate with respect to the quantity on the horizontal axis. The gradient of the graph tells you how many units the quantity on the vertical axis changes for every one unit that the quantity on the horizontal axis changes. In other words, the gradient of the graph is the rate of change of the quantity on the vertical axis with respect to the quantity on the horizontal axis. So, for example, the gradient of a distance–time graph is the rate of change of distance with respect to time, which is speed.

Here’s another example of a linear model. Consider the graph in Figure 16. It represents the change in the concentration of a chemical as it undergoes a chemical reaction over time. The concentration is denoted by \( c \) and measured in a unit called the molar (M), and time is denoted by \( t \) and measured in seconds (s). A chemical reaction in which the concentration falls linearly, as in Figure 16, is called a zero-order reaction.

![Figure 16](image)

**Figure 16** The relationship between the concentration of a reacting chemical and time

The gradient of the graph in Figure 16 is negative, because the concentration of the chemical decreases as time goes on. Notice that the graph includes values of \( t \) only between 0 and 60, and the corresponding values of \( c \). This is because the times since the beginning of the reaction are all positive, and after 60 seconds the concentration of the chemical has decreased to 0, so the model does not apply for later times.

The gradient of the graph in Figure 16 can be calculated as follows. The two points marked in Figure 16 are \((0, 2)\) and \((60, 0)\), so

\[
\text{gradient} = \frac{0 - 2}{60 - 0} = -\frac{2}{60} = -\frac{1}{30} \text{ M/s}.
\]

This tells you that the rate of change of the concentration with respect to time is approximately \(-0.03 \text{ M/s}\). In other words, in each second that passes, the concentration of the chemical decreases by approximately 0.03 M.
In the next activity, you’re asked to find and interpret the gradients of two straight-line graphs.

**Activity 9  Calculating gradients of real-life graphs**

For each of the graphs below, find the gradient in appropriate units and explain what the gradient represents.

Graph (a) represents the relationship between the number $n$ of people attending a meeting and the cost $C$ (in £) of hiring the meeting room (including lunch and refreshments for the attendees).

Graph (b) represents the relationship between the price $p$ (in £/kg) charged for building sand and the quantity $q$ (in kg) purchased by a customer.

(a) 

<table>
<thead>
<tr>
<th>Cost, $C$ (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
</tr>
<tr>
<td>200</td>
</tr>
<tr>
<td>300</td>
</tr>
<tr>
<td>400</td>
</tr>
<tr>
<td>500</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of people, $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
</tbody>
</table>

(b) 

<table>
<thead>
<tr>
<th>Price per kilogram, $p$ (£/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
</tr>
<tr>
<td>0.04</td>
</tr>
<tr>
<td>0.06</td>
</tr>
<tr>
<td>0.08</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity, $q$ (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
</tr>
<tr>
<td>400</td>
</tr>
<tr>
<td>600</td>
</tr>
<tr>
<td>800</td>
</tr>
<tr>
<td>1000</td>
</tr>
</tbody>
</table>

The intercepts of real-life graphs often have practical interpretations too. For example, look back at the graph in Figure 16, which represents the decrease in the concentration of a reacting chemical over time. The intercept on the $c$-axis is 2M, which is the concentration of the chemical at the start of the chemical reaction. The intercept on the $t$-axis is 60s, which is the time at which the concentration of the chemical falls to zero.

**Activity 10  Interpreting intercepts of real-life graphs**

Look back at the graph in Activity 9(a). State the vertical intercept, and explain what it means.
The intercepts of real-life graphs don’t always have useful meanings. For example, look back at the graph in Activity 9(b). Notice that it has been drawn as a line segment that doesn’t cross either of the axes. This is because the model isn’t valid for the values of \( q \) and \( p \) that don’t correspond to points on this line segment. In particular, it isn’t valid when \( q = 0 \) and \( p = 0 \), so the intercepts have no meaning in this case.

When you’re working with a linear model, it’s usually helpful to use the equation of the associated straight-line graph. You can often find the equation using the methods that you practised earlier in this section – you use the variables that represent the real-life quantities in place of the standard variables \( x \) and \( y \). For example, for the distance–time graph in Figure 15, the variables on the horizontal and vertical axes are \( t \) and \( s \), respectively, so you use \( t \) in place of \( x \) and \( s \) in place of \( y \). The gradient of this graph is 80 km/h and the \( y \)-intercept is 0 km, so the equation of the graph is

\[
s = 80t. \tag{5}
\]

An alternative way to obtain the equation of the graph in this case is to simply use the familiar relationship

\[
\text{distance travelled} = \text{constant speed} \times \text{time elapsed}.
\]

When you use relationships like this, it’s important to remember that they’re valid only if the units in which the quantities are measured are consistent. For example, for the relationship above, the units in which time is measured must be the same as the units of time contained within the derived units in which speed is measured. Here are some examples of consistent sets of units:

- time in seconds, speed in metres per second, distance in metres
- time in minutes, speed in metres per minute, distance in metres
- time in hours, speed in kilometres per hour, distance in kilometres.

The units used for the quantities in equation (5) are consistent since the number 80 represents the speed of the car in km/h, the variable \( t \) represents time in hours, and the variable \( s \) represents distance in km.

As another example of consistent units, consider the equation representing the line in Figure 16 on page 140, which is

\[
c = -\frac{1}{30} t + 2. \tag{6}
\]

The units used for the quantities in this equation are consistent since all three terms, \( c \), \(-\frac{1}{30} t \) and 2, have the same units, namely molars (M). Note that the gradient \(-\frac{1}{30}\) is measured in M/s, and the time \( t \) in seconds.
Activity 11  Finding equations of real-life graphs

(a) Using your answers to Activity 9, find the equations of the graphs in Activity 9.

(b) Use these equations to find the following.

(i) The maximum number of people that can be accommodated in the meeting room, if the maximum budget for the meeting is £560.

(ii) The price per kilogram of building sand that corresponds to a quantity of 500 kg.

Displacement and velocity

When you’re considering a moving object, such as a car travelling along a road, it’s often helpful to consider not the distance that the object has travelled, but its displacement from a particular point.

To see what this means, consider any object that’s moving along a straight line – we’ll consider only motion along a straight line here, for simplicity. We choose any point on the straight line to be a reference point and we choose one of the two directions along the line to be the positive direction. Then the object’s displacement from the reference point is its distance from that point, with a positive or negative sign to indicate the direction from that point.

For example, consider the straight line in Figure 17. The reference point has been chosen to be the point marked $R$, and the positive direction has been chosen to be rightwards. An object at position $A$ has a displacement of $3\text{ cm}$, while an object at position $B$ has a displacement of $-5\text{ cm}$.

![Figure 17](positions.png)

Figure 17  Positions along a straight line

Imagine placing your pen tip at $R$, moving it to $A$, and then to $B$. The distance moved by the pen tip is the distance from $R$ to $A$, plus the distance from $A$ to $B$, which is $3\text{ cm} + 8\text{ cm} = 11\text{ cm}$. However, the final displacement of the pen tip is $-5\text{ cm}$, since it is at $B$.

To see how the idea of displacement works in practice, suppose that a car is travelling along a straight road, and that a particular reference point $R$
on the road and direction along it have been chosen, as illustrated in Figure 18.

**Figure 18** A reference point and a direction on a straight road

The graph in Figure 19 shows the relationship between the time that has elapsed since the start of the car’s journey, and the car’s displacement from the reference point. You can see that the car has displacements of 60 km and −30 km at the start and end of its journey, respectively, and that it drives past the reference point 1 hour after the start of its journey. A graph like the one in Figure 19, in which displacement is plotted against time, is known as a displacement–time graph.

**Figure 19** A displacement–time graph for a car’s journey

The gradient of the graph in Figure 19 is −60 km/h. This tells you that in each hour the displacement of the car changes by −60 km. In other words, the car is moving at a speed of 60 km/h in the negative direction, that is, in the direction opposite to the direction chosen as the positive direction in Figure 18.

The gradient of the displacement–time graph of an object moving in a straight line is called the velocity of the object. In other words, the velocity of an object is its rate of change of displacement with respect to time. So the velocity of an object that’s moving along a straight line is the
same as its speed, except that it has a positive or negative sign to indicate the direction in which the object is moving along the line.

**Activity 12  Working with a displacement–time graph**

The displacement–time graph below, which consists of three line segments, represents a woman’s walk along a straight path. The woman walks at a constant speed to a bench, sits there for some minutes, and then returns, again at a constant speed, to her starting point. The reference point has been chosen to be the point where she begins her walk, the positive direction has been chosen to be the direction in which she first walks, and time is measured from the time when she begins her walk.

(a) What is the displacement of the bench from the woman’s starting point?

(b) How long does the woman remain at the bench?

(c) Calculate the woman’s velocity as she walks to the bench.

(d) Calculate her velocity as she walks back to her starting point.

(e) What is the woman’s speed as she walks to the bench, and what is her speed as she walks back?

(f) Find the equation of the line segment that represents the first part of the woman’s walk.

(g) Use the equation that you found in part (f) to find what the woman’s displacement would be after 50 minutes if she hadn’t stopped at the bench but had instead carried on walking at the same speed and in the same direction.

Derived units, such as km/h, are often written using *index notation*. For example, the unit km/h can be written as km h\(^{-1}\), since 1/h = h\(^{-1}\). This way of writing derived units makes no difference to the way that you read them: for example, km h\(^{-1}\) is read as ‘kilometres per hour’. Derived units are often written in index notation throughout the rest of this module.
3 Intersection of lines

In this section you’ll revise how to find the point at which two lines cross. This point is called the point of intersection of the lines.

3.1 Simultaneous equations

Here is an example where we want to find the point of intersection of two lines. Consider Figure 20, which shows the displacement–time graphs for two cars travelling on the same road, drawn on the same coordinate axes. Both graphs are straight lines. Notice that, as time goes on, the displacement of one car increases and the displacement of the other car decreases, which tells you that they’re travelling in opposite directions. At the point of intersection of the lines, the displacements of the two cars are equal at the same time.

So this point corresponds to the time and displacement when the two cars pass each other. How can we find this point of intersection?

Figure 20  Displacement–time graphs for two cars travelling in opposite directions on the same road

You could find approximate values for the time at which the two cars pass each other, and the displacement at which this happens, by reading off the point of intersection from the graph. Alternatively, you can find accurate values by working with the equations of the lines and using algebra.
The equation of the displacement–time graph of the first car is
\[ s = 60t - 20, \]  
where \( s \) is the displacement in km and \( t \) is the time in hours. The corresponding equation for the second car is
\[ s = -85t + \frac{395}{2}. \]  
The coordinates \((t, s)\) of the point of intersection must satisfy equation (7), since the point lies on the graph of that equation, and must also satisfy equation (8), since it also lies on the graph of this equation.

So to find the point of intersection you have to find a pair of values of \( t \) and \( s \) that satisfy both equations. The process of finding these values is known as solving the equations **simultaneously**, and in this context the two equations are called **simultaneous equations**. Since the equations are **linear** equations, they’re **simultaneous linear equations**. The variables \( t \) and \( s \) are unknowns, and any pair of values of \( t \) and \( s \) that satisfy both equations is called a **solution** of the simultaneous equations.

Simultaneous equations are not restricted to two equations and two unknowns; systems containing larger numbers of equations and unknowns frequently arise in practical situations.

In the next subsection you’ll revise two methods for solving simultaneous linear equations in two unknowns. You’ll meet a third method in Unit 9.

### 3.2 Solving simultaneous equations

In this subsection we’ll assume that the simultaneous equations that you want to solve are linear and have been rearranged so that the terms in the unknowns are on the left, and the constant terms are on the right. This is a standard way to present simultaneous equations.

For example, rearranging equations (7) and (8) in this way, and also clearing the fractions in equation (8), gives the pair of equations
\[ s - 60t = -20, \]
\[ 2s + 170t = 395. \]

Here’s the first method for solving simultaneous linear equations.
Substitution method

This is described in the following strategy and illustrated in Example 6.

**Strategy:**

To solve simultaneous equations: substitution method

1. Rearrange one of the equations, if necessary, to obtain a formula for one unknown in terms of the other.
2. Use this formula to substitute for this unknown in the other equation.
3. You now have an equation in one unknown. Solve it to find the value of that unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

---

**Example 6  Solving simultaneous equations by substitution**

Use the substitution method to solve the following simultaneous equations.

\[
\begin{align*}
    s - 60t &= -20 \\
    2s + 170t &= 395
\end{align*}
\]

**Solution**

1. Label the equations, so you can refer to them easily.

   The equations are

   \[
   \begin{align*}
   s - 60t &= -20, \quad (9) \\
   2s + 170t &= 395. \quad (10)
   \end{align*}
   \]

2. Rearrange one of the equations to express one unknown in terms of the other. The simplest choice here is to rearrange equation (9) to express \(s\) in terms of \(t\). Any other choice of equation or unknown leads to fractions.

   Rearranging equation (9) gives

   \[
   s = 60t - 20. \quad (11)
   \]

3. Use this formula to substitute for \(s\) in the other equation.

   Substituting in equation (10) gives

   \[
   2(60t - 20) + 170t = 395.
   \]
Solve this equation to find $t$.

\[120t - 40 + 170t = 395\]
\[290t = 435\]
\[t = \frac{435}{290} = \frac{3}{2}\]

Substitute this value of $t$ into an equation containing $s$, and solve it to find $s$.

Substituting into equation (11) gives

\[s = 60t - 20\]
\[= 60 \times \frac{3}{2} - 20\]
\[= 90 - 20\]
\[= 70.\]

So the solution is $s = 70$, $t = \frac{3}{2}$.

It’s a good idea to check that these values are correct by substituting them into the original equations.

When you solve simultaneous equations, it’s fine to label them (1) and (2), every time. If you want to refer to another equation in your working, then you can label it (3), and so on. (The simultaneous equations in Example 6 are labelled (9) and (10) just so that they’re in sequence with the other labelled equations in this unit.)

**Activity 13 Solving simultaneous equations by substitution**

Use the substitution method to solve the following pairs of simultaneous equations.

(a) $s - 5t = -3$
(b) $x + 4y = 2$

\[s + 3t = 13\]
\[2x + 5y = 3\]

Here’s a variation of the substitution method that’s sometimes useful. You’ve seen that the first step of the method is to rearrange one of the equations to express one unknown in terms of the other. If instead you start by rearranging both equations to obtain two formulas, each of which expresses one unknown (the same in both cases) in terms of the other unknown, then you can obtain an equation in one unknown by equating the right-hand sides of both formulas. You can then proceed in the usual way.
Unit 2  Graphs and equations

For example, notice that both the equations of the lines in Figure 20 were obtained originally in equations (7) and (8) as formulas for $s$ in terms of $t$:

$$s = 60t - 20,$$
$$s = -85t + \frac{395}{2}.$$

You can equate the two right-hand sides to give the following equation in $t$:

$$60t - 20 = -85t + \frac{395}{2}.$$

You can then solve this equation to find $t$, and substitute this value into one of the original equations to find $s$.

Elimination method

Often when you want to solve a pair of simultaneous equations, it’s difficult to use the substitution method without introducing fractions. In most cases it’s better to use the method described in the following strategy, which is illustrated in Example 7.

Strategy:
To solve simultaneous equations: elimination method

1. Multiply one or both of the equations by suitable numbers, if necessary, to obtain two equations that can be added or subtracted to eliminate one of the unknowns.

2. Add or subtract the equations to eliminate this unknown.

3. You now have an equation in one unknown. Solve it to find the value of that unknown.

4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

Example 7  Solving simultaneous equations by elimination

Use the elimination method to solve the following simultaneous equations.

$$5u - 40v = 155$$
$$2u + 9v = 12$$
### Solution

#### Label the equations.

The equations are

\[ 5u - 40v = 155, \quad (12) \]
\[ 2u + 9v = 12. \quad (13) \]

#### Multiply the first equation by 2, and multiply the second equation by 5, to obtain two equations in which \( u \) has the same coefficient.

(Alternatively, you could multiply the first equation by 9 and the second equation by 40, to obtain two equations in which \( v \) can be eliminated, but that involves harder arithmetic.)

Multiplying equation (12) by 2 and equation (13) by 5 gives

\[ 10u - 80v = 310, \quad (14) \]
\[ 10u + 45v = 60. \quad (15) \]

#### Subtract equation (14) from equation (15) to eliminate \( u \).

Subtracting equation (14) from equation (15) gives

\[ 10u - 10u + 45v + 80v = 60 - 310 \]

Solve this equation to find \( v \).

\[ 125v = -250 \]
\[ v = -2. \]

Substitute this value of \( v \) into an equation containing \( u \), say equation (12), and solve it to find \( u \).

Hence

\[ 5u = 40v + 155 \]
\[ = 40 \times (-2) + 155 \]
\[ = -80 + 155 \]
\[ = 75 \]
\[ u = \frac{1}{5} \times 75 = 15. \]

So the solution is \( u = 15, v = -2. \)

The ‘check’ referred to in the strategy has not been included in the solution to Example 7. You should check your answers wherever possible, but you don’t have to write the check down as part of your solution.

The aim of the first step in the solution to Example 7 was to find two equations in which \( u \) has the same coefficient. This was so that subtracting the equations will eliminate \( u \). An alternative strategy in the first step is
to find two equations in which the coefficients of \( u \) are \textit{negatives of each other}, such as

\[-10u + 80v = -310,\]
\[10u + 45v = 60.\]

(These equations are obtained by multiplying the first of the original equations by \(-2\) and the second by \(5\).) Then \textit{adding} the equations will eliminate \( u \).

\textbf{Activity 14} \hspace{1em} \textit{Solving simultaneous equations by elimination}

Use the elimination method to solve the following pairs of simultaneous equations.

(a) \[ s + 6t = 20 \]
\[ 2s + 7t = 35 \]
(b) \[ 2x + 3y = -5 \]
\[ 3x - 2y = 12 \]
(c) \[ 3u - v = -\frac{5}{2} \]
\[ 2u + 5v = 21 \]

\subsection*{3.3 The number of solutions of simultaneous equations}

Each pair of simultaneous linear equations in the last subsection had exactly one solution, but it’s also possible for a pair of simultaneous linear equations to have no solutions, or infinitely many solutions.

To see this, consider the lines that are the graphs of the two equations. They might intersect in a single point, or they might be parallel, or they might be the same line. The three possibilities are illustrated in Figure 21.

\textbf{Figure 21} \hspace{1em} Three possible configurations of two lines

If the two lines have different gradients, as illustrated in Figure 21(a), then they intersect at a single point. So there is exactly one solution.

If the two lines have the same gradient but different \( y \)-intercepts, as illustrated in Figure 21(b), then they are parallel and hence do not
intersect. In this case, you will not be able to solve the simultaneous equations. For example, the simultaneous equations

\[
\begin{align*}
y - 2x &= 3, \\
y - 2x &= 4
\end{align*}
\]

have no solution; no matter what values of \(x\) and \(y\) are chosen, it is impossible for \(y - 2x\) to be equal to both 3 and 4.

If the two lines have the same gradient and the same \(y\)-intercept, as illustrated in Figure 21(e), then the graphs of the two equations are identical. So every point that satisfies the first equation also satisfies the second equation, and hence there are infinitely many solutions. This situation occurs if one equation can be obtained by rearranging the other. For example, consider the simultaneous equations

\[
\begin{align*}
y - 2x &= 3, \\
2y - 4x &= 6
\end{align*}
\]

The second equation is obtained by multiplying the first equation by 2, so they represent the same line, and hence they have infinitely many solutions.

### Activity 15 Finding the number of solutions of simultaneous equations

Determine whether each of the following pairs of simultaneous equations has one, infinitely many or no solutions. If there is one solution, find it.

(a) \(y - 3x = -2\) \hspace{1cm} (b) \(y = 4x - 5\) \hspace{1cm} (c) \(4y = 2x + 6\)

\[
\begin{align*}
2y + x &= 10 \\
2y - 8x &= 10 \\
2y - x &= 3
\end{align*}
\]

In practice, when you’re solving simultaneous equations there’s no need to start by investigating the number of solutions. You can just try to solve the equations, and see what happens!

### 4 Quadratics

In this section we turn our attention to equations of the form

\[
y = ax^2 + bx + c, \tag{16}
\]

where \(a\), \(b\) and \(c\) are constants and \(a \neq 0\). An expression of the form of the right-hand side of this equation is called a \textbf{quadratic expression}, or simply a \textbf{quadratic}. An equation of the form

\[
a x^2 + bx + c = 0, \tag{17}
\]

where \(a\), \(b\) and \(c\) are constants and \(a \neq 0\), is called a \textbf{quadratic equation}. Since equation (17) is obtained by putting \(y = 0\) in equation (16), its solutions are the \(x\)-intercepts of the graph of equation (16).
In the first subsection of this section, you’ll revise the shapes of the graphs of equations of form (16). Then in the next few subsections you’ll revise two useful techniques for rearranging quadratic expressions, namely factorisation and completing the square. You’ll see how each of these techniques leads to a method for solving quadratic equations. You’ll also revise a third method for solving quadratic equations, the quadratic formula.

It’s important to make sure that you’re familiar with all three of these methods for solving quadratic equations, as they’re all used later in the module. You’ll also need to use the techniques of factorisation of quadratics and completing the square for purposes other than solving quadratic equations.

Finally in this section you’ll learn how to sketch the graphs of equations of form (16), and you’ll see a few applications of quadratics.

Note that the reason for the condition ‘\(a \neq 0\)’ in the definitions above is that, if \(a = 0\), then the expression \(ax^2 + bx + c\) reduces to \(bx + c\), which is a linear expression.

### 4.1 Quadratic graphs

You have already seen one quadratic graph in this unit. In Activity 2 on page 123 you were asked to draw the graph of \(y = x^2 + 2x + 2\) by first plotting points. In the next activity you can explore the shape of the graph of the equation \(y = ax^2 + bx + c\) for various values of \(a\), \(b\) and \(c\).

<table>
<thead>
<tr>
<th>Activity 16</th>
<th>Investigating quadratic graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use the Quadratic graphs applet to investigate how the shape of the graph of the equation (y = ax^2 + bx + c) changes as you vary the values of (a), (b) and (c).</td>
<td></td>
</tr>
</tbody>
</table>

In Activity 16, you should have seen that no matter what the values of \(a\), \(b\) and \(c\) are (as long as \(a \neq 0\)), the graph of the equation \(y = ax^2 + bx + c\) always has the same type of shape, which is called a parabola. If \(a\) is positive, then the parabola is u-shaped, as shown in Figure 22(a). If \(a\) is negative, then it is n-shaped, as shown in Figure 22(b). In both cases, the parabola has a vertical axis of symmetry.
The word ‘parabola’ was first used for curves like these by the Greek geometer and astronomer Apollonius of Perga (262BC–190BC), though the shape itself was discovered even earlier.

You should have seen that changing the values of $a$, $b$ and $c$ can change the position of the parabola, and can stretch or squash it in a direction parallel to one of the axes. In particular, as you might have expected from your study of straight lines, changing the value of $c$ on its own moves the parabola vertically, but keeps the shape the same.

The lowest point on a u-shaped parabola, or the highest point on an n-shaped parabola, is called the vertex of the parabola. The parabola gets steeper and steeper on each side of the vertex, but never becomes vertical.

A u-shaped or n-shaped parabola can have two, one or no $x$-intercepts, as illustrated for u-shaped parabolas in Figure 23. It always has exactly one $y$-intercept, because there is exactly one value of $y$ for each value of $x$, including $x = 0$. 

**Figure 22** Typical graphs of $y = ax^2 + bx + c$ when (a) $a$ is positive (b) $a$ is negative

**Figure 23** A quadratic graph can have two, one or no $x$-intercepts
Parabolas have many practical uses, mainly due to the reflecting properties of surfaces with parabolic cross-sections. If such a surface is made of a suitable reflecting material, then all light waves or radio waves that arrive at it by travelling along a direction parallel to its axis of symmetry are reflected to pass through a single point, known as the focus. Similarly, any wave that travels from the focus to the surface is reflected to travel along a line parallel to the axis of symmetry. Parabolic reflecting surfaces can be found in car headlights, reflecting and radio telescopes, and satellite dishes. The path of a ball thrown through the air is also parabolic (as long as the effects of air resistance are negligible).

The parabolic dish of a radio telescope and the parabolic path of a ball

4.2 Factorising quadratic expressions

In this subsection you’ll revise how to factorise a quadratic expression. This usually involves more than simply taking out a common factor, which you revised in Unit 1.

If you take any two linear expressions and multiply them together, then you get a quadratic expression. For example,

\[(2x + 3)(x - 2) = 2x^2 - x - 6.\]

**Factorisation** of a quadratic is the reverse of this process – it means writing the quadratic as the product of two linear expressions. For example, factorising the quadratic \(2x^2 - x - 6\) means writing it as \((2x + 3)(x - 2)\) or, equivalently, as \((x - 2)(2x + 3)\).

Here, you’ll see how to factorise certain quadratics with integer coefficients. For example, the quadratic \(2x^2 - x - 6\) is of this type, as its coefficients 2, -1 and -6 are integers.

You’ll see how to factorise such a quadratic into a product of two linear expressions whose coefficients are also integers, where this is possible. We’ll refer to factorisation of this sort as factorisation using integers. For example, you’ve just seen that the quadratic \(2x^2 - x - 6\) can be factorised
using integers, since all the coefficients in the expression \((2x + 3)(x - 2)\) are integers. However, many quadratics with integer coefficients can’t be factorised using integers.

**Factorising quadratics in which \(x^2\) has coefficient 1**

We’ll begin by looking at quadratics in which \(x^2\) has coefficient 1, such as
\[
x^2 - 2x - 15 \quad \text{and} \quad x^2 - 8x + 12.
\]

These are usually easier to factorise than other quadratics.

If it’s possible to factorise a quadratic of this type using integers, then the factorisation must be of the form
\[
(x \quad )(x \quad ),
\]
where the two gaps are filled by constant terms. Such an arrangement of brackets with gaps for the constants is called a ‘framework’ in this unit.

To factorise the quadratic, you have to find two numbers (each a positive or negative integer, or zero) to fill the gaps in the framework (18), such that when you multiply out the brackets you get the quadratic that you started with. For example, if your quadratic is \(x^2 - 2x - 15\), then the two numbers that you’re looking for are +3 and −5, because
\[
(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.
\]

You can use two facts to work out what the two numbers must be.

First, the two numbers must multiply together to give the constant term in the quadratic. To see why, consider what happens when you multiply out an expression of form (18). For example,
\[
(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.
\]

The two numbers are multiplied to give the constant term in the quadratic.

Second, the two numbers must add together to give the coefficient of \(x\) in the quadratic. To see why, again consider what happens when you multiply out an expression of form (18).

For example,
\[
(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15.
\]

The two numbers become coefficients of \(x\). They are added to give the overall coefficient of \(x\).
The next example demonstrates an efficient strategy for using these facts to find the two numbers.

This strategy is quick to use once you’ve had some practice with it, and it can be adapted to deal with quadratics in which $x^2$ does not have coefficient 1, as you’ll see shortly. However, there are other strategies – there’s a different one in the document An alternative strategy for factorising quadratics on the website that you might like to look at, and you may have learned another one elsewhere. It’s fine to use any strategy that works for you.

You might find it easier to understand the strategy if you hear someone explain it to you, so it’s a good idea to watch the tutorial clip for Example 8. The strategy is summarised after the example.

---

**Example 8  Factorising quadratics in which $x^2$ has coefficient 1**

Factorise the following quadratics.

(a) $x^2 - 8x + 12$  
(b) $x^2 + 2x - 15$

**Solution**

(a) Start by writing out the framework. The numbers to go in the gaps must have product 12 (the constant term) and sum $-8$ (the coefficient of $x$). List the factor pairs of 12:

$1, 12$; $-1, -12$; $2, 6$; $-2, -6$; $3, 4$; $-3, -4$.

Find a pair in this list with sum $-8$. The only such pair is $-2, -6$.

$x^2 - 8x + 12 = (x - 2)(x - 6)$

You can check the answer by multiplying out the brackets.

(b) Write out the framework. The numbers to go in the gaps must have product $-15$ and sum 2.

List the factor pairs of $-15$:

$1, -15$; $-1, 15$; $3, -5$; $-3, 5$.

The only pair with sum 2 is $-3, 5$.

$x^2 + 2x - 15 = (x - 3)(x + 5)$
The strategy demonstrated in Example 8 is summarised below.

**Strategy:**
To factorise a quadratic of the form $x^2 + bx + c$

1. Start by writing
   $$x^2 + bx + c = (x \quad )(x \quad ).$$
2. Find the factor pairs of $c$ (including both positive and negative factors).
3. Choose a factor pair with sum $b$.
4. Write your factor pair $p, q$ in position:
   $$x^2 + bx + c = (x + p)(x + q).$$

For some quadratics, this strategy doesn’t give a factorisation. When this happens, it means that the quadratic can’t be factorised using integers. For example, if you try to apply the strategy to the quadratic $x^2 + 2x + 3$, then you find that neither of the factor pairs of 3, namely 1, 3 and $-1, -3$, have sum 2. So this quadratic can’t be factorised using integers.

As you become more familiar with factorising quadratics, you’ll find ways to apply the strategy above more efficiently. For example, rather than writing down all the factor pairs before considering their sums, you can consider the sum of each pair as you go along, and stop once you’ve found a pair with the required sum. Similarly, if you’re trying to factorise a quadratic $x^2 + bx + c$ where both $b$ and $c$ are positive, then you need consider only positive factor pairs of $c$, since the sum of two negative numbers can’t be the positive number $b$.

**Activity 17**  Factorising quadratics of the form $x^2 + bx + c$

Factorise the following quadratics. (They can all be factorised.)

(a) $x^2 + 5x + 6$  
(b) $x^2 - 8x + 15$  
(c) $x^2 + 4x - 5$

(d) $x^2 - 2x - 35$  
(e) $x^2 - 6x + 9$  
(f) $x^2 - 6x + 8$

(g) $y^2 - 7y - 18$  
(h) $u^2 + 4u + 4$  
(i) $p^2 - 4p - 12$

(j) $s^2 + s - 30$  
(k) $v^2 + 5v - 50$  
(l) $r^2 - 10r + 16$
Factorising quadratics in which $x^2$ doesn’t have coefficient 1

We’ll now look at how to factorise quadratics in which $x^2$ doesn’t have coefficient 1, such as $6x^2 - 11x - 35$.

The first thing to do when you want to factorise a quadratic like this is to check whether there are any numerical common factors. If there are, then take them out. For example, if the quadratic is $4x^2 + 2x - 2$, then write it as

$$4x^2 + 2x - 2 = 2(2x^2 + x - 1).$$

Also, if the coefficient of $x^2$ is negative, then take out a factor of $-1$. For example, if the quadratic is $-3x^2 + x + 2$, then write it as

$$-3x^2 + x + 2 = -(3x^2 - x - 2).$$

Once you’ve done these things, you can focus on factorising the simpler quadratic inside the brackets.

Because you can take out a factor of $-1$ if necessary, you only ever need to factorise quadratics in which the coefficient of $x^2$ is positive. You can do that by using the strategy that you saw earlier, with two adaptations when the coefficient of $x^2$ is not 1.

The first adaptation is needed to deal with the fact that there can be more than one possibility for the initial framework that you write out. Each pair of positive factors of the coefficient of $x^2$ gives you a possible framework. For example, if the quadratic that you want to factorise is

$$6x^2 + 11x - 35,$$

then there are two possible frameworks, namely

$$(6x \quad )(x \quad ) \quad \text{and} \quad (2x \quad )(3x \quad ).$$

To deal with this, you apply the factorisation strategy to each of the possible frameworks in turn, until you find one that gives you a factorisation. If none of the possibilities gives you a factorisation, then it means that the quadratic can’t be factorised using integers.

The second adaptation is a little more complicated. In the earlier strategy, once you’ve written out the framework, you consider the factor pairs of the constant term, and for each factor pair you consider whether its sum is equal to the coefficient of $x$. In the adapted strategy, you still consider the factor pairs of the constant term, but you don’t consider the sum of each factor pair, as this doesn’t tell you anything useful when the coefficient of $x^2$ isn’t 1. Instead, you consider directly whether each factor pair leads to the correct term in $x$ when you multiply out. This process is explained more fully in the following example.

The adapted strategy might seem quite complicated when you first meet it, but it should seem more straightforward after you’ve practised it a few times. Again, you might find it easier to understand if you hear someone explain it to you, so it’s a good idea to watch the tutorial clip for Example 9.
Example 9  Factorising a quadratic in which $x^2$ doesn’t have coefficient 1

Factorise the quadratic $2x^2 - 7x - 15$.

Solution

There are no numerical common factors that can be taken out. The coefficient of $x^2$ is a prime number, so there’s only one possible framework.

$$2x^2 - 7x - 15 = (2x \quad )(x \quad )$$

Consider the factor pairs of the constant term $-15$:

$1, -15; \quad -1, 15; \quad 3, -5; \quad -3, 5$.

Each factor pair can go in the brackets in two different ways, giving eight possible cases, as follows.

\[
\begin{align*}
(2x + 1)(x - 15) & \quad (2x - 15)(x + 1) \\
(2x - 1)(x + 15) & \quad (2x + 15)(x - 1) \\
(2x + 3)(x - 5) & \quad (2x - 5)(x + 3) \\
(2x - 3)(x + 5) & \quad (2x + 5)(x - 3)
\end{align*}
\]

For each case, calculate the term in $x$ that you obtain when you multiply out the brackets.

\[
\begin{align*}
(2x + 1)(x - 15) & \quad -29x \\
(2x - 1)(x + 15) & \quad 29x \\
(2x + 3)(x - 5) & \quad -7x \\
(2x - 3)(x + 5) & \quad 7x
\end{align*}
\]

\[
\begin{align*}
(2x - 15)(x + 1) & \quad -13x \\
(2x + 15)(x - 1) & \quad 13x \\
(2x - 5)(x + 3) & \quad x \\
(2x + 5)(x - 3) & \quad -x
\end{align*}
\]

Identify the case that gives $-7x$.

$$2x^2 - 7x - 15 = (2x + 3)(x - 5)$$

You can check the answer to Example 9 by multiplying out the brackets.

The solution to Example 9 includes some working to help with finding the factorisation. You might find it helpful to write down working like this while you’re getting used to factorising quadratics, but after a while you’ll probably find that you can usually do it in your head. So, when you factorise a quadratic, you don’t need to write down any working – it’s fine to just write down the quadratic and its factorisation, like this:

$$2x^2 - 7x - 15 = (2x + 3)(x - 5).$$

Here’s a summary of the method demonstrated in the example above.
**Strategy:**

To factorise a quadratic of the form $ax^2 + bx + c$

1. Take out any numerical common factors. If the coefficient of $x^2$ is negative, also take out the factor $-1$. Then apply the steps below to the quadratic inside the brackets.

2. Find the positive factor pairs of $a$, the coefficient of $x^2$. For each such factor pair $d,e$ write down a framework $(dx \quad)(ex \quad)$.

3. Find all the factor pairs of $c$, the constant term (including both positive and negative factors).

4. For each framework and each factor pair of $c$, write the factor pair in the gaps in the framework in both possible ways.

5. For each of the resulting cases, calculate the term in $x$ that you obtain when you multiply out the brackets.

6. Identify the case where this term is $bx$, if there is such a case. This is the required factorisation.

As with the earlier strategy, if this strategy doesn’t lead to a factorisation, then the quadratic can’t be factorised using integers.

**Activity 18  Factorising quadratics of the form $ax^2 + bx + c$**

Factorise the following quadratics. (They can all be factorised.)

(a) $5x^2 + 13x - 6$  
(b) $3x^2 + 16x + 5$  
(c) $6x^2 - 11x + 3$

(d) $5x^2 - 8x - 21$  
(e) $18x^2 + 9x - 2$  
(f) $4x^2 - 8x + 3$

(g) $4p^2 - 19p - 5$  
(h) $6u^2 + 11u - 35$  
(i) $4t^2 + 4t + 1$

(j) $9v^2 - 12v + 4$  
(k) $-4s^2 + 4s + 3$  
(l) $12y^2 - 10y - 2$

There are two special types of quadratic that can be factorised more easily than those that you have seen so far in this subsection. You should always check whether your quadratic is one of these before you embark on either of the two strategies above.

**Quadratics with no constant term**

These can be factorised by taking $x$ out as a common factor. For example,

- $x^2 + 4x = x(x + 4)$,
- $3x^2 - 6x = 3(x^2 - 2x) = 3x(x - 2)$. 

Differences of two squares

As you saw in Unit 1, a **difference of two squares** is any expression of the form

\[ A^2 - B^2, \]

where \( A \) and \( B \) are subexpressions. You can check, by multiplying out the brackets, that


If you can recognise a quadratic as a difference of two squares, then you can use equation (19) to factorise it immediately. For example,

\[
\begin{align*}
    x^2 - 9 &= x^2 - 3^2 = (x + 3)(x - 3), \\
    x^2 - 1 &= x^2 - 1^2 = (x + 1)(x - 1), \\
    4x^2 - 1 &= (2x)^2 - 1^2 = (2x + 1)(2x - 1).
\end{align*}
\]

### Activity 19  Factorising special quadratics

Factorise the following quadratics.

(a) \( x^2 - 4 \)  
(b) \( x^2 - 4x \)  
(c) \( 2x^2 + 5x \)  
(d) \( x^2 - 25 \)

(e) \( 9y^2 - 4 \)  
(f) \( 25t^2 - 1 \)  
(g) \( 9u^2 - u \)  
(h) \( p^2 + p \)

4.3 Solving quadratic equations by factorisation

In this subsection you’ll see how you can use the method of factorisation to solve quadratic equations. Remember that a quadratic equation is an equation of the form \( ax^2 + bx + c = 0 \), where \( a \), \( b \) and \( c \) are constants with \( a \neq 0 \).

Whenever you have a quadratic equation to solve, the first thing to do is to check whether it can be simplified, as this will make it easier to deal with. Here are some things that you might be able to do.

**Simplifying a quadratic equation**

- If necessary, rearrange the equation so that all the non-zero terms are on the same side.
- If the coefficient of \( x^2 \) is negative, then multiply the equation through by \(-1\) to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.
Once you’ve simplified the quadratic equation as much as you can, you can go on to solve it, using one of the three methods covered in this section.

The method of solving quadratic equations by *factorisation* depends on the techniques that you practised in the last subsection, together with the following crucial fact.

If the product of two or more numbers is 0, then at least one of the numbers must be 0.

You can see how this fact is used in the example below.

---

**Example 10  Solving a quadratic equation by factorisation**

Solve the equation

\[ 6x^2 + 14x - 12 = 0. \]

**Solution**

1. Simplify the equation if possible. Here you can divide through by 2.

\[ 3x^2 + 7x - 6 = 0 \]

2. Factorise the quadratic expression.

\[ (3x - 2)(x + 3) = 0 \]

3. Use the fact in the box above.

\[ 3x - 2 = 0 \quad \text{or} \quad x + 3 = 0 \]

4. Solve these linear equations for \( x \).

\[ x = \frac{2}{3} \quad \text{or} \quad x = -3 \]

So the solutions are \( x = \frac{2}{3} \) and \( x = -3 \).

---

You can check the answers to Example 10 by substituting each of the solutions into the original equation.

Some quadratic equations can’t be solved using the method of Example 10, because it’s not possible to factorise their associated quadratic expressions using integers. You’ll see another method for solving quadratic equations later in this section.
Activity 20  Solving quadratic equations by factorisation

Solve the following quadratic equations by factorisation. (They can all be solved in this way.)

(a) \( x^2 + 4x - 21 = 0 \)  
(b) \( 3x^2 - 18x + 24 = 0 \)  
(c) \( -t^2 - 6t - 9 = 0 \)  
(d) \( 8x + 4x^2 - 5 = 0 \)  
(e) \( 3x^2 - x = 0 \)  
(f) \( x^2 - 16 = 0 \)  
(g) \( \frac{3}{2}u^2 + u - \frac{1}{2} = 0 \)  
(h) \( x^2 + \frac{9}{4}x + \frac{1}{2} = 0 \)  
(i) \( 10(x^2 + 1) = 29x \)

In part (c) of the activity above, you’ll have noticed that when you factorised the quadratic, the two linear factors were the same, which led to only one solution. A quadratic equation with only one solution is said to have a repeated solution.

4.4 Expressing a quadratic in completed-square form

In this subsection you’ll revise how to complete the square in a quadratic expression, and in the next subsection you’ll see how you can use this technique to solve quadratic equations. Completing the square is also useful for other purposes, as you’ll see later in the module.

Completing the square in a quadratic expression means rearranging it in a particular way. For example, the completed-square form of the quadratic \( 2x^2 - 12x + 25 \) is

\[
2(x - 3)^2 + 7.
\]

You can check that this expression is equivalent to the original quadratic by multiplying out the brackets and simplifying:

\[
2(x - 3)^2 + 7 = 2(x^2 - 6x + 9) + 7
\]

\[
= 2x^2 - 12x + 18 + 7
\]

\[
= 2x^2 - 12x + 25.
\]

In general, to complete the square in a quadratic \( ax^2 + bx + c \), you have to rearrange it into the form

\[
a(x + r)^2 + s,
\]

where \( a, r \) and \( s \) are constants. The constant \( a \) is the same as the constant \( a \) in the original expression, and \( r \) and \( s \) are new constants, each of which can be positive, negative or zero. The rearranged form of the quadratic is called its completed-square form.
The key to completing the square in any quadratic is to first learn how to do it for quadratics of the form \( x^2 + bx \). Some examples of quadratics of this form are \( x^2 + 6x \), \( x^2 - 10x \) and \( x^2 + x \).

Let’s start by considering the quadratic \( x^2 + 6x \). To see how to write it in completed-square form, consider what happens when you multiply out the expression \((x + 3)^2\). The ‘+3’ here is obtained by halving the coefficient of \( x \) in the original quadratic, which is +6. You obtain

\[
(x + 3)^2 = x^2 + 3x + 3x + 9 = x^2 + 6x + 9.
\]

So expanding \((x + 3)^2\) gives the original quadratic \(x^2 + 6x\), with an extra term, namely 9, added on. So if you subtract 9 from \((x + 3)^2\), then you’ll have an expression that’s equivalent to \(x^2 + 6x\). That is,

\[
x^2 + 6x = (x + 3)^2 - 9.
\]

The expression on the right of this equation is the completed-square form of \(x^2 + 6x\).

The method can be summarised as follows.

**Strategy:**

To complete the square in a quadratic of the form \( x^2 + bx \)

1. Write down \((x \quad)^2\), filling the gap with the number that’s half of \( b \), the coefficient of \( x \) (including its + or − sign, of course).
2. Subtract the square of the number that you wrote in the gap.

The first step of the strategy ensures that you have squared brackets that, when expanded, give the quadratic \( x^2 + bx \) together with an extra term, which is the square of the number written in the gap. In the second step of the strategy you subtract this extra term to obtain a final completed-square form that’s equivalent to \(x^2 + bx\). Here’s an example.

**Example 11 Completing the square in a quadratic of the form \( x^2 + bx \)**

Write the quadratic expression \(x^2 - 10x\) in completed-square form.

**Solution**

\[
x^2 - 10x = (x - 5)^2 - 25
\]

Halve this coefficient and write it here. Square the constant term in brackets and subtract it.
Remember that once you’ve found the completed-square form of a quadratic, you can always check that it’s correct by multiplying it out.

Here are some examples of completing the square for you to try. In this activity, try checking your answers by multiplying them out – this will help you understand how the method works.

**Activity 21 Completing the square in quadratics of the form** \(x^2 + bx\)

Write the following quadratics in completed-square form.

(a) \(x^2 + 8x\)  
(b) \(x^2 - 4x\)  
(c) \(x^2 + x\)  
(d) \(t^2 - 3t\)

Once you know how to complete the square in any quadratic of the form \(x^2 + bx\), you can also complete the square in any quadratic of the form \(x^2 + bx + c\).

**Strategy:**

To complete the square in a quadratic of the form \(x^2 + bx + c\)

1. Use the earlier strategy to complete the square in the subexpression \(x^2 + bx\).
2. Collect the constant terms.

**Example 12 Completing the square in a quadratic of the form** \(x^2 + bx + c\)

Write the quadratic \(x^2 - 8x + 2\) in completed-square form.

**Solution**

\(\) Complete the square in the subexpression \(x^2 - 8x\), leaving the \(+ 2\) unchanged. 
\[x^2 - 8x + 2 = (x - 4)^2 - 16 + 2\]

\(\) Collect the constant terms. 
\[= (x - 4)^2 - 14.\]

Here are some examples for you to try.
Activity 22  Completing the square in quadratics of the form $x^2 + bx + c$

Write the following quadratics in completed-square form.

(a) $x^2 + 2x + 2$  (b) $x^2 + 3x - 1$  (c) $y^2 + \frac{1}{2}y - \frac{1}{4}$

In fact you can use the method that you’ve seen for completing the square in quadratics of the form $x^2 + bx$ to allow you to complete the square in any quadratic at all. To complete the square in a quadratic of the form $ax^2 + bx + c$, where $a \neq 1$, you begin by factorising the coefficient $a$ out of the subexpression formed by the terms in $x^2$ and $x$. For example, if the quadratic is $2x^2 - 12x + 20$, then you begin by writing

$$2x^2 - 12x + 20 = 2(x^2 - 6x) + 20.$$ 

It doesn’t matter if the coefficient of $x^2$ in the quadratic isn’t a factor of the coefficient of $x$. For example, for the quadratic $-2x^2 + x + 1$, you begin by writing

$$-2x^2 + x + 1 = -2\left(x^2 - \frac{1}{2}x\right) + 1.$$ 

Once you’ve carried out this initial step, the quadratic in the brackets will be of the form $x^2 + bx$. You can then complete the square in this quadratic, and finally simplify the results to obtain the completed-square form of the original quadratic. Here’s an example.

Example 13  Completing the square in a quadratic of the form $ax^2 + bx + c$

Express $2x^2 - 12x + 20$ in completed-square form.

Solution

1. Factorise the coefficient of $x^2$ out of the subexpression formed by the terms in $x^2$ and $x$.

$$2x^2 - 12x + 20 = 2\left(x^2 - 6x\right) + 20$$

2. Now the brackets contain a quadratic of the form $x^2 + bx$. Complete the square in it, keeping it enclosed within its brackets.

$$= 2\left(\left(x - 3\right)^2 - 9\right) + 20$$

3. Multiply out the outer brackets. Don’t multiply out the inner brackets, because you want the square $(x - 3)^2$ to appear in the final expression.

$$= 2(x - 3)^2 - 18 + 20$$

4. Collect the constant terms.

$$= 2(x - 3)^2 + 2$$
Here’s a summary of the method demonstrated in Example 13.

**Strategy:**
To complete the square in a quadratic of the form 
\[ ax^2 + bx + c, \text{ where } a \neq 1 \]

1. Rewrite the quadratic with the coefficient \( a \) taken out of the expression \( ax^2 + bx \) as a factor. This generates a pair of brackets.
2. Use the earlier strategy to complete the square in the simple quadratic inside the brackets. This generates a second pair of brackets, inside the first pair.
3. Multiply out the outer brackets.
4. Collect the constant terms.

**Activity 23**  
Completing the square in quadratics of the form 
\[ ax^2 + bx + c \]

Write the following quadratics in completed-square form.

(a) \( 3x^2 + 6x + 5 \)  
(b) \( 2y^2 - 5y + 4 \)  
(c) \( -x^2 + x - \frac{1}{2} \)

**4.5 Solving quadratic equations by completing the square**

The example below demonstrates how you can use the technique of completing the square to solve a quadratic equation. You can use this method to solve any quadratic equation that has solutions, including those that can’t be solved by factorisation using integers.

**Example 14**  
Solving a quadratic equation by completing the square

Solve the quadratic equation \( 2x^2 - 8x + 5 = 0 \) by completing the square.

**Solution**

Check whether the equation can be simplified (see the box on page 163). Here there’s no simplification to be done. Next, complete the square on the left-hand side.

\[
2(x^2 - 4x) + 5 = 0 \\
2 \left( (x - 2)^2 - 4 \right) + 5 = 0 \\
2(x - 2)^2 - 8 + 5 = 0 \\
2(x - 2)^2 - 3 = 0
\]
Rearrange the equation so that the left-hand side is of the form \((x)^2\).

\[(x - 2)^2 = \frac{3}{2}\]

Take square roots of both sides, remembering that a positive number has both a positive and a negative square root.

\[x - 2 = \sqrt{\frac{3}{2}} \quad \text{or} \quad x - 2 = -\sqrt{\frac{3}{2}}\]

Get \(x\) by itself on the left of each equation.

\[x = 2 + \sqrt{\frac{3}{2}} \quad \text{or} \quad x = 2 - \sqrt{\frac{3}{2}}\]

So the solutions are \(x = 2 + \sqrt{3}/2\) and \(x = 2 - \sqrt{3}/2\).

Here are some examples for you to try.

**Activity 24   Solving quadratic equations by completing the square**

Solve the following quadratic equations by completing the square.

(a) \(x^2 + 4x + 1 = 0\)  
(b) \(3t^2 - 12t + 11 = 0\)  
(c) \(2x^2 + 3x - 3 = 0\)

In practice, if you can’t solve a particular quadratic equation by factorisation, then instead of solving it by completing the square, you may prefer to use the *quadratic formula*, which is given in the next subsection. This formula is obtained by completing the square in the general quadratic expression \(ax^2 + bx + c\).

### 4.6 The quadratic formula

The formula below expresses the solutions of a general quadratic equation in terms of its coefficients.

**The quadratic formula**

The solutions of the quadratic equation \(ax^2 + bx + c = 0\) are

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.\]
Recall from Unit 1 that the symbol $\pm$ means ‘plus or minus’.

Later in this subsection you’ll see how the quadratic formula is obtained by completing the square – for now you should concentrate on how to use it.

Remember that before you solve a quadratic equation you should always simplify it as much as possible, following the guidelines on page 163.

**Example 15 Using the quadratic formula**

Use the quadratic formula to solve the equation $2x^2 - 6x - 5 = 0$.

**Solution**

- Check that the equation is in the form $ax^2 + bx + c = 0$, and find the values of $a$, $b$ and $c$.
- Here $a = 2$, $b = -6$ and $c = -5$.
- Substitute these values into the quadratic formula.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 2 \times (-5)}}{2 \times 2}
\]

\[
= \frac{6 \pm \sqrt{36 + 40}}{4}
\]

\[
= \frac{6 \pm \sqrt{76}}{4}
\]

- Simplify this pair of surds.

\[
= \frac{6 \pm \sqrt{4 \times 19}}{4}
\]

\[
= \frac{6 \pm 2\sqrt{19}}{4}
\]

\[
= \frac{3 \pm \sqrt{19}}{2}
\]

So the solutions are $x = \frac{1}{2}(3 + \sqrt{19})$ and $x = \frac{1}{2}(3 - \sqrt{19})$.

As you can see from Example 15, the calculations that you have to do when you use the quadratic formula can be quite complicated, and it’s easy to make mistakes. Before you solve a quadratic equation using the quadratic formula, it’s always worth checking whether you can solve it more easily by factorisation instead.

When you use the quadratic formula, the solutions that you obtain are often surds. In fact, if you don’t obtain surds, then it means that the
quadratic equation could have been solved by factorisation instead. If you obtain surds, then you should leave them as surds, expressing them in their simplest form, unless you’ve been asked for a decimal approximation, or the solutions to the quadratic equation are the answers to a practical problem.

You can practise using the quadratic formula in the next activity. Remember to simplify the equations before you apply the formula, where possible.

**Activity 25 Using the quadratic formula**

Solve the following quadratic equations by using the quadratic formula. In each case where the solutions are not surds, try solving the quadratic equation by factorisation as well.

(a) \( x^2 - 6x - 1 = 0 \)

(b) \( 9x^2 + 15x - 6 = 0 \)

(c) \( 9x^2 + 6x = 11 \)

(d) \( t^2 + \frac{5}{2}t + \frac{3}{2} = 0 \)

(e) \( u^2 = 4u - 4 \)

To end this subsection, here’s an explanation of where the quadratic formula comes from. Given the general quadratic equation,

\[ ax^2 + bx + c = 0, \]

where \( a \neq 0 \), the first step is to complete the square:

\[
\begin{align*}
    a \left( x^2 + \frac{b}{a}x \right) + c &= 0 \\
    a \left( x + \frac{b}{2a} \right)^2 - \left( \frac{b}{2a} \right)^2 + c &= 0 \\
    a \left( x + \frac{b}{2a} \right)^2 &= \left( \frac{b}{2a} \right)^2 + c \\\n    a \left( x + \frac{b}{2a} \right)^2 &= \frac{b^2}{4a^2} + c \\\n    a \left( x + \frac{b}{2a} \right)^2 &= \frac{b^2 - 4ac}{4a^2}. 
\end{align*}
\]

Next rearrange the equation so that you get the constant terms on the right, and combine them into a single fraction:

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}. 
\]
Now take square roots of both sides:

\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \]

\[
= \pm \frac{\sqrt{b^2 - 4ac}}{2a}.
\]

The last step is to get \(x\) by itself on the left-hand side:

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]

\[
= -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

This is the quadratic formula!

### 4.7 The number of solutions of a quadratic equation

Earlier in this unit, you saw that the graph of an equation of the form

\[
y = ax^2 + bx + c,
\]

where \(a\), \(b\) and \(c\) are constants with \(a \neq 0\), can have two, one or no \(x\)-intercepts. Since the \(x\)-intercepts of the graph of

\[
y = ax^2 + bx + c
\]

are the solutions of the quadratic equation

\[
ax^2 + bx + c = 0,
\]

this means that a quadratic equation can have two, one or zero solutions that are real numbers.

In fact every quadratic equation has at least one solution if we allow solutions that are *complex numbers* – these numbers were mentioned in Unit 1, and they include all the real numbers and also many ‘imaginary’ numbers, such as the square root of \(-1\). You’ll see more about this in Unit 12, but until then you only need to consider the solutions of quadratic equations that are real numbers, which are known as their *real solutions*.

When you use the quadratic formula to solve a quadratic equation, the number of real solutions of the equation quickly becomes clear. For example, consider the equation

\[
x^2 + 2x + 3 = 0.
\]

Here \(a = 1\), \(b = 2\) and \(c = 3\). Substituting these values into the quadratic formula, we obtain

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2 \times 1}
\]

\[
= \frac{-2 \pm \sqrt{4 - 12}}{2}
\]

\[
= \frac{-2 \pm \sqrt{-8}}{2}.
\]

This expression involves \(\sqrt{-8}\), but there is no such number in the set of real numbers. So this equation has no real solutions. This is confirmed by the graph of \(y = x^2 + 2x + 3\) in Figure 24, which does not cross or touch the \(x\)-axis and so has no \(x\)-intercepts.

**Figure 24** The graph of \(y = x^2 + 2x + 3\)
Similarly, consider the equation

\[ x^2 + 2x + 1 = 0. \]

Here \( a = 1, \ b = 2 \) and \( c = 1 \). Substituting these values into the quadratic formula, we obtain

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 4}}{2} = \frac{-2 \pm \sqrt{0}}{2} = -1.
\]

In this calculation the number under the square root sign turns out to be zero, and this leads to just one solution of the equation. This is confirmed by the graph of \( y = x^2 + 2x + 1 \) in Figure 25, which touches the \( x \)-axis but does not cross it. From these examples you can see that it’s the value of the expression \( b^2 - 4ac \), which appears under the square root sign in the quadratic formula, that determines how many solutions a quadratic equation has. The different possibilities are set out in the box below. The value \( b^2 - 4ac \) is called the **discriminant** of the quadratic expression \( ax^2 + bx + c \).

**The number of real solutions of a quadratic equation**

The quadratic equation \( ax^2 + bx + c = 0 \) has:

- two real solutions if \( b^2 - 4ac > 0 \) (the discriminant is positive)
- one real solution if \( b^2 - 4ac = 0 \) (the discriminant is zero)
- no real solutions if \( b^2 - 4ac < 0 \) (the discriminant is negative).

**Activity 26 Predicting the number of real solutions of a quadratic equation**

Use the discriminant to determine whether each of the following quadratic equations has two, one or no real solutions. Find any real solutions.

(a) \( 4x^2 - 20x + 25 = 0 \)  
(b) \( 2x^2 + 6x + 5 = 0 \)  
(c) \( 4x^2 = 4x + 5 \)

The discriminant of a quadratic also tells you whether the quadratic can be factorised using integers. You can see that if the discriminant of a quadratic with integer coefficients is a perfect square, then you won’t get surds when you use the quadratic formula to solve the corresponding quadratic equation. This tells you that the expression can be factorised using integers.
4.8 Solving equations related to quadratic equations

In this section you’ll see how to use the techniques for solving quadratic equations to solve some equations that don’t look like quadratic equations at first sight, and to solve some equations that are not quadratic.

First, remember that in Unit 1 you saw how to solve some equations containing algebraic fractions. Usually the first step is to clear the fractions, by multiplying through by a suitable expression. All the equations containing algebraic fractions in Unit 1 turned out to be linear equations, but it’s also possible for an equation containing algebraic fractions to turn out to be a quadratic equation. Here’s an example.

<table>
<thead>
<tr>
<th>Example 16</th>
<th>Solving an equation involving algebraic fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Solution</strong></td>
<td>There is a fraction with denominator $x + 2$, so multiply through by $x + 2$ to clear it. For this to be guaranteed to give an equivalent equation, you have to assume that $x + 2 \neq 0$; that is, $x \neq -2$.</td>
</tr>
<tr>
<td></td>
<td>Assume that $x \neq -2$.</td>
</tr>
<tr>
<td></td>
<td>$4 + x(x + 2) = 3(x + 2)$</td>
</tr>
<tr>
<td></td>
<td>Multiply out the brackets.</td>
</tr>
<tr>
<td></td>
<td>$4 + x^2 + 2x = 3x + 6$</td>
</tr>
<tr>
<td></td>
<td>Get all the non-zero terms on the left-hand side.</td>
</tr>
<tr>
<td></td>
<td>$x^2 - x - 2 = 0$</td>
</tr>
<tr>
<td></td>
<td>This is a quadratic equation. Solve it by factorising.</td>
</tr>
<tr>
<td></td>
<td>$(x + 1)(x - 2) = 0$</td>
</tr>
<tr>
<td></td>
<td>So $x + 1 = 0$ or $x - 2 = 0$; that is, $x = -1$ or $x = 2$.</td>
</tr>
<tr>
<td></td>
<td>These values satisfy the assumption $x \neq -2$, so the solutions of the original equation are $x = -1$ and $x = 2$.</td>
</tr>
</tbody>
</table>

You can sometimes use the methods for solving quadratic equations to solve an equation containing a power of $x$ greater than 2, if you start by factorising the equation. This is illustrated in the next example.
Example 17 Factorising to reveal a quadratic

Solve the equation $3x^3 - 12x^2 + 3x = 0$.

Solution

- Simplify the equation by dividing through by 3. $x^3 - 4x^2 + x = 0$
- There is a common factor, $x$. Take it out. $x(x^2 - 4x + 1) = 0$
- This gives two expressions whose product is zero, so at least one of the expressions is zero. $x = 0$ or $x^2 - 4x + 1 = 0$
- Now solve the quadratic equation. It cannot be factorised using integers, so use the quadratic formula, with $a = 1$, $b = -4$, $c = 1$.

The solutions of the quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$
$$= \frac{4 \pm 2\sqrt{3}}{2}$$
$$= 2 \pm \sqrt{3}.$$ 

So the solutions of the original equation are $x = 0$, $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

The equation in Example 17 is an example of a cubic equation. A cubic equation is an equation of the form $ax^3 + bx^2 + cx + d = 0$, where $a$, $b$, $c$ and $d$ are constants, with $a \neq 0$. You’ll see in Unit 3 that every cubic equation has at most three solutions.

In the next example an equation is transformed into a quadratic equation by making a substitution.
Example 18  Substituting to reveal a quadratic

Solve the equation $x^4 - 5x^2 + 4 = 0$.

Solution

1. Substitute another letter, say $X$, for $x^2$.

Let $X = x^2$. The equation becomes

$$X^2 - 5X + 4 = 0$$

2. This is a quadratic equation in $X$. It can be solved by factorisation.

$$(X - 1)(X - 4) = 0$$

$X = 1$ or $X = 4$

3. Use the fact that $X = x^2$.

$x^2 = 1$ or $x^2 = 4$

4. Take square roots of both sides of each equation.

$x = \pm 1$ or $x = \pm 2$.

So the solutions of the original equation are $x = 1$, $x = -1$, $x = 2$ and $x = -2$.

The method in Example 18 can be used to solve any equation of the form

$$ax^4 + bx^2 + c = 0,$$

where $a$, $b$ and $c$ are constants, with $a \neq 0$.

Activity 27  Solving equations related to quadratic equations

Solve the following equations.

(a) $\frac{4}{x} = \frac{3x}{x + 1}$

(b) $\frac{1}{x - 2} = 1 + 4x$

(c) $2x^3 - 2x^2 - 12x = 0$

(d) $x^4 - 2x^2 - 8 = 0$

(e) $u^4 - 4 = 0$

(f) $s^5 - 9s^3 = 0$

(g) $(t^2 - 3)(t^2 - 3t + 2) = 0$
4.9 Sketching quadratic graphs

In Subsection 4.1 you saw how the graph of the equation \( y = ax^2 + bx + c \) changes when you change the values of \( a \), \( b \) and \( c \). In this subsection you’ll revise how to sketch the graph of any equation of this form. Note that a sketch of a graph is a diagram that gives an impression of its shape, with key points marked and positioned approximately correctly relative to a pair of coordinate axes. It’s different from a plot of a graph, which is a more accurate diagram obtained by precisely plotting a reasonably large number of points on the graph. For example, you were asked to plot a graph in Activity 2 on page 123.

Although a sketch of a graph is not an accurate representation of the graph, it should be sufficiently correct to convey the main properties. The axis scales need not be marked, but there should be some indication of scale – for example, the key points can be labelled with their coordinates. In the case of a parabola, the key points are the points where it crosses the axes, and the vertex. When you’re sketching a quadratic graph, it’s helpful to remember the properties summarised below. You met these properties in Subsection 4.1.

Properties of the graph of \( y = ax^2 + bx + c \), where \( a \neq 0 \)
1. The graph is a parabola with a vertical axis of symmetry.
2. If \( a \) is positive it is u-shaped; if \( a \) is negative it is n-shaped.
3. It has two, one or no \( x \)-intercepts.
4. It has one \( y \)-intercept.

Here’s a strategy for sketching a quadratic graph. It’s illustrated in the next example.

Strategy:
To sketch the graph of \( y = ax^2 + bx + c \), where \( a \neq 0 \)
1. Find whether the parabola is u-shaped or n-shaped.
2. Find its intercepts, axis of symmetry and vertex.
3. Plot the features found, and hence sketch the parabola.
4. Label the parabola with its equation, intercepts and the coordinates of the vertex.

It’s fine to label a parabola with the coordinates of the points where it crosses the axis, rather than with its intercepts. For example, the parabola in Example 19 is labelled with \((0,5)\) rather than with 5. Also, you can choose whether or not to draw the axis of symmetry. If you do include it, you should draw it as a dashed line.
Example 19  Sketching a quadratic graph

Sketch the graph of \( y = -x^2 - 4x + 5 \).

Solution

1. Find whether the parabola is u-shaped or n-shaped.

The coefficient of \( x^2 \) is negative, so the graph is n-shaped.

2. Find the y-intercept.

Putting \( x = 0 \) gives \( y = 5 \), so the y-intercept is 5.

3. Find any x-intercepts.

Putting \( y = 0 \) gives

\[
-x^2 - 4x + 5 = 0
\]
\[
x^2 + 4x - 5 = 0
\]
\[
(x - 1)(x + 5) = 0
\]
\[
x - 1 = 0 \quad \text{or} \quad x + 5 = 0
\]
\[
x = 1 \quad \text{or} \quad x = -5.
\]

So the x-intercepts are 1 and \(-5\).

4. The axis of symmetry lies halfway between the x-intercepts. The number halfway between any two numbers \( p \) and \( q \) is their mean, \((p + q)/2\).

The value of \( x \) halfway between \(-5\) and 1 is

\[
(-5 + 1)/2 = -4/2 = -2.
\]

So the axis of symmetry is the line \( x = -2 \).

5. The vertex lies on the axis of symmetry.

The point with \( x \)-coordinate \(-2\) on the graph has \( y \)-coordinate \( y = -(-2)^2 - 4 \times (-2) + 5 = 9 \). So the vertex is \((-2, 9)\).

6. Plot the features found. Hence sketch the parabola as a smooth curve, and label it with its equation, intercepts and vertex.

\[
(-2, 9)
\]
\[
(0, 5)
\]
\[
(-5, 0)
\]
\[
(1, 0)
\]
You can use the method demonstrated in Example 19 to sketch the graph of any equation of the form \( y = ax^2 + bx + c \), where \( a \neq 0 \), provided that there are two \( x \)-intercepts or one \( x \)-intercept. If there is just one \( x \)-intercept, then the vertex of the parabola is at this \( x \)-intercept. The graphs of both equations in the activity below have one or two \( x \)-intercepts.

**Activity 28  Sketching quadratic graphs**

Sketch the graphs of the following equations.

(a) \( y = -2x^2 + 3x - 1 \)  
(b) \( y = 2x^2 + 8x + 8 \)

To sketch the graph of an equation of the form \( y = ax^2 + bx + c \) when there are no \( x \)-intercepts, you need a different method for finding the equation of the axis of symmetry. One method is to first find any two points on the parabola that have the same \( y \)-coordinate, and use the fact that the axis of symmetry lies halfway between them.

The next example demonstrates a neat way to find two such points.

**Example 20  Sketching a quadratic graph with no \( x \)-intercepts**

Sketch the graph of \( y = 2x^2 - 12x + 20 \).

**Solution**

1. Find whether the parabola is u-shaped or n-shaped. 🌑
   
The coefficient of \( x^2 \) is positive, so the graph is u-shaped.

2. Find the \( y \)-intercept. 🌑
   
   Putting \( x = 0 \) gives \( y = 20 \), so the \( y \)-intercept is 20.

3. Find any \( x \)-intercepts. 🌑
   
   Putting \( y = 0 \) gives
   
   \[
   2x^2 - 12x + 20 = 0 \\
   x^2 - 6x + 10 = 0
   \]
   
The discriminant of \( x^2 - 6x + 10 \) is
   
   \[
   b^2 - 4ac = (-6)^2 - 4 \times 1 \times 10 = -4.
   \]
   
The discriminant is negative, so the quadratic equation above has no solutions, and hence the graph has no \( x \)-intercepts.
To find the axis of symmetry, start by taking out the common factor $x$ from the terms in $x^2$ and $x$ in the equation whose graph you’re trying to sketch. It’s convenient to take out any numerical common factors too.

The equation $y = 2x^2 - 12x + 20$ can be rearranged as

$$y = 2x(x - 6) + 20.$$  

From this form of the equation you can see that the points with $x$-coordinates 0 and 6 have the same $y$-coordinate, namely 20. The axis of symmetry lies halfway between these two points.

The points $(0, 20)$ and $(6, 20)$ lie on the graph. The number halfway between 0 and 6 is 3, so the axis of symmetry is the line $x = 3$.

The vertex lies on the axis of symmetry.

The point on the graph with $x$-coordinate 3 has $y$-coordinate $y = 2 \times 3^2 - 12 \times 3 + 20 = 2$, so the vertex is $(3, 2)$.

Plot the features found. Hence sketch the parabola as a smooth curve, and label it with its equation, intercepts and vertex.

An alternative way to find the vertex of the graph of an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, is to complete the square in the quadratic expression. For example, consider again the equation in Example 20:

$$y = 2x^2 - 12x + 20.$$  

Completing the square in the right-hand side (which was done for this particular expression in Example 13) gives

$$y = 2(x - 3)^2 + 2.$$  

Think about what this form of the equation tells you. Since $(x - 3)^2$ is a square, it can never be negative, no matter what the value of $x$ is. The least value that it can take is 0, and it takes this value when $x = 3$. So the
least value that \( y \) can take is \( y = 2 \times 0 + 2 = 2 \), and \( y \) takes this value when \( x = 3 \). This tells you that the vertex is \((3, 2)\).

In general, when you want to sketch a graph, there is no single correct approach. You can use any appropriate means to find the information that you need.

**Activity 29**  *Sketching a quadratic graph with no \( x \)-intercepts*

Sketch the graph of \( y = x^2 - 2x + 4 \).

### 4.10 Applications of quadratics

Quadratic equations occur frequently when you model real-life problems involving objects that are falling vertically under the influence of gravity, or have been thrown, particularly when the effects of air resistance are small enough to be ignored. Here’s an example.

**Example 21**  *Solving a quadratic equation in a real-life problem*

A boy standing on a vertical cliff 150 m above sea level kicks a ball into the sea. Let \( x \) be the horizontal displacement (in metres) of the ball from the base of the cliff, and \( y \) be the vertical displacement (in metres) of the ball measured upwards from sea level. Then the curve followed by the ball can be modelled by the equation

\[
y = 150 + x - \frac{x^2}{40}.
\]

Find the horizontal distance from the cliff at which the ball hits the sea.
Solution

The ball hits the sea when the vertical displacement \( y \) is 0. The corresponding horizontal displacement \( x \) satisfies the equation

\[
0 = 150 + x - \frac{x^2}{40}.
\]

Clear the fraction and simplify the equation.

Rearranging gives

\[
0 = 6000 + 40x - x^2
\]

\[
x^2 - 40x - 6000 = 0
\]

Factorise.

\[
(x - 100)(x + 60) = 0
\]

\[
x - 100 = 0 \quad \text{or} \quad x + 60 = 0
\]

\[
x = 100 \quad \text{or} \quad x = -60.
\]

The negative answer is meaningless in the context of the question. So the ball hits the sea 100 m from the cliff.

Activity 30 Using quadratics

A ball is thrown vertically upwards. Its vertical displacement \( s \) (in metres, measured upwards from the point of release) is modelled by the equation

\[
s = 12t - 5t^2,
\]

where \( t \) is the time in seconds since the moment of release.

(a) How high will the ball be after 2 seconds?

(b) At what times after the ball is thrown will it be at a height of 5 m above its starting position?

(c) After what time will the ball return to its starting position?

(d) By completing the square in the expression for \( s \), find the maximum height that the ball will reach above its starting position.

The techniques that you need to create models like the ones in Example 21 and Activity 30 are taught in the module MST125.
5 Using the computer for graphs and equations

The following activity completes your study of Unit 2.

**Activity 31 Using the computer algebra system**

Work through Sections 2 and 3 of the MST124 *Computer algebra guide*, where you will learn how to use the computer to manipulate algebraic expressions, solve equations and plot lines and curves.

You will need to use your computer while you work through these sections.

**Learning outcomes**

After studying this unit, you should be able to:

- plot the graph of an equation by constructing a table of values and plotting points
- find the gradient and intercepts of a straight line from its equation
- interpret the gradient and intercepts of a straight line in real-life situations, where possible
- find the equation of a straight line from its gradient and \( y \)-intercept, from its gradient and a point on the line, or from two points on the line
- use the equation of a straight line to draw the line
- find the gradient of a line perpendicular to another line whose gradient you know
- solve simultaneous linear equations, and hence find the point of intersection of two straight lines
- factorise a quadratic expression using integers, where possible
- complete the square in a quadratic expression
- solve quadratic equations by factorising, by completing the square and by using the quadratic formula
- sketch the graph of an equation of the form \( y = ax^2 + bx + c \)
- use the module computer algebra system to manipulate mathematical expressions and plot lines and curves.
Solutions to activities

Solution to Activity 1
(a) When \(x = 6\) and \(y = 20\),
   \[\text{LHS} = y - 2 = 20 - 2 = 18\]
   and \(\text{RHS} = 3x = 3 \times 6 = 18\).
   So the point \((6, 20)\) satisfies the equation.
(b) When \(x = -2\) and \(y = 8\),
   \[\text{LHS} = y - 2 = 8 - 2 = 6\]
   and \(\text{RHS} = 3x = 3 \times (-2) = -6\).
   So the point \((-2, 8)\) does not satisfy the equation.

Solution to Activity 2
A table of values for the equation \(y = x^2 + 2x + 2\) is as follows.
\[
\begin{array}{c|cccc}
  x & -2 & -1 & 0 & 1 & 2 \\
  \hline
  y & 2 & 1 & 2 & 5 & 10
\end{array}
\]
Drawing a smooth curve through these points gives the graph below.

Solution to Activity 3
(a) For every unit moved to the right, the pen tip moves up 3 units, so the gradient is 3.
(b) For every unit moved to the right, the pen tip moves up \(\frac{1}{2}\) unit, so the gradient is \(\frac{1}{2}\).
(c) For every unit moved to the right, the pen tip moves down 4 units, so the gradient is -4.
(d) For every unit moved to the right, the pen tip moves down \(\frac{1}{3}\) unit, so the gradient is -\(\frac{1}{3}\).
(e) For every unit moved to the right, the pen tip moves up 1 unit, so the gradient is 1.
(f) For every unit moved to the right, the pen tip moves down 1 unit, so the gradient is -1.

Solution to Activity 4
(a) \(A\) is \((-4,5)\); \(B\) is \((1,3)\); \(C\) is \((-1,-2)\); \(D\) is \((-6,0)\).
(b) (i) For the gradient of the line through \(A\) and \(B\), take \(A\) to be the first point and \(B\) the second:
   \[
   \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{1 - (-4)} = \frac{-2}{5} = -\frac{2}{5}.
   \]
(ii) For the gradient of the line through \(A\) and \(D\), take \(A\) to be the first point and \(D\) the second:
   \[
   \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 5}{-6 - (-4)} = \frac{-5}{-2} = \frac{5}{2}.
   \]
(iii) For the gradient of the line through \(B\) and \(C\), take \(B\) to be the first point and \(C\) the second:
   \[
   \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 3}{-1 - 1} = \frac{-5}{-2} = \frac{5}{2}.
   \]

Solution to Activity 5
(a) The coefficient of \(x\) is -4, so the gradient is -4.
The constant term is 3, so the \(y\)-intercept is 3.
To find the \(x\)-intercept, put \(y = 0\), which gives
\[
0 = -4x + 3.
\]
Solving this equation gives \(4x = 3\), so \(x = \frac{3}{4}\).
Hence the \(x\)-intercept is \(\frac{3}{4}\).
(b) Rearranging the equation in the form \(y = mx + c\) gives \(y = \frac{1}{3}x - \frac{2}{3}\). The coefficient of \(x\) is \(\frac{1}{3}\), so the gradient is \(\frac{1}{3}\).
The constant term is \(-\frac{2}{3}\), so the \(y\)-intercept is \(-\frac{2}{3}\).
To find the \(x\)-intercept, put \(y = 0\), which gives
\[
0 = \frac{1}{3}x - \frac{2}{3}.
\]
Solving this equation gives \(\frac{1}{3}x = \frac{2}{3}\), so \(x = 2\).
Hence the \(x\)-intercept is 2.
Unit 2  Graphs and equations

Solution to Activity 6

(a) Putting $x = 0$ gives $y = \frac{1}{3} \times 0 + 2 = 2$, so $(0, 2)$ lies on the line. (You might also notice that this point lies on the line since the $y$-intercept is 2.)

Putting $x = 3$ gives $y = \frac{1}{3} \times 3 + 2 = 1 + 2 = 3$, so $(3, 3)$ lies on the line.

Solution to Activity 7

(a) Using the equation $y - y_1 = m(x - x_1)$ with $m = 3$, $x_1 = 2$ and $y_1 = 1$ gives

$$y - 1 = 3(x - 2).$$

Expanding the brackets and rearranging gives the equation of the line as

$$y = 3x - 5.$$
So the equation of the line is
\[ y = x + 1. \]
(c) Using the equation \( y = mx + c \) with \( m = 2 \) and \( c = 3 \) gives the equation of the line as
\[ y = 2x + 3. \]
(d) The point at which the line crosses the x-axis is (2,0). Using the equation \( y - y_1 = m(x - x_1) \) with \( m = -3 \), \( x_1 = 2 \) and \( y_1 = 0 \) gives
\[ y - 0 = -3(x - 2). \]
So the equation of the line is
\[ y = -3x + 6. \]
(e) Each point on a vertical line has the same x-coordinate, so the equation of the line is \( x = 1 \).
(f) The y-coordinates of the two points are the same, so the line is horizontal with equation \( y = 3 \).

**Solution to Activity 8**

(a) The gradient of the line \( y = 3x + 5 \) is 3, so the gradient of a line perpendicular to this line is \(-\frac{1}{3}\).

(b) The gradient of the required line is \(-\frac{1}{3}\) and it passes through the point (2,1). Using the equation \( y - y_1 = m(x - x_1) \) gives
\[ y - 1 = -\frac{1}{3}(x - 2) \]
\[ y - 1 = -\frac{1}{3}x + \frac{2}{3}. \]
So the required line is
\[ y = -\frac{1}{3}x + \frac{2}{3}. \]

**Solution to Activity 9**

(a) The line passes through the points (0,200) and (40,500), so its gradient is
\[ \frac{500 - 200}{40 - 0} = \frac{300}{40} = 7.5 \text{ £/person}. \]
The gradient is the rate of increase of the cost of the room with respect to the number of people attending the meeting. Each additional person attending raises the cost by £7.50.

(b) The line passes through the points (200,0.07) and (1000,0.04), so its gradient is
\[ \frac{0.04 - 0.07}{1000 - 200} = -0.003 = -\frac{3}{80000} \text{ £/kg}. \]

The units can be simplified:
\[ \frac{\text{£/kg}}{\text{kg}} = \frac{\text{£}}{\text{kg}} \div \frac{\text{kg}}{\text{kg}} \times \frac{1}{\text{kg}} = \frac{\text{£}}{\text{kg}^2}. \]
So the gradient is \(-\frac{3}{80000} \text{ £/kg}^2\). The gradient is the rate of change of the price (in £/kg) with respect to the quantity (in kg). For each extra kg of sand, the price (per kg) decreases by approximately \( 3 \times 10^{-5} \text{ £/kg} \).

**Solution to Activity 10**

The vertical intercept is £200. This represents the basic cost of the meeting room without the additional cost for each person attending.

**Solution to Activity 11**

(a) The gradient of the graph of the cost of the meeting room is 7.5 £/person. From the graph, the vertical intercept is £200. So the equation of the graph is
\[ C = 7.5n + 200, \]
where \( C \) represents the cost (in £) of the meeting room and \( n \) represents the number of people attending. The gradient of the building sand graph is \(-\frac{3}{80000} \text{ £/kg}^2\). The graph passes through the point (200,0.07), so its equation is given by
\[ p - 0.07 = -\frac{3}{80000}(q - 200); \]
that is,
\[ p = -\frac{3}{80000}q + \frac{600}{80000} + 0.07 \]
\[ p = -\frac{3}{80000}q + \frac{31}{400}. \]

(b) (i) Using \( C = 7.5n + 200 \) with \( C = 560 \) gives
\[ 560 = 7.5n + 200 \]
\[ 7.5n = 360 \]
\[ n = 360 \div 7.5 = 48. \]
So the maximum number of people that can be accommodated is 48.

(ii) Using \( p = -\frac{3}{80000}q + \frac{31}{400} \) with \( q = 500 \) gives
\[ p = -\frac{3}{80000} \times 500 + \frac{31}{400} = \frac{47}{800}, \]
so the price per kilogram is approximately £0.059.
Solution to Activity 12
(a) The displacement is 3 km.
(b) The woman remains at the bench for 10 minutes.
(c) On the way to the bench her velocity is the gradient of the first line segment of the graph:
\[
\frac{3 - 0}{30 - 0} = 0.1 \text{ km/min}.
\]
(d) On the way back her velocity is the gradient of the third line segment of the graph:
\[
\frac{0 - 3}{90 - 40} = \frac{-3}{50} = -0.06 \text{ km/min}.
\]
(e) On the way to the bench her speed is 0.1 km/min. On the way back her speed is 0.06 km/min.
(f) Let \( s \) be the displacement in kilometres and \( t \) be the time in minutes. The equation of the line segment representing the first part of her walk is \( s = 0.1t \).
(g) Her displacement after 50 minutes if she hadn’t stopped would be \( s = 0.1 \times 50 = 5 \text{ km} \).

Solution to Activity 13
(a) The equations are
\[
\begin{align*}
\text{(20)} & \quad s - 5t = -3, \\
\text{(21)} & \quad s + 3t = 13.
\end{align*}
\]
Making \( s \) the subject of equation (20) gives
\[
\text{(22)} \quad s = 5t - 3.
\]
Substituting for \( s \) in equation (21) gives
\[
\begin{align*}
5t - 3 + 3t & = 13, \\
8t & = 16, \\
t & = 2.
\end{align*}
\]
Substituting this value for \( t \) in equation (22) gives
\[
s = 5 \times 2 - 3 = 7.
\]
So the solution is \( s = 7, t = 2 \).
(b) The equations are
\[
\begin{align*}
\text{(23)} & \quad x + 4y = 2, \\
\text{(24)} & \quad 2x + 5y = 3.
\end{align*}
\]
Making \( x \) the subject of equation (23) gives
\[
\text{(25)} \quad x = 2 - 4y.
\]
Substituting for \( x \) in equation (24) gives
\[
\begin{align*}
2(2 - 4y) + 5y & = 3, \\
4 - 8y + 5y & = 3, \\
-3y & = -1, \\
y & = \frac{1}{3}.
\end{align*}
\]
Substituting this value for \( y \) in equation (25) gives
\[
x = 2 - 4 \times \frac{1}{3} = \frac{2}{3}.
\]
So the solution is \( x = \frac{2}{3}, y = \frac{1}{3} \).

Solution to Activity 14
(a) The equations are
\[
\begin{align*}
\text{(26)} & \quad s + 6t = 20, \\
\text{(27)} & \quad 2s + 7t = 35.
\end{align*}
\]
Multiplying equation (26) by 2 (to make the coefficients of \( s \) the same) gives
\[
\begin{align*}
2s + 12t & = 40, \\
2s + 7t & = 35.
\end{align*}
\]
Subtracting equation (29) from equation (28) gives
\[
\begin{align*}
12t - 7t & = 40 - 35, \\
5t & = 5, \\
t & = 1.
\end{align*}
\]
Substituting this value for \( t \) in equation (26) gives
\[
\begin{align*}
s + 6 \times 1 &= 20, \\
s + 6 &= 20, \\
s & = 14.
\end{align*}
\]
So the solution is \( s = 14, t = 1 \).
(b) The equations are
\[
\begin{align*}
\text{(30)} & \quad 2x + 3y = -5, \\
\text{(31)} & \quad 3x - 2y = 12.
\end{align*}
\]
Multiplying equation (30) by 3 and equation (31) by 2 (to make the coefficients of \( x \) the same) gives
\[
\begin{align*}
6x + 9y & = -15, \\
6x - 4y & = 24.
\end{align*}
\]
Subtracting equation (33) from equation (32) gives
\[
\begin{align*}
9y - (-4y) & = -15 - 24, \\
13y & = -39, \\
y & = -3.
\end{align*}
\]
Substituting this value for $y$ in equation (30) gives
\[
2x + 3 \times (-3) = -5
\]
\[
2x - 9 = -5
\]
\[
2x = 4
\]
\[
x = 2.
\]
So the solution is $x = 2, y = -3$.

(c) The equations are
\[
3u - v = -\frac{5}{2}, \tag{34}
\]
\[
2u + 5v = 21. \tag{35}
\]
Multiplying equation (34) by 5 (to make the coefficients of $v$ have the same magnitude) gives
\[
15u - 5v = -\frac{25}{2}. \tag{36}
\]
Adding equation (36) and equation (35) gives
\[
17u = \frac{17}{2}
\]
\[
u = \frac{1}{2}.
\]
Substituting this value for $u$ in equation (34) gives
\[
3 \times \frac{1}{2} - v = -\frac{5}{2}
\]
\[
-v = -\frac{5}{2} - \frac{3}{2} = -4
\]
\[
v = 4.
\]
So the solution is $u = \frac{1}{2}, v = 4$.

Solution to Activity 15

(a) The equations are
\[
y - 3x = -2,
\]
\[
2y + x = 10,
\]
which can be rearranged as
\[
y = 3x - 2, \tag{37}
\]
\[
y = -\frac{1}{2}x + 5. \tag{38}
\]
So the gradient of the line in equation (37) is 3 and the gradient of the line in equation (38) is $-\frac{1}{2}$. Since the gradients are different, the lines must intersect and there is exactly one solution.

The equations can be solved by eliminating $y$ from equations (37) and (38) to give
\[
3x - 2 = -\frac{1}{2}x + 5
\]
\[
\frac{7}{2}x = 7
\]
\[
x = 2.
\]

Substituting this value for $x$ in equation (37) gives
\[
y = 3 \times 2 - 2
\]
\[
y = 4.
\]
So the solution is $x = 2, y = 4$.

(b) The equations are
\[
y = 4x - 5,
\]
\[
2y - 8x = 10.
\]
Writing the equations with both unknowns on the LHS, and dividing the second equation by the common factor 2 gives
\[
y - 4x = -5
\]
\[
y - 4x = 5.
\]
Since $y - 4x = -5$ and $y - 4x = 5$ cannot both be true, the equations have no solutions.

(c) The equations are
\[
4y = 2x + 6,
\]
\[
2y - x = 3.
\]
Writing the equations with both unknowns on the LHS, and dividing the first equation by the common factor 2 gives
\[
2y - x = 3,
\]
\[
2y - x = 3.
\]
The two equations are identical, so their graphs are identical. So all values of $x$ and $y$ that satisfy the first equation also satisfy the second equation, and hence there are infinitely many solutions.

Solution to Activity 16

(The effects that you should have seen are described in the text that follows this activity.)
Unit 2  Graphs and equations

Solution to Activity 17
(a) \( x^2 + 5x + 6 = (x + 2)(x + 3) \)
(b) \( x^2 - 8x + 15 = (x - 3)(x - 5) \)
(c) \( x^2 + 4x - 5 = (x - 1)(x + 5) \)
(d) \( x^2 - 2x - 35 = (x - 5)(x + 7) \)
(e) \( x^2 - 6x + 9 = (x - 3)^2 \)
(f) \( x^2 - 6x + 8 = (x - 2)(x - 4) \)
(g) \( y^2 - 7y - 18 = (y + 2)(y - 9) \)
(h) \( u^2 + 4u + 4 = (u + 2)^2 \)
(i) \( p^2 - 4p - 12 = (p + 2)(p - 6) \)
(j) \( s^2 + s - 30 = (s - 5)(s + 6) \)
(k) \( v^2 + 5v - 50 = (v - 5)(v + 10) \)
(l) \( r^2 - 10r + 16 = (r - 2)(r - 8) \)

Solution to Activity 18
(a) \( 5x^2 + 13x - 6 = (5x - 2)(x + 3) \)
(b) \( 3x^2 + 16x + 5 = (3x + 1)(x + 5) \)
(c) \( 6x^2 - 11x + 3 = (2x - 3)(3x - 1) \)
(d) \( 5x^2 - 8x - 21 = (5x + 7)(x - 3) \)
(e) \( 18x^2 + 9x - 2 = (6x - 1)(3x + 2) \)
(f) \( 4x^2 - 8x + 3 = (2x - 3)(2x - 1) \)
(g) \( 4p^2 - 19p + 5 = (4p - 1)(p - 5) \)
(h) \( 6u^2 + 11u - 35 = (2u + 7)(3u - 5) \)
(i) \( 4t^2 + 4t + 1 = (2t + 1)(2t - 1) = (2t + 1)^2 \)
(j) \( 9v^2 - 12v + 4 = (3v - 2)^2 \)
(k) \( -4s^2 + 4s + 3 = -(4s^2 - 4s - 3) \)
\( = -(2s + 1)(2s - 3) \)
(l) \( 12y^2 - 10y - 2 = 2(6y^2 - 5y - 1) \)
\( = 2(6y + 1)(y - 1) \)

(In part (l), if you forget to take the common factor 2 out of the quadratic, then you obtain)
\( 12y^2 - 10y - 2 = (12y + 2)(y - 1) \)
or
\( 12y^2 - 10y - 2 = (6y + 1)(2y - 2) \).

These answers are fine, though they can be factorised further by taking the common factor 2 out of one of the brackets, which gives the answer above. It’s better to take the common factor out at the beginning, as this makes it easier to factorise.)

Solution to Activity 19
(a) \( x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2) \)
(b) \( x^2 - 4x = x(x - 4) \)
(c) \( 2x^2 + 5x = x(2x + 5) \)
(d) \( x^2 - 25 = x^2 - 5^2 = (x + 5)(x - 5) \)
(e) \( 9y^2 - 4 = (3y)^2 - 2^2 = (3y + 2)(3y - 2) \)
(f) \( 25t^2 - 1 = (5t)^2 - 1^2 = (5t + 1)(5t - 1) \)
(g) \( 9u^2 - u = u(9u - 1) \)
(h) \( p^2 + p = p(p + 1) \)

Solution to Activity 20
(a) \( x^2 + 4x - 21 = 0 \)
\( (x - 3)(x + 7) = 0 \)
\( x - 3 = 0 \) or \( x + 7 = 0 \)
\( x = 3 \) or \( x = -7 \)
(b) \( 3x^2 - 18x + 24 = 0 \)
\( x^2 - 6x + 8 = 0 \)
\( (x - 2)(x - 4) = 0 \)
\( x - 2 = 0 \) or \( x - 4 = 0 \)
\( x = 2 \) or \( x = 4 \)
(c) \( -t^2 - 6t - 9 = 0 \)
\( t^2 + 6t + 9 = 0 \)
\( (t + 3)(t + 3) = 0 \)
\( t + 3 = 0 \) or \( t + 3 = 0 \)
\( t = -3 \)
(d) \( 8x + 4x^2 - 5 = 0 \)
\( 4x^2 + 8x - 5 = 0 \)
\( (2x - 1)(2x + 5) = 0 \)
\( 2x - 1 = 0 \) or \( 2x + 5 = 0 \)
\( x = \frac{1}{2} \) or \( x = -\frac{5}{2} \)
(e) \( 3x^2 - x = 0 \)
\( x(3x - 1) = 0 \)
\( x = 0 \) or \( 3x - 1 = 0 \)
\( x = 0 \) or \( x = \frac{1}{3} \)
(f) \( x^2 - 16 = 0 \)
\( (x - 4)(x + 4) = 0 \)
\( x - 4 = 0 \) or \( x + 4 = 0 \)
\( x = 4 \) or \( x = -4 \)
Solution to Activity 21
(a) \( x^2 + 8x = (x + 4)^2 - 4^2 \)
\( = (x + 4)^2 - 16 \)

(b) \( x^2 - 4x = (x - 2)^2 - (-2)^2 \)
\( = (x - 2)^2 - 4 \)

(c) \( x^2 + x = (x + \frac{1}{2})^2 - (\frac{1}{2})^2 \)
\( = (x + \frac{1}{2})^2 - \frac{1}{4} \)

(d) \( t^2 - 3t = (t - \frac{3}{2})^2 - (\frac{3}{2})^2 \)
\( = (t - \frac{3}{2})^2 - \frac{9}{4} \)

Solution to Activity 22
(a) \( x^2 + 2x + 2 = (x + 1)^2 - 1 + 2 \)
\( = (x + 1)^2 + 1 \)

(b) \( x^2 + 3x - 1 = (x + \frac{3}{2})^2 - \frac{9}{4} - 1 \)
\( = (x + \frac{3}{2})^2 - \frac{13}{4} \)

(c) \( y^2 + \frac{1}{2}y - \frac{1}{4} = (y + \frac{1}{4})^2 - \frac{1}{16} - \frac{1}{4} \)
\( = (y + \frac{1}{4})^2 - \frac{5}{16} \)

Solution to Activity 23
(a) \( 3x^2 + 6x + 5 = 3(x^2 + 2x) + 5 \)
\( = 3((x + 1)^2 - 1) + 5 \)
\( = 3(x + 1)^2 - 3 + 5 \)
\( = 3(x + 1)^2 + 2 \)

(b) \( 2y^2 - 5y + 4 = 2(y^2 - \frac{5}{2}y) + 4 \)
\( = 2((y - \frac{5}{4})^2 - \frac{25}{16}) + 4 \)
\( = 2(y - \frac{5}{4})^2 - \frac{25}{8} + 4 \)
\( = 2(y - \frac{5}{4})^2 + \frac{7}{8} \)

(c) \( -x^2 + x - \frac{1}{2} = -(x^2 - x) - \frac{1}{2} \)
\( = -((x - \frac{1}{2})^2 - \frac{1}{4}) - \frac{1}{2} \)
\( = -(x - \frac{1}{2})^2 + \frac{1}{4} - \frac{1}{2} \)
\( = -(x - \frac{1}{2})^2 - \frac{1}{4} \)

Solution to Activity 24
(a) First complete the square on the left-hand side.
\( x^2 + 4x + 1 = 0 \)
\( (x + 2)^2 - 4 + 1 = 0 \)
\( (x + 2)^2 - 3 = 0 \)
Now solve this equation.
\( (x + 2)^2 = 3 \)
\( x + 2 = \sqrt{3} \) or \( x + 2 = -\sqrt{3} \)
\( x = \sqrt{3} - 2 \) or \( x = -\sqrt{3} - 2 \)
So the solutions are \( x = \sqrt{3} - 2 \) and \( x = -\sqrt{3} - 2 \).

(b) First complete the square on the left-hand side.
\( 3t^2 - 12t + 11 = 0 \)
\( 3(t^2 - 4t) + 11 = 0 \)
\( 3((t - 2)^2 - 4) + 11 = 0 \)
\( 3(t - 2)^2 - 1 = 0 \)
Now solve this equation.
\( (t - 2)^2 = \frac{1}{3} \)
\( t - 2 = \frac{1}{\sqrt{3}} \) or \( t - 2 = -\frac{1}{\sqrt{3}} \)
\( t = \frac{1}{\sqrt{3}} + 2 \) or \( t = -\frac{1}{\sqrt{3}} + 2 \)
So the solutions are \( t = 1/\sqrt{3} + 2 \) and \( t = -1/\sqrt{3} + 2 \).
Unit 2  Graphs and equations

(c) First complete the square on the left-hand side.

\[ 2x^2 + 3x - 3 = 0 \]
\[ 2 \left( x^2 + \frac{3}{2}x \right) - 3 = 0 \]
\[ 2 \left( x + \frac{3}{4} \right)^2 - \frac{9}{16} - 3 = 0 \]
\[ 2 \left( x + \frac{3}{4} \right)^2 - \frac{33}{8} = 0 \]

Now solve this equation.

\[ \left( x + \frac{3}{4} \right)^2 = \frac{33}{16} \]
\[ x + \frac{3}{4} = \frac{\sqrt{33}}{4} \quad \text{or} \quad x + \frac{3}{4} = -\frac{\sqrt{33}}{4} \]
\[ x = \frac{\sqrt{33} - 3}{4} \quad \text{or} \quad x = -\frac{\sqrt{33} + 3}{4} \]

So the solutions are

\[ x = \frac{\sqrt{33} - 3}{4} \quad \text{and} \quad x = -\frac{\sqrt{33} + 3}{4}. \]

Solution to Activity 25

(a) The equation is

\[ x^2 - 6x - 1 = 0, \]

so \( a = 1, \ b = -6 \) and \( c = -1. \)

The quadratic formula gives

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times (-1)}}{2 \times 1} \]
\[ = \frac{6 \pm \sqrt{36 + 4}}{2} \]
\[ = \frac{6 \pm \sqrt{40}}{2} \]
\[ = \frac{6 \pm 2\sqrt{10}}{2} \]
\[ = 3 \pm \sqrt{10}. \]

The solutions are \( x = 3 + \sqrt{10} \) and \( x = 3 - \sqrt{10}. \)

(b) The equation is

\[ 9x^2 + 15x - 6 = 0, \]

which can be simplified to

\[ 3x^2 + 5x - 2 = 0. \]

So \( a = 3, \ b = 5 \) and \( c = -2. \)

The quadratic formula gives

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-2)}}{2 \times 3} \]
\[ = \frac{-5 \pm \sqrt{25 + 24}}{6} \]
\[ = \frac{-5 \pm \sqrt{49}}{6} \]
\[ = \frac{-5 \pm 7}{6} \]
\[ = \frac{2}{6} \quad \text{or} \quad -\frac{12}{6} \]
\[ = \frac{1}{3} \quad \text{or} \quad -2. \]

Alternatively, factorisation gives

\[ 3x^2 + 5x - 2 = 0 \]
\[ (3x - 1)(x + 2) = 0 \]
\[ 3x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \]
\[ x = \frac{1}{3} \quad \text{or} \quad x = -2. \]

The solutions are \( x = \frac{1}{3} \) and \( x = -2. \)

(c) The equation is

\[ 9x^2 + 6x = 11, \]

which can be rearranged as

\[ 9x^2 + 6x - 11 = 0. \]

So \( a = 9, \ b = 6 \) and \( c = -11. \)
Solutions to activities

The quadratic formula gives
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times (-11)}}{2 \times 9} \]
\[ = \frac{-6 \pm \sqrt{36 + 396}}{18} \]
\[ = \frac{-6 \pm \sqrt{432}}{18} \]
\[ = \frac{-6 \pm 12\sqrt{3}}{18} \]
\[ = \frac{-1 \pm 2\sqrt{3}}{3}. \]
The solutions are \( x = \frac{1}{3}(-1 + 2\sqrt{3}) \) and \( x = \frac{1}{3}(-1 - 2\sqrt{3}). \)

(d) The equation is
\[ t^2 + \frac{5}{2}t + \frac{3}{2} = 0, \]
which can be simplified as
\[ 2t^2 + 5t + 3 = 0. \]
So \( a = 2, \ b = 5 \) and \( c = 3. \)
The quadratic formula gives
\[ t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times 3}}{2 \times 2} \]
\[ = \frac{-5 \pm \sqrt{25 - 24}}{4} \]
\[ = \frac{-5 \pm 1}{4} \]
\[ = \frac{-5 + 1}{4} \quad \text{or} \quad \frac{-5 - 1}{4} \]
\[ = \frac{-4}{4} \quad \text{or} \quad \frac{-6}{4} \]
\[ = -1 \quad \text{or} \quad -\frac{3}{2}. \]
Alternatively, factorisation gives
\[ 2t^2 + 5t + 3 = 0 \]
\[ (t + 1)(2t + 3) = 0 \]
\[ t + 1 = 0 \quad \text{or} \quad 2t + 3 = 0 \]
\[ t = -1 \quad \text{or} \quad t = -\frac{3}{2}. \]
The solutions are \( t = -1 \) and \( t = -\frac{3}{2}. \)

(e) The equation is \( u^2 = 4u - 4, \) which can be rearranged as \( u^2 - 4u + 4 = 0. \) So \( a = 1, \ b = -4 \) and \( c = 4. \)
The quadratic formula gives
\[ u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 4}}{2 \times 1} \]
\[ = \frac{4 \pm \sqrt{16 - 16}}{2} \]
\[ = \frac{4 \pm 0}{2} \]
\[ = 2. \]
Alternatively, factorisation gives
\[ u^2 - 4u + 4 = 0 \]
\[ (u - 2)(u - 2) = 0 \]
\[ u - 2 = 0 \quad \text{or} \quad u - 2 = 0 \]
\[ u = 2. \]
The only solution is \( u = 2. \)

Solution to Activity 26

(a) The equation is
\[ 4x^2 - 20x + 25 = 0, \]
so \( a = 4, \ b = -20 \) and \( c = 25. \)
The discriminant is
\[ b^2 - 4ac = (-20)^2 - 4 \times 4 \times 25 \]
\[ = 400 - 400 \]
\[ = 0. \]
Since the discriminant is 0, there is one solution.
The equation can be solved by factorising, as follows. Since there is only one solution, the two linear expressions in the factorisation must be the same (or one linear expression must be the other multiplied through by some number).
\[ 4x^2 - 20x + 25 = 0 \]
\[ (2x - 5)(2x - 5) = 0 \]
\[ (2x - 5)^2 = 0 \]
\[ 2x - 5 = 0 \]
\[ x = \frac{5}{2} \]
The only solution is \( x = \frac{5}{2}. \)
(b) The equation is
\[ 2x^2 + 6x + 5 = 0, \]
so \( a = 2, \ b = 6 \) and \( c = 5. \)
The discriminant is
\[ b^2 - 4ac = 6^2 - 4 \times 2 \times 5 = 36 - 40 = -4. \]
Since the discriminant is negative, there are no real solutions.

(c) The equation can be rearranged as follows:
\[ 4x^2 = 4x + 5, \]
\[ 4x^2 - 4x - 5 = 0. \]
So \( a = 4, \ b = -4 \) and \( c = -5. \)
The discriminant is
\[ b^2 - 4ac = (-4)^2 - 4 \times 4 \times (-5) = 16 + 80 = 96. \]
Since the discriminant is positive, there are two real solutions.
The equation cannot be factorised using integers, so we solve it by using the quadratic formula. This gives
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{96}}{2 \times 4} = \frac{4 \pm 4\sqrt{6}}{8} = \frac{1}{2}(1 \pm \sqrt{6}). \]
The solutions are \( x = \frac{1}{2}(1 + \sqrt{6}) \) and \( x = \frac{1}{2}(1 - \sqrt{6}). \)
(The value of \( b^2 - 4ac \) was already worked out before the quadratic formula was used, so this value was substituted in, instead of working it out again.)

Solution to Activity 27

(a) The equation is \( \frac{4}{x} = \frac{3x}{x+1}. \)
Assume that \( x \neq 0 \) and \( x \neq -1. \)
Multiply through by \( x(x+1) \) to clear the fractions.
\[ 4(x + 1) = 3x \times x \]
Rearrange this equation.
\[ 4x + 4 = 3x^2 \]
\[ 3x^2 - 4x - 4 = 0 \]
\[ (3x + 2)(x - 2) = 0 \]
\[ 3x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \]
\[ x = -\frac{2}{3} \quad \text{or} \quad x = 2 \]
Neither value is 0 or \(-1\), so the solutions of the original equation are \( x = -\frac{2}{3} \) and \( x = 2. \)

(b) The equation is \( \frac{1}{x - 2} = 1 + 4x. \)
Assume that \( x \neq 2. \)
Multiply through by \( x - 2 \) to clear the fraction.
\[ 1 = (1 + 4x)(x - 2) \]
Rearrange this equation.
\[ 4x^2 - 7x - 2 = 0 \]
This quadratic cannot be factorised, so use the quadratic formula with \( a = 4, \ b = -7, \ c = -3. \)
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 4 \times (-3)}}{2 \times 4} = \frac{7 \pm \sqrt{49 + 48}}{8} = \frac{7 \pm \sqrt{97}}{8} \]
Neither value is 2, so the solutions of the original equation are \( x = \frac{1}{8}(7 + \sqrt{97}) \) and \( x = \frac{1}{8}(7 - \sqrt{97}). \)

(c) The equation is \( 2x^3 - 2x^2 - 12x = 0. \)
Take out the common factor \( 2x. \)
\[ 2x(x^2 - x - 6) = 0 \]
So \( 2x = 0 \) or \( x^2 - x - 6 = 0. \)
The equation \( 2x = 0 \) gives \( x = 0. \)
Factorise the quadratic \( x^2 - x - 6. \)
\[ (x - 3)(x + 2) = 0 \]
So \( x - 3 = 0 \) or \( x + 2 = 0, \) giving \( x = 3 \) or \( x = -2. \)
So the solutions of \( 2x^3 - 2x^2 - 12x = 0 \) are \( x = 0, \ x = 3 \) and \( x = -2. \)
(d) The equation is \(x^4 - 2x^2 - 8 = 0\).

It includes only even powers of \(x\), so let \(X = x^2\).

The equation becomes
\[X^2 - 2X - 8 = 0.\]

Factorise the quadratic.
\[(X - 4)(X + 2) = 0\]

So \(X = 4\) or \(X = -2\).

Now \(X = x^2\), so \(x\) satisfies \(x^2 = 4\) or \(x^2 = -2\).

The equation \(x^2 = -2\) has no real solutions.

The equation \(x^2 = 4\) gives \(x = \pm 2\).

So the solutions are \(x = -2\) and \(x = 2\).

(e) The equation is \(u^4 - 4 = 0\), which can be rearranged as \(u^4 = 4\). Taking the square root of both sides gives \(u^2 = \pm 2\).

There are no real solutions satisfying \(u^2 = -2\).

The equation \(u^2 = 2\) gives two solutions,
\[u = \sqrt{2}\] and \(u = -\sqrt{2}\).

(f) The equation is \(s^5 - 9s^3 = 0\).

Taking out the common factor \(s^3\) gives
\[s^3(s^2 - 9) = 0.\]

So \(s^3 = 0\) or \(s^2 - 9 = 0\). The first equation has just one solution, \(s = 0\). The second equation has two solutions, \(s = \pm 3\).

So the solutions are \(s = 0\), \(s = 3\) and \(s = -3\).

(g) The equation is \((t^2 - 3)(t^2 - 3t + 2) = 0\).

So \(t^2 - 3 = 0\) or \(t^2 - 3t + 2 = 0\).

If \(t^2 - 3 = 0\), then \(t^2 = 3\), giving \(t = \pm \sqrt{3}\).

If \(t^2 - 3t + 2 = 0\), then \((t - 1)(t - 2) = 0\), giving \(t = 1\) or \(t = 2\).

So the solutions are \(t = 1\), \(t = 2\), \(t = \sqrt{3}\) and \(t = -\sqrt{3}\).

**Solution to Activity 28**

(a) The equation is \(y = -2x^2 + 3x - 1\).

Since the coefficient of \(x^2\) is negative, the graph is \(u\)-shaped.

Putting \(x = 0\) gives
\[y = -2 \times 0^2 + 3 \times 0 - 1 = -1,\]
so the \(y\)-intercept is \(-1\).

To find the \(x\)-intercepts (if any), solve
\[0 = -2x^2 + 3x - 1.\]

Simplify by multiplying through by \(-1\):
\[2x^2 - 3x + 1 = 0.\]

Factorising gives
\[(2x - 1)(x - 1) = 0.\]

So \(2x - 1 = 0\) or \(x - 1 = 0\), giving \(x = \frac{1}{2}\) or \(x = 1\). The \(x\)-intercepts are \(\frac{1}{2}\) and 1.

The axis of symmetry lies halfway between the points \((\frac{1}{2}, 0)\) and \((1, 0)\), so its equation is
\[x = \frac{1/2 + 1}{2} = \frac{3}{4}.\]

The vertex lies on the axis of symmetry, so it has \(y\)-coordinate
\[y = -2 \left(\frac{3}{4}\right)^2 + 3 \times \frac{3}{4} - 1 = \frac{1}{8}.\]

So the vertex is \((\frac{3}{4}, \frac{1}{8})\).

This is shown in the following graph.

(b) The equation is \(y = 2x^2 + 8x + 8\).

Since the coefficient of \(x^2\) is positive, the graph is \(u\)-shaped.

Putting \(x = 0\) gives \(y = 2 \times 0^2 + 8 \times 0 + 8 = 8\), so the \(y\)-intercept is 8.

To find the \(x\)-intercepts (if any), solve
\[0 = 2x^2 + 8x + 8.\]

Simplify by factoring out the common factor 2:
\[x^2 + 4x + 4 = 0.\]

Factorising gives
\[(x + 2)(x + 2) = 0.\]

So \(x + 2 = 0\), giving \(x = -2\) as the only \(x\)-intercept. Hence \((-2, 0)\) is the only point of the curve on the \(x\)-axis, so this is the vertex.
The graph is shown below.

Solution to Activity 29
The equation is \( y = x^2 - 2x + 4 \).
Since the coefficient of \( x^2 \) is positive, the graph is u-shaped.
Putting \( x = 0 \) gives \( y = 0^2 - 2 \times 0 + 4 = 4 \), so the \( y \)-intercept is 4 and the point (0, 4) lies on the curve.
To find the \( x \)-intercepts (if any), solve
\[
0 = x^2 - 2x + 4.
\]
The discriminant of this quadratic is
\[
b^2 - 4ac = (-2)^2 - 4 \times 1 \times 4 = 4 - 16 = -12,
\]
which is negative. So the quadratic equation above has no real solutions and there are no \( x \)-intercepts.
Rearrange the equation \( y = x^2 - 2x + 4 \) as
\[
y = x(x - 2) + 4.
\]
It follows that \( x = 0 \) and \( x = 2 \) have the same \( y \)-coordinate 4, so the points (0, 4) and (2, 4) lie on the graph.
The axis of symmetry lies halfway between the points (0, 4) and (2, 4), so its equation is
\[
x = \frac{1}{2}(0 + 2) = 1.
\]
The vertex lies on the axis of symmetry, so it has \( y \)-coordinate \( y = 1^2 - 2 \times 1 + 4 = 3 \).
So the vertex is (1, 3).

Solution to Activity 30
(a) Substitute \( t = 2 \) into \( s = 12t - 5t^2 \) to give
\[
s = 12 \times 2 - 5 \times 2^2 = 24 - 20 = 4.
\]
So the ball will be at height 4 m after 2 seconds.
(b) To find when the ball is at height 5 m, solve
\[
5 = 12t - 5t^2.
\]
Rearranging gives
\[
5t^2 - 12t + 5 = 0.
\]
Use the quadratic formula with \( a = 5 \), \( b = -12 \) and \( c = 5 \) to give
\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 5 \times 5}}{2 \times 5}
\]
\[
= \frac{12 \pm \sqrt{144}}{10}
\]
\[
= \frac{12 \pm 12}{10}
\]
\[
= 0.54 \text{ or } 1.86 \text{ (to 2 d.p.)}.
\]
So the ball is at height 5 m on two occasions, namely 0.54 s and 1.86 s after it was thrown (once on the way up and once on the way down).
(c) The ball returns to its starting position when its vertical displacement is zero, that is, when \( s = 0 \). Substituting \( s = 0 \) in \( s = 12t - 5t^2 \) gives
\[
0 = 12t - 5t^2.
\]
Factorising gives
\[
t(12 - 5t) = 0.
\]
So \( t = 0 \) or \( 12 - 5t = 0 \), giving \( t = 0 \) or \( t = \frac{12}{5} = 2.4 \).

The solution \( t = 0 \) corresponds to the time when the ball is at its starting position, at the beginning of its motion, so the ball returns to
this position 2.4 s later.

(d) Completing the square in the quadratic expression for $s$ gives

$$s = -5 \left( t^2 - \frac{12}{5} t \right)$$

$$= -5 \left( \left( t - \frac{6}{5} \right)^2 - \frac{36}{25} \right)$$

$$= -5 \left( t - \frac{6}{5} \right)^2 + \frac{36}{5}.$$

The maximum value of $s$ occurs when $5 \left( t - \frac{6}{5} \right)^2$ is zero, that is, when $t = \frac{6}{5} = 1.2$. So the maximum height that a ball will reach above its starting position is 1.2 m.

When $t = \frac{6}{5}$ the value of $s$ is $\frac{36}{5} = 7.2$, so the maximum height of the ball is 7.2 m.
Acknowledgements

Grateful acknowledgement is made to the following sources:

Page 147: Taken from:
www.metoffice.gov.uk/media/image/4/1/forecast_chart.jpg

Page 156: Credit: Jodrell Bank Centre for Astrophysics, University of Manchester

Page 156: Michael Mags.

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