Unit A2

Complex functions
Introduction

In Unit A1 we introduced complex numbers and described a number of fundamental operations using them; in particular, we investigated the solution of equations involving complex numbers. We now study complex functions and we find that many of the standard real functions (polynomial, rational, exponential, trigonometric and hyperbolic) have complex analogues with remarkable geometric properties.

In Section 1 we establish the basic language associated with complex functions, such as sums, products, quotients and composites of functions. We also discuss the problems associated with forming inverses of complex functions.

In Section 2 we discuss two special types of complex function, namely those with codomain $\mathbb{R}$ and those with domain a subset of $\mathbb{R}$. Each of these types of function has a role to play in our understanding of the geometric effect of a complex function, which we study in Section 3. There we examine some particular complex functions in detail and sketch the images of various ‘grids’ under these functions.

In Section 4 we introduce the complex exponential function and describe its geometric properties. The complex trigonometric and hyperbolic functions are then defined in terms of this exponential function.

Finally, in Section 5, we introduce the complex logarithm function, which will have an important part to play in complex integration; we also discuss complex powers.

Unit guide

You should find that Section 1 is mainly revision and you should aim to work through it fairly quickly. Sections 2 and 3 are closely related, so you might try to study them in one session. Sections 4 and 5 are also closely related, and they contain a good deal of basic technical material which will be used throughout the module.

1 Complex functions and their properties

After working through this section, you should be able to:

- determine the domain and rule of the sum, product and quotient of two complex functions
- determine the domain and rule of the composite of two complex functions
- determine whether a given complex function has an inverse function, and find that inverse function in suitable cases.
1.1 Defining complex functions

The main aim of complex analysis is to extend the theory of calculus to include functions of a complex variable; such functions are called *complex functions*.

**Definitions**

A *complex function* $f$ is defined by specifying:

- two sets $A$ and $B$ in the complex plane $\mathbb{C}$
- a rule that associates with each number $z$ in $A$ a unique number $w$ in $B$; we write $w = f(z)$.

The set $A$ is called the **domain** of the function $f$, and the set $B$ is called the **codomain** of $f$.

The number $w$ is called the **image of $z$ under $f$**, or the **value of $f$ at $z$**, and we say that $f$ maps $z$ to $w$.

For example, consider the expression

$$f : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto z^2,$$

(1.1)

which defines a complex function with $A = \mathbb{C}$ and $B = \mathbb{C}$, and with rule given by $f(z) = z^2$. Under this function $f$, the image of $z = 2i$ is $w = f(2i) = (2i)^2 = -4$, and, similarly, the image of $1 - i$ is $(1 - i)^2 = -2i$.

These values are plotted in Figure 1.1; the points $2i$ and $1 - i$ are shown in the domain (the $z$-plane) and the corresponding images are shown in the codomain (the $w$-plane).

![Figure 1.1 Images of points under the function $f(z) = z^2$](image)

We will often use this type of diagrammatic representation for specific complex functions, whereas a general complex function $f : A \to B$ may be represented by a diagram in which the sets $A$ and $B$ appear as ‘blobs’ (Figure 1.2).
1 Complex functions and their properties

Figure 1.2 Representation of a complex function \( f: A \rightarrow B \)

These diagrams make it clear why some texts refer to the codomain \( B \) as the **range** or **target** of \( f \).

Observe that in Figure 1.1 we have labelled the axes of the \( z \)-plane with \( x \) and \( y \) and labelled the axes of the \( w \)-plane with \( u \) and \( v \), to help distinguish the two planes. We retain this convention for labelling the axes of the complex plane for many of the figures in this unit (where it seems helpful). In other units, however, we do not generally label the axes of the complex plane.

Usually we will refer to a complex function \( f \) simply as a ‘function’, unless there is some particular reason to emphasise that \( f \) is a complex function. Other commonly used words for function are **map**, **mapping** and **transformation**.

The notation (1.1) can be written more concisely in one of the forms

\[
  f(z) = z^2 \quad (z \in \mathbb{C}) \quad \text{or} \quad z \mapsto z^2 \quad (z \in \mathbb{C}),
\]

where it is assumed that the codomain is \( \mathbb{C} \), or in one of the forms

\[
  f(z) = z^2 \quad \text{or} \quad z \mapsto z^2,
\]

where it is assumed that both the domain and codomain are \( \mathbb{C} \). To avoid uncertainty, we adopt the following convention.

**Convention**

When a function \( f \) is specified *just* by its rule, it is to be understood that the domain of \( f \) is the set of all complex numbers to which the rule is applicable, and the codomain of \( f \) is \( \mathbb{C} \).

For example, the function

\[
  f(z) = \frac{1}{z-i}
\]

has domain \( \mathbb{C} - \{i\} \) because \( 1/(z - i) \) is defined for all complex numbers \( z \) other than \( z = i \). Its codomain is \( \mathbb{C} \).
Exercise 1.1

Using the convention above, write down the domain and codomain of each of the following functions.

(a) \( f(z) = z + 2 \)
(b) \( f(z) = \frac{z}{z + 2} \)
(c) \( f(z) = \text{Arg } z \)
(d) \( f(z) = \frac{1}{z^2 + 1} \)

1.2 The image set of a function

If a function \( f \) has domain \( A \) and codomain \( B \), then for each \( z \) in \( A \) the image \( w = f(z) \) is in \( B \). However, it is not necessarily true that for each \( w \) in \( B \), there is some \( z \) in \( A \) such that \( f(z) = w \). For example, if \( f(z) = 1/(z - i) \), then the domain of \( f \) is \( \mathbb{C} \setminus \{i\} \) and the codomain is \( \mathbb{C} \) (by the convention). However, there is no point \( z \) in the domain of \( f \) which maps to the point 0 (in the codomain); in other words, there is no \( z \) in \( \mathbb{C} \setminus \{i\} \) such that \( f(z) = 0 \).

Definitions

Given a function \( f : A \rightarrow B \), the image set of \( f \), written \( f(A) \), is the set of all values \( f(z) \), where \( z \in A \). Thus

\[
f(A) = \{ f(z) : z \in A \}.
\]

If \( f(A) = B \), then the function \( f \) is said to be onto.

Note that some texts use surjective rather than onto.

Figure 1.3 depicts a function that is not onto because \( f(A) \neq B \).

---

Example 1.1

Determine the image set of the function

\[
f(z) = \frac{1}{z - i}.
\]
Solution
The domain of $f$ is $A = \mathbb{C} - \{i\}$ and $f(z) = \frac{1}{z - i}$, so we have

\[
f(A) = \left\{ \frac{1}{z - i} : z \in \mathbb{C} - \{i\} \right\}
= \left\{ w = \frac{1}{z - i} : z \neq i \right\}
= \left\{ w : z = i + \frac{1}{w} \neq i \right\},
\]
where the last line was obtained by rearranging $w = 1/(z - i)$ to give $z = i + 1/w$. Since $z = i + 1/w$ exists and is not equal to $i$ if and only if $w \neq 0$, we see that

\[
f(A) = \{ w : w \neq 0 \} = \mathbb{C} - \{0\}.
\]

In the preceding example, we found $f(A)$ by defining $w = f(z)$ and then expressing $f(A)$ in the form $\{ w : \text{condition on } w \}$. This is a useful strategy to adopt when approaching similar examples (even if you can ‘see’ what the image set is, as you may have done in Example 1.1).

Exercise 1.2

For each of the following functions $f$, determine the image set of $f$.

(a) $f(z) = 3iz$  
(b) $f(z) = \frac{3z + 1}{z + i}$  
(c) $f(z) = \text{Im } z$

Several important functions, such as

\[
z \mapsto \text{Re } z, \quad z \mapsto \text{Im } z, \quad z \mapsto |z|, \quad z \mapsto \text{Arg } z,
\]
have image sets that are subsets of the real line. For example, the function $f(z) = \text{Re } z$ has image set $f(\mathbb{C}) = \mathbb{R}$, the whole real line (Figure 1.4). Such functions are called real-valued functions. Be careful not to confuse real-valued functions with real functions, which are functions whose domain and codomain consist of real numbers.

Definitions

A function $f : A \rightarrow B$ is called a \textbf{real-valued function} (of a complex variable) if $f(A) \subseteq \mathbb{R}$.

The function $f$ is called a \textbf{real function} if $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. 

\[f(\mathbb{C}) = \mathbb{R}\]
Exercise 1.3

Write down the image sets of each of the following functions.

(a) \( f(z) = |z| \)  
(b) \( f(z) = \text{Arg } z \)

1.3 Sums, products and quotients of functions

Let \( f \) and \( g \) be the functions

\[
 f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = z^2 + 1 \quad (z \in \mathbb{C}).
\]

The expressions \( f + g \), \( fg \) and \( f/g \) are used to denote the following functions:

\[
 (f + g)(z) = f(z) + g(z) = \frac{1}{z} + (z^2 + 1) \quad (z \in \mathbb{C} - \{0\}),
\]

\[
 (fg)(z) = f(z)g(z) = \frac{1}{z}(z^2 + 1) \quad (z \in \mathbb{C} - \{0\}),
\]

\[
 (f/g)(z) = f(z)/g(z) = \frac{1}{z} \left( \frac{1}{z^2 + 1} \right) \quad (z \in \mathbb{C} - \{0, i, -i\}).
\]

The domains of \( f + g \) and \( fg \) include only those points at which both \( f \) and \( g \) are defined. When forming the quotient \( f/g \), we must also exclude from the domain those points \( z \) at which \( g(z) = 0 \). The points at which a function takes the value zero are called the zeros of the function.

Definitions

Let \( f: A \rightarrow \mathbb{C} \) and \( g: B \rightarrow \mathbb{C} \) be functions.

The sum \( f + g \) is the function with domain \( A \cap B \) and rule

\[
 (f + g)(z) = f(z) + g(z).
\]

The multiple \( \lambda f \), where \( \lambda \in \mathbb{C} \), is the function with domain \( A \) and rule

\[
 (\lambda f)(z) = \lambda f(z).
\]

The product \( fg \) is the function with domain \( A \cap B \) and rule

\[
 (fg)(z) = f(z)g(z).
\]

The quotient \( f/g \) is the function with domain \( A \cap B - \{z: g(z) = 0\} \) and rule

\[
 (f/g)(z) = f(z)/g(z).
\]

The next exercise gives you practice at combining functions.
Exercise 1.4

Let $f$ and $g$ be the functions
\[ f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = \frac{z + 3i}{z^2 - z} \quad (z \in \mathbb{C} - \{0, 1\}). \]

Determine the domain and rule of each of the following functions.

(a) $f + g$  
(b) $fg$  
(c) $f/g$

Starting from the two basic functions $z \mapsto 1$ and $z \mapsto z$, we can build up any polynomial function of degree $n$
\[ p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \]
where $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$, by forming suitable sums, multiples and products. The domain of any polynomial function is $\mathbb{C}$. Allowing quotients also, we can build up any rational function, that is, any function of the form
\[ f(z) = \frac{p(z)}{q(z)}, \]
where $p$ and $q$ are polynomial functions. It follows from the definition of quotient that the domain of such a rational function is
\[ \mathbb{C} - \{z : q(z) = 0\}. \]

For example, the rational function
\[ f(z) = \frac{z}{z^2 + 1} \]
has domain $\mathbb{C} - \{i, -i\}$.

1.4 Composite functions

Let $f$ and $g$ be functions. The composite function $g \circ f$ is obtained by applying first $f$ and then $g$. Thus the rule of $g \circ f$ is
\[ (g \circ f)(z) = g(f(z)). \]

The process of forming $g \circ f$ is called ‘composition of functions’ or ‘composing functions’.

For example, if
\[ f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = z^2 + 1 \quad (z \in \mathbb{C}), \]
then the rule of $g \circ f$ is
\[ (g \circ f)(z) = g(1/z) = (1/z)^2 + 1, \]
whereas the rule of $f \circ g$ is
\[ (f \circ g)(z) = f(z^2 + 1) = 1/(z^2 + 1). \]
But what is the domain of a composite function? In general, if \( f : A \rightarrow \mathbb{C} \) and \( g : B \rightarrow \mathbb{C} \), then the value \( g(f(z)) \) is defined if and only if 

\[
\begin{align*}
z & \text{ lies in } A \\
f(z) & \text{ lies in } B.
\end{align*}
\]

Thus if \( z_1 \) and \( z_2 \) are elements of \( A \) such that \( f(z_1) \in B \) but \( f(z_2) \notin B \), then \( g(f(z)) \) is defined for \( z = z_1 \) but not for \( z = z_2 \), as indicated in Figure 1.5.

![Figure 1.5 Composition of two functions \( f \) and \( g \)](image)

We define the domain of \( g \circ f \) to be consistent with this.

**Definition**

Let \( f : A \rightarrow \mathbb{C} \) and \( g : B \rightarrow \mathbb{C} \) be complex functions. Then the **composite function** \( g \circ f \) has domain

\[
\{ z \in A : f(z) \in B \}
\]

and rule

\[
(g \circ f)(z) = g(f(z)).
\]

To find the domain of \( g \circ f \), we remove from \( A \) (the domain of \( f \)) each point whose image under \( f \) is not in \( B \) (the domain of \( g \)). Thus the domain of \( g \circ f \) can be written as

\[
A - \{ z : f(z) \notin B \}.
\]

For example, if

\[
\begin{align*}
f(z) &= z^2 + i \quad (z \in \mathbb{C}) \\
g(z) &= \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}),
\end{align*}
\]

then the domain of \( g \circ f \) is

\[
\mathbb{C} - \{ z : z^2 + i \notin \mathbb{C} - \{i\} \} = \mathbb{C} - \{ z : z^2 + i = i \} = \mathbb{C} - \{0\}.
\]

In practice, it often happens that the image set of \( f \) is contained in \( B \) (that is, \( f(A) \subseteq B \)) and in this case the domain of \( g \circ f \) is \( A \) itself. (This always happens when the domain of \( g \) is \( \mathbb{C} \), of course.) For example, if

\[
\begin{align*}
f(z) &= \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}) \\
g(z) &= \operatorname{Arg} z \quad (z \in \mathbb{C} - \{0\}),
\end{align*}
\]
then $A = \mathbb{C} - \{i\}$, $B = \mathbb{C} - \{0\}$. Also, as you saw in Example 1.1,

$$f(\mathbb{C} - \{i\}) = \mathbb{C} - \{0\},$$

which is (contained in) the domain of $g$. Thus the domain of $g \circ f$ is $\mathbb{C} - \{i\}$. In fact, $g(f(z)) = \text{Arg}(1/(z - i))$ and the only complex number $z$ for which $\text{Arg}(1/(z - i))$ is not defined is $z = i$.

We remark that, in contrast to our approach, some texts require $f(A) \subseteq B$ in the definition of $g \circ f$.

**Exercise 1.5**

Let $f$ and $g$ be the functions

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = \frac{z + 3i}{z^2 - z} \quad (z \in \mathbb{C} - \{0, 1\}).$$

Determine the domain and rule of each of the following functions.

(a) $g \circ f$  
(b) $f \circ g$

### 1.5 Inverse functions

Let $f$ be the function

$$f(z) = 3z \quad (z \in \mathbb{C}).$$

Then for each number $w$ in $\mathbb{C}$, there is a unique number $z = \frac{1}{3}w$ in the domain of $f$ such that

$$f(z) = f\left(\frac{1}{3}w\right) = 3 \times \frac{1}{3}w = w.$$

The corresponding function $g: w \mapsto \frac{1}{3}w$ is called the *inverse function* of $f$ because it ‘undoes’ the effect of $f$. To be precise,

$$g(f(z)) = \frac{1}{3}f(z) = \frac{1}{3} \times 3z = z \quad (z \in \mathbb{C}).$$

Similarly, $f$ undoes $g$:

$$f(g(w)) = 3g(w) = 3 \times \frac{1}{3}w = w \quad (w \in \mathbb{C}).$$

The inverse function of $f$ is denoted by $f^{-1}$ (see Figure 1.6).

**Figure 1.6** Images of points under the functions $f(z) = 3z$ and $f^{-1}(w) = \frac{1}{3}w$
Not every function has an inverse function. For example, consider the function

\[ f(z) = z^2 \quad (z \in \mathbb{C}). \]

Since \( f(2) = 4 \) and \( f(-2) = 4 \), we cannot assign a unique value \( z \) in the domain of \( f \) such that \( f(z) = 4 \) (Figure 1.7).

![Figure 1.7](image)

The problem here is that the function \( f \) is not one-to-one. In general, it is possible to define the inverse of a function only if that function is one-to-one.

**Definition**

The function \( f: A \rightarrow B \) is one-to-one if the images under \( f \) of distinct points in \( A \) are also distinct; that is,

if \( z_1, z_2 \in A \) and \( z_1 \neq z_2 \), then \( f(z_1) \neq f(z_2) \).

Note that some texts use injective rather than one-to-one.

An equivalent statement of the one-to-one condition is that

if \( w \in f(A) \), then there is a unique \( z \) in \( A \) such that \( f(z) = w \).

This principle is illustrated in Figure 1.8, in which there is only one point \( z \) in \( A \) whose image under \( f \) is \( w \).

![Figure 1.8](image)
The uniqueness statement of the second version of the one-to-one condition makes it possible to define the inverse function \( f^{-1} \) of \( f \) with domain \( f(A) \). This is illustrated in Figure 1.9.

\[ z = f^{-1}(w) \]

**Figure 1.9** The inverse function \( f^{-1} : f(A) \rightarrow A \)

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**Definition**

Let \( f : A \rightarrow B \) be a one-to-one function. Then the **inverse function** \( f^{-1} \) of \( f \) has domain \( f(A) \) and rule

\[ f^{-1}(w) = z, \]

where \( w = f(z) \).

Thus there are two ways of proving that a function has an inverse function.

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**Strategy for proving that an inverse function exists**

To prove that a function \( f \) has an inverse function:

- **either** prove that \( f \) is one-to-one directly by showing that if \( z_1 \neq z_2 \), then \( f(z_1) \neq f(z_2) \)
- **or** determine the image set \( f(A) \) and show that for each \( w \in f(A) \) there is a unique \( z \in A \) such that \( f(z) = w \).

Notice that the statement of the first strategy ‘if \( z_1 \neq z_2 \), then \( f(z_1) \neq f(z_2) \)’ is equivalent to

\[ f(z_1) = f(z_2) \implies z_1 = z_2. \]

For the second strategy, one way of demonstrating that ‘there is a unique \( z \in A \) such that \( f(z) = w \)’ is to **find** the rule for \( f^{-1} \) (if this is possible). For some functions this can be done by solving the equation \( w = f(z) \) to obtain a unique \( z \) in terms of \( w \). We adopt the second strategy (rather than the first strategy) with this approach whenever it is possible, because it has the advantage that we thereby specify the function \( f^{-1} \). This strategy is illustrated in the next example.
Example 1.2

Prove that the function

\[ f(z) = \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}) \]

has an inverse function, and determine the domain and rule of \( f^{-1} \).

Solution

First we determine the image set of \( f \). This is

\[ f(\mathbb{C} - \{i\}) = \mathbb{C} - \{0\} \quad \text{(from Example 1.1)}. \]

Now, for each \( w \in \mathbb{C} - \{0\} \), we wish to solve the equation

\[ w = \frac{1}{z - i} \]

to obtain a unique solution \( z \) in \( \mathbb{C} - \{i\} \). This is achieved by the rearrangement

\[ z = i + \frac{1}{w}. \]

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} - \{0\} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} - \{0\} \) and rule

\[ f^{-1}(w) = i + \frac{1}{w}. \]

Remark

Usually when defining a function, we write \( z \) for the domain variable. To conform to this practice, we could rewrite the inverse function \( f^{-1} \) in Example 1.2 in the form

\[ f^{-1}(z) = i + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}), \]

with \( z \) in place of \( w \), after the required algebraic manipulations have been completed. It is not necessary to do this as a matter of routine.

Exercise 1.6

Prove that the function

\[ f(z) = \frac{3z + 1}{z + i} \quad (z \in \mathbb{C} - \{-i\}) \]

has an inverse function \( f^{-1} \), and determine the domain and rule of \( f^{-1} \). (See Exercise 1.2(b).)

As we pointed out earlier, the function \( f(z) = z^2 \) is not one-to-one (on \( \mathbb{C} \)), so it does not have an inverse function. One way round this difficulty is to
reduce the domain of \( f \) (without changing the rule) so as to make the
resulting function one-to-one. Note that reducing the domain of a function
leads to a new function – a restriction of the original function – which
should really be denoted by a different letter. In practice, we usually retain
the same letter, particularly if the original domain is no longer under
discussion. Here is an example of this.

**Example 1.3**

Let \( A = \{0\} \cup \{z : -\pi/2 < \text{Arg} \ z \leq \pi/2\} \), as shown in Figure 1.10.
Prove that the function

\[ f(z) = z^2 \quad (z \in A) \]

has an inverse function \( f^{-1} \), and determine the domain and rule
of \( f^{-1} \).

**Solution**

Let us first determine the image set of \( f \). By writing
\( z = r(\cos \theta + i \sin \theta) \), we obtain

\[ f(A) = \{z^2 : z \in A\} \]
\[ = \{0\} \cup \{w = z^2 : -\pi/2 < \text{Arg} \ z \leq \pi/2\} \]
\[ = \{0\} \cup \{w = r^2(\cos 2\theta + i \sin 2\theta) : r > 0, -\pi/2 < \theta \leq \pi/2\} \]
\[ = \{0\} \cup \{w = \rho(\cos \phi + i \sin \phi) : \rho > 0, -\pi < \phi \leq \pi\}, \]

where \( \rho = r^2 \) and \( \phi = 2\theta \). Thus \( f(A) = \mathbb{C} \).

Now, for each \( w \in \mathbb{C} \), we wish to solve the equation
\[ w = z^2 \] (1.2)

to obtain a unique solution \( z \) in \( A \). If \( w = 0 \), then equation (1.2) has
the unique solution \( z = 0 \). If \( w \neq 0 \), then \( w \) can be written in the form
\[ w = \rho(\cos \phi + i \sin \phi), \]
where \( \rho > 0 \) and \( -\pi < \phi \leq \pi \), and equation (1.2) has exactly two
solutions:
\[ z_0 = \rho^{1/2}(\cos(\phi/2) + i \sin(\phi/2)), \]
\[ z_1 = \rho^{1/2}(\cos(\phi/2 + \pi) + i \sin(\phi/2 + \pi)) \]
(by Theorem 3.1 of Unit A1), which are shown in Figure 1.11.

Clearly, \( z_0 \in A \), since \( -\pi/2 < \phi/2 \leq \pi/2 \), whereas \( z_1 \notin A \).

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} \). Hence \( f \) has an
inverse function \( f^{-1} \) with domain \( \mathbb{C} \) and rule given by \( f^{-1}(0) = 0 \) and,
for \( w \neq 0 \),
\[ f^{-1}(w) = \rho^{1/2}(\cos(\phi/2) + i \sin(\phi/2)), \]

where \( w = \rho(\cos \phi + i \sin \phi), \rho > 0, -\pi < \phi \leq \pi \).
Remark
In this solution we chose \( \phi \) to satisfy \(-\pi < \phi \leq \pi\) so that
\[-\frac{\pi}{2} < \phi / 2 \leq \frac{\pi}{2},\]
and hence \( z_0 \in A \). Since \( \phi = \text{Arg} \, w \), for \( w \neq 0 \), it follows that \( z_0 \) is the principal square root \( \sqrt{w} \) of \( w \). Furthermore, because we defined \( \sqrt{0} = 0 \) (see Subsection 3.1 of Unit A1), the rule for \( f^{-1} \) can be written in the form
\[ f^{-1}(w) = \sqrt{w} \quad (w \in \mathbb{C}). \]
The set \( A \) in Example 1.3 is not the only one on which the function
\( z \mapsto z^2 \) is one-to-one with image set \( \mathbb{C} \). In the following exercise, you are asked to investigate another such set.

**Exercise 1.7**
Let \( A = \{0\} \cup \{z : 0 \leq \text{Arg} \, z < \pi\} \). Prove that the function
\[ f(z) = z^2 \quad (z \in A) \]
has an inverse function \( f^{-1} \), and determine the domain and rule of \( f^{-1} \).

**Further exercises**

**Exercise 1.8**
Write down the domain of each of the following functions.
(a) \( f(z) = (z - 1)^2 \)  
(b) \( f(z) = \frac{1}{z - 1} \)  
(c) \( f(z) = \frac{z}{z^2 + 1} \)  
(d) \( f(z) = \frac{1}{\Re z} \)  
(e) \( f(z) = \frac{1}{|z| - 1} \)  
(f) \( f(z) = \frac{1}{z^3 + 1} \)

**Exercise 1.9**
Determine the image set of each of the following functions.
(a) \( f(z) = 2z + 1 \)  
(b) \( f(z) = \frac{1}{z - 1} \)  
(c) \( f(z) = \frac{z}{z - 1} \)  
(d) \( f(z) = |z - 1| \)  
(e) \( f(z) = \Re(z + i) \)  
(f) \( f(z) = |\text{Arg} \, z| \)

**Exercise 1.10**
Let
\[ f(z) = \frac{z - 1}{z} \quad \text{and} \quad g(z) = \frac{z}{z - 1}. \]
Determine the domain and rule of each of the following functions.
(a) \( f + g \)  
(b) \( 3f - 2ig \)  
(c) \( fg \)  
(d) \( f/g \)
Exercise 1.11

For the functions $f$ and $g$ of Exercise 1.10, write down the domain and rule of each of the following functions.

(a) $f \circ g$
(b) $g \circ f$
(c) $f \circ f$

Exercise 1.12

Determine whether or not each of the functions $f$ in Exercise 1.9 is one-to-one, and write down the inverse function of $f$, where possible.

Exercise 1.13

Let $A = \{0\} \cup \{z : -\pi/3 < \text{Arg} \ z \leq \pi/3\}$. Prove that the function

$$f(z) = z^3 \quad (z \in A)$$

has an inverse function $f^{-1}$, and determine the domain and rule of $f^{-1}$.

2 Special types of complex function

After working through this section, you should be able to:

• find the real and imaginary parts of a complex function
• sketch a path
• obtain (where possible) the equation of a path by eliminating the parameter from its parametrisation
• find the image under a function of a path in simple cases
• use the table of standard parametrisations.

2.1 Real-valued functions

In Section 1 we pointed out that various common functions, such as $z \mapsto |z|$, have image sets in $\mathbb{R}$ and are therefore called real-valued functions. Because the image sets of such functions lie in $\mathbb{R}$, we do not have to resort to the $z$-plane/w-plane representation of the functions. In fact, we can sketch the graph of a real-valued function by introducing a third axis, which is at right angles to the complex plane ($z$-plane). We call this third axis the $s$-axis (rather than the $z$-axis because of possible confusion with the complex variable $z = x + iy$). The graph of the function $z \mapsto |z|$ is shown in Figure 2.1; it is the surface with equation $s = |z| = \sqrt{x^2 + y^2}$. 
The surface is an infinite cone with apex at the origin and with axis the positive \( s \)-axis. Any horizontal plane with equation \( s = c \), where \( c \) is a positive constant, intersects the cone in a circle of radius \( c \) at height \( c \) above the \((x, y)\)-plane, as shown in Figure 2.1.

Notice that the equation \( s = c \) represents a plane in three dimensions – the plane perpendicular to the \( s \)-axis through the point \((0, 0, c)\) – and likewise the equations \( x = c \) and \( y = c \) also represent planes in three dimensions. In contrast, in the two-dimensional complex plane, the equation \( x = c \) represents the line parallel to the imaginary axis made up of complex numbers with real part \( c \).

Figure 2.2 shows the spiral-like surface \( s = \text{Arg } z \). It is obtained by lifting (that is, translating vertically) each of the rays \( \{ z : \text{Arg } z = c \} \), where \( -\pi < c \leq \pi \), in the \( z \)-plane to height \( c \). Two such lifted rays, with \( c = \pi/3 \) and \( c = -\pi/4 \), are shown in the figure. Each lifted ray is the intersection of the surface with a plane with equation \( s = c \), for some constant \( c \).
Sketching such surfaces is rather demanding from an artistic point of view, and we do not expect you to be skilled at it. Do not spend more than a few minutes on the following exercise.

**Exercise 2.1**

Sketch the following surfaces.

(a) \( s = \text{Re } z \)  
(b) \( s = \text{Im } z \)

Real-valued functions of a complex variable arise naturally when we study complex functions. For example, if \( f(z) = z^2 \), where \( z = x + iy \), then

\[
f(z) = (x + iy)^2 = (x^2 - y^2) + 2ixy;
\]

thus

\[
f(z) = u + iv,
\]

where

\[
u = x^2 - y^2, \quad v = 2xy.
\]

Here both \( z \mapsto u \) and \( z \mapsto v \) are real-valued functions of the complex variable \( z \).

In general, for any complex function \( f \), we can write \( f(z) \) in the form

\[
f(z) = u + iv,
\]

where \( u \) and \( v \) are real. The values of these real numbers change as \( z \) changes, giving rise to two real-valued functions \( z \mapsto u \) and \( z \mapsto v \) with the same domain as \( f \). They are called the **real** and **imaginary parts** of \( f \), written \( \text{Re } f \) and \( \text{Im } f \), respectively. Using the notation for the real and imaginary parts of a complex number introduced in Unit A1, we have

\[
\text{Re } f: z \mapsto \text{Re}(f(z)) \quad \text{and} \quad \text{Im } f: z \mapsto \text{Im}(f(z)).
\]

Thus

\[
(\text{Re } f)(z) = \text{Re}(f(z)) \quad \text{and} \quad (\text{Im } f)(z) = \text{Im}(f(z)).
\]

**Exercise 2.2**

Determine the functions \( \text{Re } f \) and \( \text{Im } f \) for the function \( f(z) = 1/z \).

If we are given two real-valued functions \( g \) and \( h \) with the same domain \( A \) in \( \mathbb{C} \), then we can combine them to obtain a function \( f: A \rightarrow \mathbb{C} \) by writing

\[
f(z) = g(z) + ih(z) \quad (z \in A).
\]
For example, consider
\[ g(z) = \log |z| \]
(where \( \log x \) denotes the real-valued logarithm to base \( e \) of a positive number \( x \), sometimes written as \( \ln x \) or \( \log_e x \) in other texts) and
\[ h(z) = \text{Arg} \, z. \]

Both \( g \) and \( h \) are real-valued functions with domain \( \mathbb{C} - \{0\} \), and the function
\[ f(z) = g(z) + ih(z) = \log |z| + i \text{Arg} \, z \]
also has domain \( \mathbb{C} - \{0\} \).

Figure 2.2 gave the graph of the surface \( s = \text{Arg} \, z \), and the graph of the surface \( s = \log |z| \) is shown in Figure 2.3; for each real number \( c \), the horizontal plane \( s = c \) intersects the surface in a circle of radius \( e^c \). This is because
\[ s = c \text{ and } s = \log |z| \iff \log |z| = c \iff |z| = e^c. \]

We will discuss the function \( f(z) = \log |z| + i \text{Arg} \, z \) and some of its properties later in the unit.

Figure 2.3  Graph of \( s = \log |z| \)

### 2.2 Functions with domains in the real numbers

In order to gain insight into the geometric effect of a given complex function \( f \), it is helpful to be able to picture how the image point \( w = f(z) \) behaves as \( z \) moves around the domain of \( f \). This subsection is about making these intuitive geometric ideas precise.

When a point moves in a plane, it traces a curve or path as time passes. The position of the point on this path can be described by giving both the \( x \)- and \( y \)-coordinates of the point as functions of time, \( t \). In this context, the real variable \( t \) is called a parameter.
For example, suppose that the $x$- and $y$-coordinates are given by the equations

$$x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]).$$

Then, as time $t$ increases from 0 to $2\pi$, the point

$$z = x + iy = \cos t + i \sin t$$

moves around the circle $\Gamma$ with centre 0 and radius 1, starting (when $t = 0$) and finishing (when $t = 2\pi$) at the point 1, as indicated in Figure 2.4.

If we introduce the function

$$\gamma(t) = \cos t + i \sin t \quad (t \in [0, 2\pi]),$$

then the circle $\Gamma$ is the image set of $\gamma$; that is, $\Gamma = \gamma([0, 2\pi])$. The function $\gamma$ describes a mode of traversing the circle $\Gamma$. In general, a set $\Gamma$ and the associated function $\gamma$ are the ingredients in our definition of the term ‘path’.

Definitions

A **path** is a subset $\Gamma$ of $\mathbb{C}$ which is the image set of an associated continuous function $\gamma : I \rightarrow \mathbb{C}$, where $I$ is a real interval. In this context, the function $\gamma$ is called a **parametrisation** (of $\Gamma$). If

$$\gamma(t) = \phi(t) + i \psi(t) \quad (t \in I),$$

where $\phi$ and $\psi$ are real functions, then the equations

$$x = \phi(t), \quad y = \psi(t) \quad (t \in I)$$

are called **parametric equations** (of $\Gamma$).

If $I$ is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of $\Gamma$, respectively.

Figure 2.5 illustrates these definitions in the case of a closed interval $I = [a, b]$. 

[Diagram of a path $\Gamma$ with parametrisation $\gamma : I \rightarrow \mathbb{C}$]
Remarks

1. We often speak of ‘the path $\Gamma$’ without referring specifically to the associated parametrisation $\gamma$. Sometimes it is convenient to speak of ‘the path $\Gamma : \gamma(t) = \ldots$’.

2. The condition that the function $\gamma : I \rightarrow \mathbb{C}$ be continuous is included to ensure that the path $\Gamma$ has no gaps in it. We will define the notion of continuity precisely in Unit A3, but meanwhile we point out that all functions $\gamma$ considered in this context are continuous.

3. As in Figures 2.4 and 2.5, a path $\Gamma$ is usually marked with an arrow (or arrows, if necessary) to show the direction in which it is traversed. (The arrow points in the direction of increasing values of $t$.)

4. Observe that the initial point and final point can be equal. For example, if

$$\gamma(t) = \cos t + i \sin t \quad (t \in [0, 2\pi]),$$

then the initial point $\gamma(0)$ and the final point $\gamma(2\pi)$ both equal 1, as shown in Figure 2.4.

5. It is sometimes possible to eliminate the parameter $t$ from the parametric equations to obtain the equation of the path in terms of $x$ and $y$ alone. For example, if

$$x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]),$$

then

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

However, unlike the parametric equations, the equation $x^2 + y^2 = 1$ does not tell us, for example, which way the arrow goes on the path.

6. It is often useful to plot a few points of the path to help us to understand the shape of a given path. This is done in the following example.

---

**Example 2.1**

Let $\gamma(t) = t^2 + it^3 \quad (t \in \mathbb{R})$.

Plot the points $\gamma(-1), \gamma(-\frac{1}{2}), \gamma(0), \gamma(\frac{1}{2}), \gamma(1)$ and hence sketch the path $\Gamma$ with parametrisation $\gamma$. Determine the equation of the path $\Gamma$ in terms of $x$ and $y$.

**Solution**

First we compile a table of the required values of $x = t^2$ and $y = t^3$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$-1$</th>
<th>$-\frac{1}{2}$</th>
<th>$0$</th>
<th>$\frac{1}{2}$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$1$</td>
<td>$\frac{1}{4}$</td>
<td>$0$</td>
<td>$\frac{1}{4}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$y$</td>
<td>$-1$</td>
<td>$-\frac{1}{8}$</td>
<td>$0$</td>
<td>$\frac{1}{8}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
We plot the points $x + iy$ and hence sketch the path $\Gamma$ in Figure 2.6.

In order to eliminate $t$ from the parametric equations $x = t^2$ and $y = t^3$, we note that
\[
x^3 = (t^2)^3 = t^6 = (t^3)^2 = y^2,
\]
so $\Gamma$ has the equation $y^2 = x^3$.

In the following exercise, you are asked to sketch various paths. For each path, you may find it helpful to compile a table of values (as we did in Example 2.1), but you may find that you can just use the equation of the path in terms of $x$ and $y$ to create your sketch.

**Exercise 2.3**

Sketch the paths $\Gamma$ with the following parametrisations.

(a) $\gamma(t) = 1 + it \quad (t \in \mathbb{R})$

(b) $\gamma(t) = t^2 + it \quad (t \in [-1, 1])$

(c) $\gamma(t) = 1 - t + it \quad (t \in [0, 1])$

(d) $\gamma(t) = 2 \cos t + 5i \sin t \quad (t \in [0, 2\pi])$

In each case determine the equation of $\Gamma$ in terms of $x$ and $y$.

It is important to realise that a given set can be considered as many different paths by using different parametrisations. For example, the functions
\[
\gamma(t) = t \quad (t \in [0, 1]) \quad \text{and} \quad \gamma(t) = t^2 \quad (t \in [0, 1])
\]
are both parametrisations of the real interval $[0, 1]$ in $\mathbb{C}$. As indicated in Figure 2.7, for the parametrisation $\gamma(t) = t$, the progress of the point along the interval $[0, 1]$ is uniform – for example, it is halfway along at time $t = \frac{1}{2}$. But for the parametrisation $\gamma(t) = t^2$, the speed of the point varies with $t$ – for example, in the time interval $0 \leq t \leq \frac{1}{2}$, the point has travelled one-quarter of the distance along the interval $[0, 1]$, and in the time interval $\frac{1}{2} \leq t \leq 1$, it travels the next three-quarters. (In fact, in Unit A4, you will see that the speed of the point at time $t$ is given by the modulus of the derivative of $\gamma$ at $t$.)

$\Gamma : \gamma(t) = t$

$\Gamma : \gamma(t) = t^2$

**Figure 2.7** Two parametrisations of $\Gamma$
Various types of sets (such as line segments and arcs of circles) occur frequently in this module as paths. We will normally use a *standard parametrisation* for each of these, as indicated in the following table.

<table>
<thead>
<tr>
<th>Set</th>
<th>Standard parametrisation</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line through $\alpha$ and $\beta$</td>
<td>$\gamma(t) = (1 - t)\alpha + t\beta \quad (t \in \mathbb{R})$</td>
<td><img src="image1" alt="Line through $\alpha$ and $\beta$" /></td>
</tr>
<tr>
<td>Line segment from $\alpha$ to $\beta$</td>
<td>$\gamma(t) = (1 - t)\alpha + t\beta \quad (t \in [0, 1])$</td>
<td><img src="image2" alt="Line segment from $\alpha$ to $\beta$" /></td>
</tr>
<tr>
<td>Circle with centre $\alpha$, radius $r$:</td>
<td>$\gamma(t) = \alpha + r(\cos t + i \sin t) \quad (t \in [0, 2\pi])$</td>
<td><img src="image3" alt="Circle with centre $\alpha$, radius $r$:" /></td>
</tr>
<tr>
<td>Arc of circle with centre $\alpha$, radius $r$</td>
<td>$\gamma(t) = \alpha + r(\cos t + i \sin t) \quad (t \in [t_1, t_2])$</td>
<td><img src="image4" alt="Arc of circle with centre $\alpha$, radius $r$" /></td>
</tr>
<tr>
<td>Ellipse in standard form:</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b &gt; 0$</td>
<td><img src="image5" alt="Ellipse in standard form:" /></td>
</tr>
<tr>
<td></td>
<td>$\gamma(t) = a \cos t + ib \sin t \quad (t \in [0, 2\pi])$</td>
<td><img src="image6" alt="Ellipse in standard form:" /></td>
</tr>
</tbody>
</table>
Parabola in standard form:
\[ y^2 = 4ax, \]
where \( a > 0 \)

\[ \gamma(t) = at^2 + 2iat \quad (t \in \mathbb{R}) \]

Right half of hyperbola in standard form:
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \]
where \( a, b > 0 \)

\[ \gamma(t) = a \cosh t + ib \sinh t \quad (t \in \mathbb{R}) \]

---

**Exercise 2.4**

For each of the following paths, write down the standard parametrisation and obtain the corresponding parametric equations:

(a) the line through \(-2\) and \(i\)
(b) the line segment from 1 to 1 + \(i\)
(c) the circle with centre 1 + \(i\) and radius 1
(d) the parabola \(y^2 = x\).

Let us now consider how the image point \(w = f(z)\) behaves as the point \(z\) moves around the domain of the function \(f\). We make the following definition.

**Definition**

Given a function \(f: A \rightarrow B\) and a subset \(S\) of \(A\), the **image under \(f\) of \(S\)**, written \(f(S)\), is

\[ f(S) = \{ f(z) : z \in S \}. \]

**Remarks**

1. We often write ‘\(f\) maps \(S\) to \(T\)’ to mean \(f(S) = T\).
2. If \(S = A\), then, as noted earlier, \(f(S)\) is also described as the image set of \(f\).
Suppose that \( f(z) = z^2 \) and that the point \( z \) moves around the unit circle \( |z| = 1 \). In this case, our knowledge of the function \( f \) tells us that the image \( w \) of \( z \) satisfies \( |w| = |z^2| = |z|^2 = 1 \), and

if \( \theta \) is an argument of \( z \), then \( 2\theta \) is an argument of \( w \)

(see Figure 2.8). Thus if \( z \) moves once around the circle \( |z| = 1 \) anticlockwise, then \( w \) moves twice around the circle \( |w| = 1 \) anticlockwise.

\[ f(z) = z^2 \]

\[ \gamma(t) = \cos t + i\sin t \quad (t \in [0, 2\pi]) \]

of the unit circle \( |z| = 1 \) traversed once anticlockwise. Now, from equations (2.1) in Subsection 2.1 we know that for \( f(z) = z^2 \), with \( z = x + iy \) and \( w = f(z) = u + iv \),

\[ u = x^2 - y^2, \quad v = 2xy. \]  

(2.2)

The parametric equations corresponding to the parametrisation \( \gamma \) are

\[ x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]), \]  

(2.3)

and, on substituting these in equations (2.2), we obtain

\[ u = \cos^2 t - \sin^2 t, \quad v = 2\cos t \sin t; \]

that is,

\[ u = \cos 2t, \quad v = \sin 2t \quad (t \in [0, 2\pi]). \]  

(2.4)

An alternative way to obtain these equations is to observe that

\[ u + iv = (x + iy)^2 = (\cos t + i\sin t)^2 = \cos 2t + i\sin 2t, \]

by De Moivre’s Theorem, and then equate real and imaginary parts.

Equations (2.4) are the parametric equations for the image circle \( |w| = 1 \). From these equations we can verify our earlier assertion that as \( t \) increases from 0 to \( 2\pi \), \( z \) moves \( once \) around the circle \( |z| = 1 \) anticlockwise while \( w \) moves \( twice \) around the circle \( |w| = 1 \) anticlockwise.
In the next definition we assume that a complex function $f$ is ‘continuous’. This is to ensure that if $\gamma$ is the parametrisation of a path, then $f \circ \gamma$ is also the parametrisation of a path. Continuous functions and their properties are discussed in Unit A3.

Definition

Let $f$ be a continuous function, and let $\Gamma$ be a path in the domain of $f$. Then $f(\Gamma)$ is called the image path (under $f$ of $\Gamma$). If $\Gamma$ has parametrisation $\gamma$, then $f(\Gamma)$ has parametrisation $f \circ \gamma$, which is the function with rule $t \mapsto f(\gamma(t))$.

Remark

If $\gamma$ is the standard parametrisation of $\Gamma$, then $f \circ \gamma$ is certainly a parametrisation of $f(\Gamma)$, but it need not be the standard parametrisation of $f(\Gamma)$. For example, consider again the function $f(z) = z^2$, which maps the unit circle $\Gamma$ to itself. The standard parametrisation of $\Gamma$ is $\gamma(t) = \cos t + i \sin t$ ($t \in [0, 2\pi]$), but, as we have seen, $(f \circ \gamma)(t) = \cos 2t + i \sin 2t$ ($t \in [0, 2\pi]$) is a parametrisation of $\Gamma$, but not the standard one.

The two approaches for finding image paths that we applied to the function $f(z) = z^2$ and the unit circle are summarised in the following strategy.

Strategy for determining an image path

Let $f$ be a continuous function, and let $\Gamma$ be a path with parametrisation

$$\gamma(t) = \phi(t) + i \psi(t) \quad (t \in I).$$

To find the image path $f(\Gamma)$

- either use the geometric properties of $f$
- or substitute $x = \phi(t)$, $y = \psi(t)$ into the equation

$$u + iv = f(x + iy),$$

and then, by equating real parts and imaginary parts, obtain expressions for $u$ and $v$ in terms of $t$. (These expressions are the parametric equations of the image path $f(\Gamma)$ associated with the parametrisation $f \circ \gamma$.)

You will practise applying this strategy in the next section.
Further exercises

Exercise 2.5
Determine the real and imaginary parts, \( \text{Re} \) and \( \text{Im} \), of each of the following functions.
(a) \( f(z) = z \)  \( \) (b) \( f(z) = iz \)  \( \) (c) \( f(z) = z^3 \)  \( \) (d) \( f(z) = |z| \)

Exercise 2.6
Sketch each of the paths \( \Gamma \) with the following parametrisations.
(a) \( \gamma(t) = 1 - it \quad (t \in \mathbb{R}) \)  \( \) (b) \( \gamma(t) = i + (1 - i)t \quad (t \in [0, 1]) \)  \( \) (c) \( \gamma(t) = \cos t - i \sin t \quad (t \in [0, 2\pi]) \)

Exercise 2.7
For each of the following parametrisations \( \gamma \), find the equation of the corresponding path \( \Gamma \) in terms of \( x \) and \( y \) only. Sketch and classify the path in each case.
(a) \( \gamma(t) = (1 - t)(1 + i) + ti \quad (t \in \mathbb{R}) \)  \( \) (b) \( \gamma(t) = 2 \cos t + 3i \sin t \quad (t \in [0, 2\pi]) \)  \( \) (c) \( \gamma(t) = 1 + 2 \cos t - (1 - 2 \sin t)i \quad (t \in [0, 2\pi]) \)

Exercise 2.8
Determine the standard parametrisation for each of the following sets:
(a) the circle with centre \( 1 - i \) and radius 3
(b) the ellipse \( 2x^2 + 3y^2 = 6 \)
(c) the parabola \( 8y^2 = x \).

Exercise 2.9
Sketch the path with parametrisation
\[
\gamma(t) = \frac{1}{2}(\cos t + i \sin t) - \frac{1}{4}(\cos 2t + i \sin 2t) \quad (t \in [-\pi, \pi])
\]
by first plotting \( \gamma(t) \) for \( t = 0, \pm \frac{\pi}{4}, \pm \frac{\pi}{2}, \pm \frac{3\pi}{4}, \pm \pi \).
Verify that the equation of the path is
\[
4(x^2 + y^2)^2 - \frac{3}{2}(x^2 + y^2) + \frac{1}{8}x = \frac{3}{64}.
\]

Exercise 2.10
Determine the image under the function \( f(z) = \sqrt{z} \) of each of the following sets:
(a) the negative real axis  \( \) (b) the circle \( |z| = 1 \).
3 Images of grids

After working through this section, you should be able to:

- sketch the image of a given Cartesian grid under various complex functions
- sketch the image of a given polar grid under various complex functions.

3.1 Cartesian and polar grids

To obtain a clear picture of the geometric effect of a given complex function, we can consider the images of many lines or paths in the domain of the function. In order to do this in a systematic way, we introduce two types of grid.

The first type of grid is a Cartesian grid consisting of lines of the form \( x = a \) and \( y = b \), usually evenly spaced in each direction (Figure 3.1(a)).

The second type consists of circles with centre 0 and rays emerging from 0; it is called a polar grid because of the connection with the polar form \( z = r \cos \theta + i \sin \theta \). Each of the circles has an equation of the form \( r = a \), where \( a \) is a positive constant, and each of the rays has an equation of the form \( \theta = b \), where \( b \) is a constant in the interval \((-\pi, \pi]\) (see Figure 3.1(b)).

![Cartesian grid](a) ![Polar grid](b)

**Figure 3.1** (a) A Cartesian grid (b) A polar grid

**Exercise 3.1**

Plot the polar grid consisting of the circles \( r = 1, r = \frac{1}{2}, r = \frac{1}{3} \) and the rays \( \theta = 0, \theta = \pm \pi/3, \theta = \pm 2\pi/3, \theta = \pi \).
3.2 Images of Cartesian and polar grids

In this subsection we examine the images of Cartesian and polar grids under three different complex functions to give us insight into the geometric effects of these functions. The images are found by using the strategy for determining an image path given at the end of Subsection 2.2.

In each case we have also highlighted (by using shading) the effect of the function on a particular set bounded by parts of the grid.

Before we consider these examples, you should try the following exercise, the results of which will be used in the examples to follow.

**Exercise 3.2**

Eliminate \( t \) from each of the following pairs of parametric equations. (In each case, \( a \) is a real constant, with \( a \neq 0 \) in parts (b) and (c).)

(a) \( u = a - t, \quad v = a + t \)
(b) \( u = a^2 - t^2, \quad v = 2at \)
(c) \( u = \frac{a}{a^2 + t^2}, \quad v = \frac{-t}{a^2 + t^2} \)

**Images of grids under \( f(z) = (1 + i)z \)**

The first function we consider is the linear function \( f(z) = (1 + i)z \). For convenience we define \( w = f(z) \). According to the strategy for determining an image path, one method for finding images of sets under \( f \) involves splitting both \( z \) and \( w = f(z) \) into real and imaginary parts, say \( z = x + iy \) and \( w = u + iv \). The equation \( w = (1 + i)z \) then becomes

\[
u + iv = (1 + i)(x + iy) = (x - y) + i(x + y).
\]

Equating real and imaginary parts gives

\[
\begin{align*}
u &= x - y, \\
v &= x + y.
\end{align*}
\]  

(3.1)

We wish to find the image of a Cartesian grid under \( f \), and to this end let us first work out the image of a vertical line \( x = a \), where \( a \) is a real constant. It is convenient to think of this line as a path, and represent it by the parametric equations

\[
x = a, \quad y = t \quad (t \in \mathbb{R}),
\]

so that the line now has a direction: ‘pointing upwards’. Substituting the expressions for \( x \) and \( y \) into equations (3.1) gives

\[
\begin{align*}
u &= a - t, \\
v &= a + t.
\end{align*}
\]

We can now eliminate \( t \) from these equations, as you were asked to do in Exercise 3.2(a), to obtain

\[
u + v = 2a,
\]

which is the equation of a line in the \( w \)-plane.
The line $x = a$ and its image line $u + v = 2a$ are illustrated in Figure 3.2.

![Figure 3.2](image.png)

Figure 3.2  Image of a vertical line under $f(z) = (1 + i)z$

The next exercise asks you to find the images of horizontal lines under $f$.

**Exercise 3.3**

Determine the image of the line $y = b$, where $b \in \mathbb{R}$, under the function $f(z) = (1 + i)z$.

Sketch the images of the lines $y = 1$ and $y = 0$.

Now that we have worked out the images of horizontal and vertical lines under $f$, we can find the image of a Cartesian grid under $f$. The result is illustrated in Figure 3.3.

![Figure 3.3](image.png)

Figure 3.3  Image of a Cartesian grid under $f(z) = (1 + i)z$

The vertical lines in the grid in the $z$-plane all have equations of the form $x = a$, for various values of $a$. As we have seen, these lines are mapped by $f$ to lines that slope from top-left to bottom-right (negative gradient) in the $w$-plane. The horizontal lines in the $z$-plane have equations $y = b$, for various values of $b$, and these are mapped by $f$ to lines that slope from bottom-left to top-right (positive gradient) in the $w$-plane.
Each line in the $w$-plane in Figure 3.3 is labelled by the equation of the corresponding line in the $z$-plane to help you appreciate how the grid is transformed. We continue this convention whenever we sketch images of grids.

By studying Figure 3.3, you can see that the image of the grid in the $z$-plane is another grid in the $w$-plane, but one that is rotated anticlockwise about the origin by $\pi/4$. Furthermore, the grid is scaled by the factor $\sqrt{2}$, as shown in Figure 3.3. The figure illustrates the effect of the transformation on a shaded square, which is scaled and rotated under $f$.

Another way to understand the behaviour of $f$ is to write $1 + i$ in polar form as

$$1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4).$$

The geometric effect of multiplying $z$ by $1 + i$, which is what $f$ does, is to scale $z$ by the modulus of $1 + i$ (namely $\sqrt{2}$) and rotate $z$ anticlockwise about the origin by the principal argument of $1 + i$ (namely $\pi/4$).

Figure 3.4 shows the image of a polar grid under the same function $f$. Each circle $r = a$ in the $z$-plane, where $a$ is a positive constant, is scaled by a factor $\sqrt{2}$, and each ray $\theta = b$, where $b \in (-\pi, \pi]$, is rotated anticlockwise by $\pi/4$.

Figure 3.4 shows the image of a polar grid under $f(z) = (1 + i)z$.
Images of grids under \( f(z) = z^2 \)

Our next example is the function \( f(z) = z^2 \). As before, we let \( w = f(z) \) and write \( z = x + iy \) and \( w = u + iv \). From equations (2.1) in Subsection 2.1, we know that
\[
 u = x^2 - y^2, \quad v = 2xy. \tag{3.2}
\]
Let us use these equations to find the image under \( f \) of the vertical line \( x = a \), where \( a \) is a positive constant. The line \( x = a \) has parametric equations
\[
x = a, \quad y = t \quad (t \in \mathbb{R}),
\]
so the parametric equations of the image of \( x = a \) are
\[
u = a^2 - t^2, \quad v = 2at \quad (t \in \mathbb{R}).
\]
In Exercise 3.2(b) you were asked to eliminate \( t \) from this pair of equations, to obtain
\[
v^2 = 4a^2(a^2 - u).
\]
This is the equation of a parabola, provided that \( a \neq 0 \), as illustrated in Figure 3.5.

![Image of a vertical line under \( f(z) = z^2 \)](image)

**Figure 3.5** Image of a vertical line under \( f(z) = z^2 \)

In the exceptional case when \( a = 0 \), we obtain the parametric equations \( u = -t^2 \) and \( v = 0 \), so the image of the vertical line in this case is the negative real axis together with the origin. As \( t \) increases, the point \( u + iv \) moves along the negative real axis in the \( w \)-plane, first from left to right (until it reaches 0), and then back again. This is illustrated in Figure 3.6.

![Image of the line \( x = 0 \) under \( f(z) = z^2 \)](image)

**Figure 3.6** Image of the line \( x = 0 \) under \( f(z) = z^2 \)

Now try the next exercise, which asks you to find the images of horizontal lines under \( f \).

**Exercise 3.4**

Determine the image of the line \( y = b \), where \( b \in \mathbb{R} \), under the function \( f(z) = z^2 \).

Sketch the images of the lines \( y = 1 \) and \( y = 0 \).
Figure 3.7 shows the image of a Cartesian grid under \( f \). Notice that for \( a \neq 0 \), the vertical lines \( x = a \) and \( x = -a \) both map to the same parabola \( v^2 = 4a^2(a^2 - u) \), because \((-a)^2 = a^2\). Likewise, for \( b \neq 0 \), the horizontal lines \( y = b \) and \( y = -b \) both map to the parabola \( v^2 = 4b^2(u + b) \).

We can see the effect of the function on the shaded square in the \( z \)-plane: it is mapped to the shaded set in the \( w \)-plane bounded by parts of four parabolas.

**Figure 3.7** Image of a Cartesian grid under \( f(z) = z^2 \)

As with the previous example, we can better understand the geometric effect of \( f \) by writing \( z \) in polar form as \( z = r(\cos \theta + i \sin \theta) \), so

\[
 w = z^2 = r^2(\cos 2\theta + i \sin 2\theta).
\]

From this formula we see that \( f \) squares the modulus of each complex number and doubles the argument. These properties are demonstrated using a polar grid in Figure 3.8. The radii of the concentric circles in the \( z \)-plane are all squared, and the arguments of the rays are all doubled. For example, the ray in the \( z \)-plane with argument \( \pi/4 \) is mapped to the ray in the \( w \)-plane with argument \( \pi/2 \), the positive imaginary axis. In fact, because arguments are doubled, any ray in the \( w \)-plane is the image of precisely two rays in the \( z \)-plane. For instance, the positive imaginary axis is also the image of the ray with argument \(-3\pi/4\), because doubling \(-3\pi/4\) gives \(-3\pi/2 = \pi/2 - 2\pi\).
In Figure 3.8 and some other similar figures later in the unit we have used different scales for the $z$-plane and $w$-plane in order to display the features of the grids clearly.

**Images of grids under** \( f(z) = 1/z \)

Our last example is the function \( f(z) = 1/z \), which has domain \( \mathbb{C} - \{0\} \). It is useful to observe that if we multiply the top and bottom of \( 1/z \) by \( \bar{z} \), then we obtain another expression for \( f(z) \), namely \( f(z) = \bar{z}/|z|^2 \). Next, if we let \( w = f(z) \), and write \( z = x + iy \) and \( w = u + iv \) as usual, then we see that

\[
    u + iv = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.
\]

Equating real and imaginary parts gives

\[
    u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.
\]  

(3.3)

We can now find the parametric equations of the image of a vertical line \( x = a \), where \( a \) is a non-zero real constant. The parametric equations of the line \( x = a \) are

\[
    x = a, \quad y = t \quad (t \in \mathbb{R}),
\]

so the parametric equations of the image of \( x = a \) are

\[
    u = \frac{a}{a^2 + t^2}, \quad v = -\frac{t}{a^2 + t^2} \quad (t \in \mathbb{R}).
\]

Eliminating \( t \) from these equations, as in Exercise 3.2(c), gives

\[
    u^2 + v^2 = u/a.
\]
We can rewrite this equation as
\[(u - 1/(2a))^2 + v^2 = 1/(2a)^2,\]
which is the equation of a circle with centre \(1/(2a)\) and radius \(1/(2|a|)\).
This circle passes through the origin and the point \(1/a\), as shown in Figure 3.9. As \(t\) increases from \(-\infty\) to \(\infty\), the image point \(u + iv\) moves clockwise around the circle, starting and finishing at the origin. However, the origin itself is excluded from the image, because as \(t\) approaches \(-\infty\) or \(\infty\), the point \(u + iv\) on the circle gets closer and closer to the origin without actually reaching it.

**Figure 3.9** Image of a vertical line under \(f(z) = 1/z\)

It remains to consider the image of the line \(x = a\), when \(a = 0\). This line is split into two parts, the positive and negative imaginary axes, by the point \(0\), which is excluded from the domain of \(f\). You can check that \(f\) maps the positive imaginary axis in the \(z\)-plane to the negative imaginary axis in the \(w\)-plane, and it maps the negative imaginary axis in the \(z\)-plane to the positive imaginary axis in the \(w\)-plane. The image of the line \(x = 0\) under \(f\) is shown in Figure 3.10.

In the following exercise, you are asked to find the images of horizontal lines under \(f\).

**Exercise 3.5**

Determine the image of the line \(y = b\), where \(b \in \mathbb{R}\), under the function \(f(z) = 1/z\).

Sketch the images of the lines \(y = 1\) and \(y = 0\).

Figure 3.11 shows the image of a Cartesian grid under \(f(z) = 1/z\), with different scales for the \(z\)-plane and the \(w\)-plane. The shaded square in the \(z\)-plane is mapped to a set in the \(w\)-plane bounded by parts of four circles.

Once again, we can gain insight into the geometric effect of \(f\) by writing \(z\) in polar form as \(z = r(\cos \theta + i \sin \theta)\), so
\[w = \frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i \sin(-\theta)).\]

This formula shows that \(f\) changes the modulus of \(z\) to its reciprocal, and reverses the sign of the argument of \(z\). This is demonstrated by the transformation of the polar grid in Figure 3.12.
Figure 3.11  Image of a Cartesian grid under $f(z) = 1/z$

In Figure 3.12 a circle centred at the origin in the $z$-plane of radius $a$ is mapped by $f$ to a circle centred at the origin in the $w$-plane of radius $1/a$, and the ray with argument $b$ is mapped to the ray with argument $-b$.

Figure 3.12  Image of a polar grid under $f(z) = 1/z$
The following exercises provide some practice in determining images of Cartesian grids and polar grids for complex functions. In such exercises, your geometric knowledge of the function $f$ may save you from getting involved in parametrisations.

**Exercise 3.6**

For the function $f(z) = iz + 1$, sketch the images of

(a) $S = \{z : 0 \leq \text{Re} z \leq 1, \ 0 \leq \text{Im} z \leq 1\}$

(b) the polar grid of Figure 3.1(b).

**Exercise 3.7**

Sketch the image of the polar grid of Figure 3.1(b) under the function $f(z) = z^3$, omitting the circle with equation $r = 3$. (Its image is rather large.)

**Exercise 3.8**

Find the image of the polar grid of Figure 3.1(b) under the function $f(z) = \sqrt{z}$.

**Further exercises**

**Exercise 3.9**

Sketch the images under each of the following functions $f$ of the Cartesian grid and polar grid shown in Figure 3.1.

(a) $f(z) = z + i$    (b) $f(z) = 2z$    (c) $f(z) = 2 - iz$

(d) $f(z) = iz^2$

**Vector fields**

In this unit we have seen how to illustrate complex functions using two copies of the complex plane, one representing the domain of a function and the other representing the codomain. There is an alternative method for representing a complex function $f$ geometrically using only a single copy of the complex plane. In this method, the value $f(z)$ is marked by an arrow emanating from the point $z$ that has magnitude $|f(z)|$ and that makes an angle $\text{Arg} f(z)$ with the positive horizontal direction. The resulting collection of points and arrows is called the *vector field* of $f$. 
For instance, Figure 3.13 illustrates the vector field of the function \( f(z) = z^2 \).

![Figure 3.13 Vector field of \( f(z) = z^2 \)](image)

Vector fields are widely used in science to represent physical phenomena. For example, the arrows in Figure 3.13 might represent the velocities of particles in a flowing fluid. The particles on the positive real axis are propelled rightwards along the positive real axis. Other particles follow trajectories that guide them towards the origin. The arrows shrink as they approach the origin, indicating that the origin is fixed by the function. You will learn about fluid flows in Unit D1.

## 4 Exponential, trigonometric and hyperbolic functions

After working through this section, you should be able to:

- state and use the definition and basic algebraic properties of the exponential function and describe its geometric properties
- state and use the definitions and basic algebraic properties of the trigonometric functions and hyperbolic functions
- prove simple identities involving the exponential function, the trigonometric functions and the hyperbolic functions.

### 4.1 The exponential function

The real exponential function \( x \mapsto e^x \), which is illustrated by the graph in Figure 4.1, plays a key role in a wide range of mathematical subjects. It is natural to ask whether this function has a complex analogue; that is, how is \( e^z \) defined where \( z \) is a complex number?
A fundamental property of the real exponential function is that
\[ e^{x_1} e^{x_2} = e^{x_1 + x_2}, \quad \text{for all } x_1, x_2 \in \mathbb{R}. \]

Ideally, then, the complex exponential function should satisfy
\[ e^{z_1} e^{z_2} = e^{z_1 + z_2}, \quad \text{for all } z_1, z_2 \in \mathbb{C}. \] (4.1)

In particular, if \( z = x + iy \), then it should be true that
\[ e^z = e^{x+iy} = e^x e^{iy}. \]

Since \( e^x \) is defined, because \( x \) is real, it remains only to define \( e^{iy} \), where \( y \) is real. The following manipulation of power series suggests a definition of \( e^{iy} \). The power series for \( e^x \), where \( x \in \mathbb{R} \), is
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \]

Let us now try replacing \( x \) with \( iy \) in this formula. We cannot justify doing so at this stage because \( iy \) is not a real number; nonetheless, the outcome is illuminating.

We obtain
\[ e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots = \left(1 - \frac{y^2}{2!} + \cdots\right) + i \left(y - \frac{y^3}{3!} + \cdots\right). \]

The expressions in parentheses are the power series for \( \cos y \) and \( \sin y \), respectively. Thus it seems plausible to define \( e^{iy} \) by
\[ e^{iy} = \cos y + i \sin y; \]
this formula is known as **Euler’s Identity** (or ‘Euler’s Formula’), named after the Swiss mathematician Leonhard Euler (pronounced ‘oiler’), whom we met in Unit A1 and whom we will meet again later.

**Definition**

For all \( z = x + iy \) in \( \mathbb{C} \),
\[ e^z = e^x (\cos y + i \sin y). \]

The function
\[ z \mapsto e^z \quad (z \in \mathbb{C}) \]

is called the **exponential function**, and is denoted by \( \exp \).
Thus \( \exp z = e^z \).

Before verifying that the exponential function does indeed satisfy equation (4.1), we make some remarks about the definition and then evaluate \( e^z \) for several complex numbers \( z \).
4 Exponential, trigonometric and hyperbolic functions

Remarks

1. Notice that if \( z \) is real, so \( z = x + 0i \) (that is, \( z = x \)), then
\[
e^z = e^x (\cos 0 + i \sin 0) = e^x.
\]
Thus the restriction of the (complex) exponential function to \( \mathbb{R} \) gives the real exponential function, as we would expect. In particular, for \( 0 \in \mathbb{C} \),
\[
e^0 = 1.
\]

2. Also, if \( z = 0 + iy \), then
\[
e^z = e^0 (\cos y + i \sin y),
\]
which gives Euler’s Identity
\[
e^{iy} = \cos y + i \sin y.
\]
Thus the number \( e^{iy} \) has modulus 1 and argument \( y \), and so lies on the unit circle \( \{ z : |z| = 1 \} \). Some important examples are given in Figure 4.2.

In particular, \( e^{i\pi} = -1 \); that is,
\[
e^{i\pi} + 1 = 0.
\]
This striking equation contains five of the most important numbers in mathematics, together with two of the most important symbols. It is known as ‘Euler’s Equation’, or even sometimes ‘Euler’s Identity’, although it is not to be confused with the more general formula \( e^{iy} = \cos y + i \sin y \) that we call Euler’s Identity.

Example 4.1

Express each of the following numbers in Cartesian form.

(a) \( e^{i\pi/3} \)  (b) \( e^{(1-i\pi)/2} \)  (c) \( e^{-1+i\pi/4} \)

Solution

(a) \( e^{i\pi/3} = e^0 (\cos \pi/3 + i \sin \pi/3) = 1/2 + i\sqrt{3}/2 \)

(b) \( e^{(1-i\pi)/2} = e^{1/2} (\cos(-\pi/2) + i \sin(-\pi/2)) = -ie^{1/2} \)

(c) \( e^{-1+i\pi/4} = e^{-1} (\cos \pi/4 + i \sin \pi/4) \)
\[
= e^{-1} (1/\sqrt{2} + i/\sqrt{2}) = \frac{1 + i}{e\sqrt{2}}
\]

Exercise 4.1

Express each of the following numbers in Cartesian form.

(a) \( e^{2\pi i} \)  (b) \( e^{2+i\pi/3} \)  (c) \( e^{-(1+i\pi)} \)
The next result gives a number of basic identities involving the exponential function, including equation (4.1). Here and subsequently we adopt the convention that, unless otherwise stated, identities hold for all values of the variables in the identity for which the identity has meaning. For example, the first identity below holds for all \( z_1 \) and \( z_2 \) in \( \mathbb{C} \).

### Theorem 4.1 Exponential Identities

(a) **Addition**  \( e^{z_1 + z_2} = e^{z_1}e^{z_2} \)

(b) **Modulus**  \( |e^z| = e^{Re\,z} \)

(c) **Negatives**  \( e^{-z} = 1/e^z \)

(d) **Periodicity**  \( e^{z + 2\pi i} = e^z \)

### Remarks

1. One consequence of part (a) is that
   \[
   (e^{i\theta})^n = e^{i\theta n}, \quad \text{for } \theta \in \mathbb{R}, \ n \in \mathbb{Z}.
   \]  
   This is a restatement of De Moivre’s Theorem (Theorem 2.2 of Unit A1) in a concise form.

2. Since the real exponential function is always positive, part (b) shows that \( e^z \neq 0 \) for all \( z \in \mathbb{C} \).

3. Part (d) shows that the exponential function is periodic with period \( 2\pi i \), meaning that \( \exp \) takes the same value at \( z \) and \( z + 2\pi i \), for any complex number \( z \).

### Proof

We prove parts (a), (b) and (c), leaving part (d) as an exercise.

(a) Let \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \). Then
   
   \[
e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2)
   = e^{x_1}e^{x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2)
   = e^{x_1+x_2}((\cos y_1 \cos y_2 - \sin y_1 \sin y_2)
   + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2))
   = e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2))
   = e^{x_1+x_2}e^{i(y_1+y_2)}
   = e^{z_1+z_2},
   \]
   since \( z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \).

(b) Let \( z = x + iy \). Then
   
   \[
   |e^z| = |e^x(\cos y + i \sin y)|
   = |e^x||\cos y + i \sin y|
   = e^x,
   \]
   since \( e^x > 0 \) and \( \cos^2 y + \sin^2 y = 1 \). Hence \( |e^z| = e^{Re\,z} \).

(c) By part (a),
   
   \[
   e^z e^{-z} = e^{z-z} = e^0 = 1.
   \]
   Hence \( e^{-z} = 1/e^z \).
Exercise 4.2

(a) Prove part (d) of Theorem 4.1.
(b) Prove that

$$|e^z| \leq |z|$$

for all $z \in \mathbb{C}$.
(c) Is the function exp one-to-one?
(d) Determine each of the following sets.

(i) $\{z : e^z = 1\}$
(ii) $\{z : e^z = -1\}$

The result of Exercise 4.2(b) will prove to be useful later in the module.
The exponential function provides an alternative, concise notation for expressing a non-zero complex number in polar form.

Given a non-zero complex number $z$ with modulus $r$ and argument $\theta$, both

$$z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad z = re^{i\theta}$$

are acceptable ways of writing $z$ in polar form.

For example,

$$1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4) = \sqrt{2}e^{i\pi/4}.$$

Using this concise version of polar form, it is easy to compute powers such as $(1 + i)^{10}$ as follows:

$$(1 + i)^{10} = (\sqrt{2}e^{i\pi/4})^{10}
= (\sqrt{2})^{10}(e^{i\pi/4})^{10}
= 32e^{10i\pi/4} \quad \text{(by equation (4.2))}
= 32e^{5\pi/2}
= 32e^{i\pi/2} \quad \text{(exp has period } 2\pi i)$$

$$= 32i.$$

Compare this calculation with that of Exercise 2.12(c) of Unit A1.

Exercise 4.3

By writing $\sqrt{3} + i$ in polar form, using the exponential function, evaluate $(\sqrt{3} + i)^{-6}$.
The geometric effect of the exponential function

Theorem 4.1(d) shows that the (complex) exponential function is not one-to-one, since

\[ e^{z+2\pi i} = e^z \]  \hspace{1cm} (4.3)

but \( z + 2\pi i \neq z \). Repeated application of equation (4.3) gives the following fact.

\[ e^{z+2n\pi i} = e^z \quad (n \in \mathbb{Z}). \]

Therefore each of the points

\[ z + 2n\pi i, \quad n \in \mathbb{Z}, \]

has the same image under the exponential function. These points lie on the vertical line through \( z \), as shown in Figure 4.3. We will use this observation to investigate geometric properties of the exponential function.

As in Section 3, the aim is to plot the image of a grid of lines of the form \( x = a \) and \( y = b \), for suitable real constants \( a \) and \( b \). To do this, let \( w = e^z \), where \( z = x + iy \) and \( w = u + iv \), so

\[ u + iv = e^z = e^x (\cos y + i \sin y). \]

Hence

\[ u = e^x \cos y, \quad v = e^x \sin y. \]  \hspace{1cm} (4.4)

First we consider the image of a line of the form \( x = a \), for a real constant \( a \), with parametrisation \( \gamma(t) = a + it \ (t \in \mathbb{R}) \). Substituting \( x = a \) and \( y = t \) into equations (4.4), we obtain the following property.

The function \( \exp \) maps the line \( x = a \) to the path with parametric equations

\[ u = e^a \cos t, \quad v = e^a \sin t \quad (t \in \mathbb{R}). \]

This is the circle with centre 0 and radius \( e^a \) (Figure 4.4).
Notice that

- as $t$ increases, the image point $w$ moves anticlockwise around the image circle, passing through $e^{it}$ whenever $t$ is an integer multiple of $2\pi$
- the image of the line $\{x + iy : x = 0\}$ is the unit circle $\{w : |w| = 1\}$
- as $a$ increases, the image circle of the line $x = a$ expands, the centre remaining fixed at 0.

Next consider the image of a line of the form $y = b$, for a real constant $b$, with parametrisation $\gamma(t) = t + ib$ ($t \in \mathbb{R}$). Substituting $x = t$ and $y = b$ into equations (4.4), we obtain the following property.

The function $\exp$ maps the line $y = b$ to the path with parametric equations

$$u = e^t \cos b, \quad v = e^t \sin b \quad (t \in \mathbb{R}).$$

This is the ray from 0 (excluded) through $\cos b + i \sin b$ (Figure 4.5).

![Image](image.png)

**Figure 4.5** The line $\{x + iy : y = b\}$ and its image ray $\{w = re^{i\phi} : \phi = b, \ r > 0\}$

Notice that

- as $t$ increases, the image point $w$ moves outwards along the image ray
- the image of the line $y = 0$ is the positive real axis
- as $b$ increases, the image ray of the line $y = b$ rotates anticlockwise about 0.

Combining these observations, we can now plot the image of a grid of lines of the form $x = a$ and $y = b$. For our grid, we choose the values of $a$ to be integers (as usual) but, because trigonometric functions are involved, it is convenient to choose the values of $b$ to be integer multiples of $\pi/2$.

In Figure 4.6 the image circles of the lines $x = -2, -1, 0, 1, 2$ are shown, as are the image rays of the lines $y = -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi$. (Note that the lines $y = -\pi$ and $y = \pi$, for example, have the same image.)
One effect of this choice of values for $b$ is that the image of each grid rectangle in the $z$-plane is a quarter-annulus in the $w$-plane. In particular, notice that the two shaded rectangles in Figure 4.6 map to the same quarter-annulus.

Notice also that, since $|e^z| = e^{\Re z}$, points in the right half-plane \( \{ z : \Re z > 0 \} \) have images lying outside the circle \( \{ w : |w| = 1 \} \), whereas points in the left half-plane \( \{ z : \Re z < 0 \} \) have images lying inside this circle.

Finally, notice that Figure 4.6 reveals an important property of exp, illustrated in Figure 4.7, which will prove useful in Section 5, where we discuss possible inverse functions for the exponential function.

The image of the strip \( \{ x + iy : -\pi < y \leq \pi \} \) under \( f(z) = e^z \) is \( \mathbb{C} - \{0\} \).
4 Exponential, trigonometric and hyperbolic functions

Figure 4.7 The image of the strip \( \{x + iy : -\pi < y \leq \pi\} \) under \( f(z) = e^z \) is \( \mathbb{C} - \{0\} \)

Exercise 4.4

Sketch the image of each of the following sets under the exponential function.

(a) \( \{x + iy : -1 \leq x \leq 0, -\pi/4 \leq y \leq \pi/4\} \)
(b) \( \{x + iy : -1 \leq x \leq 1, \pi \leq y \leq 2\pi\} \)
(c) \( \{x + iy : 0 < y < 2\pi\} \)

4.2 Trigonometric functions

In the study of real functions there seems, at first sight, to be no connection between the trigonometric functions \( x \mapsto \sin x \) and \( x \mapsto \cos x \), defined geometrically, and the exponential function \( x \mapsto e^x \). Certainly, their graphs (Figure 4.8) do not suggest any connection.

Figure 4.8 Graphs of \( y = \sin x \), \( y = \cos x \) and \( y = e^x \)

However, the definition of the complex exponential function makes use of real trigonometric functions, and it turns out that this complex exponential function can be used to define complex trigonometric functions.
The key to such a definition is Euler’s Identity
\[ e^{i\theta} = \cos \theta + i \sin \theta, \quad \theta \in \mathbb{R}, \]
together with the consequence
\[ e^{-i\theta} = \cos \theta - i \sin \theta, \quad \theta \in \mathbb{R}, \]
which holds because \( \cos(-\theta) = \cos \theta \) and \( \sin(-\theta) = -\sin \theta \). Eliminating first \( \sin \theta \) and then \( \cos \theta \) from these equations, we obtain
\[ \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \quad (4.5) \]
These equations suggest the following definitions.

**Definitions**

For all \( z \) in \( \mathbb{C} \),
\[ \cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \]
These functions are called the **cosine function** and **sine function**, respectively.

**Remarks**

1. We occasionally use the phrase ‘complex cosine function’, rather than just ‘cosine function’, to emphasise that we are working with a complex function. (A similar comment applies to the sine function.)
2. If \( z \) is real, then these definitions give equations (4.5).

With these definitions, the functions \( \cos \) and \( \sin \) enjoy most (but not all) of the properties of the corresponding real trigonometric functions. Before stating these properties, we determine some values of these functions.

**Example 4.2**

Evaluate each of the following numbers in Cartesian form.
(a) \( \sin i \)  
(b) \( \cos(\pi + i) \)

**Solution**

(a) \( \sin i = \frac{1}{2i}(e^{i^2} - e^{-i^2}) \)
\[ = \frac{1}{2i}(e^{-1} - e^1) = \frac{1}{2}(e - e^{-1})i \]

(b) \( \cos(\pi + i) = \frac{1}{2}(e^{i(\pi+i)} + e^{-i(\pi+i)}) \)
\[ = \frac{1}{2}(e^{i\pi-1} + e^{-i\pi+1}) \]
\[ = \frac{1}{2}(e^{i\pi}e^{-1} + e^{-i\pi}e^1) \]
\[ = \frac{1}{2}(-e^{-1} - e^1) = -\frac{1}{2}(e + e^{-1}), \]
using the fact that \( e^{i\pi} = e^{-i\pi} = -1 \).
4 Exponential, trigonometric and hyperbolic functions

Remarks

1. Notice that
   \[ |\sin i| = \frac{1}{2}(e - e^{-1}) \approx 1.175 > 1 \]
   and
   \[ |\cos(\pi + i)| = \frac{1}{2}(e + e^{-1}) \approx 1.543 > 1. \]
   Thus the well-known properties of the real sine and cosine functions
   \[ |\sin x| \leq 1 \quad \text{and} \quad |\cos x| \leq 1, \quad \text{where} \quad x \in \mathbb{R}, \]
   do not always hold when \( x \) is replaced by a complex number \( z \).

2. The solutions to Example 4.2 suggest that there is a connection between the complex trigonometric functions and the hyperbolic functions. This will be made clear later in the section.

Exercise 4.5

Evaluate each of the following complex numbers in Cartesian form.
(a) \( \sin(\pi/2 + i) \)    (b) \( \cos i \)

In order to describe the algebraic properties of the complex sine and cosine functions, we need to introduce the full range of complex trigonometric functions. First, however, we determine the zeros of \( \sin \) and \( \cos \).

Theorem 4.2

(a) The set of zeros of the sine function is
   \[ \{ z : \sin z = 0 \} = \{ n\pi : n \in \mathbb{Z} \}. \]
(b) The set of zeros of the cosine function is
   \[ \{ z : \cos z = 0 \} = \{ (n + \frac{1}{2})\pi : n \in \mathbb{Z} \}. \]

The theorem says that the only zeros of the complex sine and cosine functions are those of the real sine and cosine functions.

Proof

(a) Using the definition of \( \sin z \), we have
   \[ \sin z = 0 \iff \frac{1}{2i}(e^{iz} - e^{-iz}) = 0 \]
   \[ \iff e^{iz} = e^{-iz} \]
   \[ \iff e^{2iz} = 1 \]
   \[ \iff 2iz \in \{ 2n\pi i : n \in \mathbb{Z} \} \quad \text{(see Exercise 4.2(d))} \]
   \[ \iff z \in \{ n\pi : n \in \mathbb{Z} \}, \]
   which proves part (a).
(b) Using the definition of \( \cos z \), we have
\[
\cos z = 0 \iff \frac{1}{2}(e^{iz} + e^{-iz}) = 0 \\
\iff e^{iz} = -e^{-iz} \\
\iff e^{2iz} = -1 \\
\iff 2iz \in \{(2n + 1)\pi : n \in \mathbb{Z}\} \quad \text{(see Exercise 4.2(d))} \\
\iff z \in \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\},
\]
which proves part (b).

The other complex trigonometric functions \( \tan, \sec, \cot \) and \( \cosec \) are defined as in the real case.

**Definitions**

For all \( z \) in \( \mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\} \),
\[
\tan z = \frac{\sin z}{\cos z} \quad \text{and} \quad \sec z = \frac{1}{\cos z}.
\]

For all \( z \) in \( \mathbb{C} - \{n\pi : n \in \mathbb{Z}\} \),
\[
\cot z = \frac{\cos z}{\sin z} \quad \text{and} \quad \cosec z = \frac{1}{\sin z}.
\]

We now record the basic algebraic identities satisfied by these complex trigonometric functions. All of these identities are the same as identities satisfied by the real trigonometric functions.

**Theorem 4.3 Trigonometric Identities**

(a) **Addition**
\[
\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}
\]

(b) **Squares**
\[
\cos^2 z + \sin^2 z = 1 \\
\sec^2 z = 1 + \tan^2 z \\
\cosec^2 z = 1 + \cot^2 z
\]

(c) **Negatives**
\[
\sin(-z) = -\sin z \\
\cos(-z) = \cos z \\
\tan(-z) = -\tan z
\]
We prove three of these identities in the next example, and ask you to check some more of them in Exercise 4.6.

**Example 4.3**

Prove the following identities.

(a) \( \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \)

(b) \( \cos(-z) = \cos z \)

(c) \( \sin(z + 2\pi) = \sin z \)

**Solution**

(a) Starting with the right-hand side, we have

\[
\begin{align*}
\sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
&= \frac{1}{2i}(e^{iz_1} - e^{-iz_1}) \left( \frac{1}{2}(e^{iz_2} + e^{-iz_2}) \right) + \frac{1}{2}(e^{iz_1} + e^{-iz_1}) \left( \frac{1}{2i}(e^{iz_2} - e^{-iz_2}) \right) \\
&= \frac{1}{4i} \left( e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1+z_2)} - e^{-i(z_1-z_2)} \right) \\
&\quad + \left( e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1+z_2)} - e^{-i(z_1-z_2)} \right) \\
&= \frac{1}{2i} \left( e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} \right) \\
&= \sin(z_1 + z_2),
\end{align*}
\]

as required.

(b) We have

\[
\cos(-z) = \frac{1}{2}(e^{i(-z)} + e^{-i(-z)}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z,
\]

as required.

(c) We have

\[
\begin{align*}
\sin(z + 2\pi) &= \frac{1}{2i}(e^{i(z+2\pi)} - e^{-i(z+2\pi)}) \\
&= \frac{1}{2i}(e^{iz} e^{2\pi i} - e^{-iz} e^{-2\pi i}) \\
&= \frac{1}{2i}(e^{iz} - e^{-iz}) \quad (\text{since } e^{2\pi i} = e^{-2\pi i} = 1) \\
&= \sin z,
\end{align*}
\]

as required.
Theorem 4.3 is by no means an exhaustive list of trigonometric identities. For example, we have not included identities such as
\[
\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2, \\
\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2
\]
and
\[
\sin 2z = 2 \sin z \cos z.
\]
However, these can readily be deduced from the identities in Theorem 4.3.

Exercise 4.6

(a) Prove the following identities.
(i) \(\sin(-z) = -\sin z\)     (ii) \(\cos(z + 2\pi) = \cos z\)

(b) Deduce the following identities from Theorem 4.3.
(i) \(\cos 2z = 2 \cos^2 z - 1\)     (ii) \(\tan(z_1 - z_2) = \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}\)

4.3 Hyperbolic functions

Earlier in this section we referred to a relationship between complex trigonometric functions and the real hyperbolic functions
\[
\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \cosh x = \frac{1}{2}(e^x + e^{-x}),
\]
whose graphs appear in Figure 4.9.

The complex hyperbolic functions are defined by the same formulas as for the real hyperbolic functions.

**Definitions**

For all \(z\) in \(\mathbb{C}\),
\[
\sinh z = \frac{1}{2}(e^z - e^{-z}) \quad \text{and} \quad \cosh z = \frac{1}{2}(e^z + e^{-z}).
\]

For all \(z\) in \(\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}\),
\[
\tanh z = \frac{\sinh z}{\cosh z} \quad \text{and} \quad \sech z = \frac{1}{\cosh z}.
\]

For all \(z\) in \(\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}\),
\[
\coth z = \frac{\cosh z}{\sinh z} \quad \text{and} \quad \cosech z = \frac{1}{\sinh z}.
\]

In these definitions we have used the facts that
\[
\{z : \sinh z = 0\} = \{n\pi i : n \in \mathbb{Z}\}
\]
and
\[
\{z : \cosh z = 0\} = \{n + \frac{1}{2} \pi i : n \in \mathbb{Z}\}.
\]
All these zeros lie on the imaginary axis. These ‘zero sets’ are readily deduced from the zero sets of the sine and cosine functions by using the following result, which shows the close relationship between the complex hyperbolic functions and the complex trigonometric functions.

**Theorem 4.4**
For all $z$ in $\mathbb{C}$,

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z.$$ 

**Proof** For $z \in \mathbb{C}$,

$$\sin(iz) = \frac{1}{2i}(e^{i(iz)} - e^{-i(iz)}) = -\frac{1}{2}i(e^{-z} - e^{z}) = i \sinh z$$

and

$$\cos(iz) = \frac{1}{2}(e^{i(iz)} + e^{-i(iz)}) = \frac{1}{2}(e^{-z} + e^{z}) = \cosh z. \quad \blacksquare$$

The hyperbolic functions satisfy a number of basic identities, summarised in Theorem 4.5. We omit the proofs, which can all be deduced either from Theorem 4.3, by using the identities in Theorem 4.4, or directly from the definitions.

**Theorem 4.5  Hyperbolic Identities**

(a) **Addition**

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\tanh(z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$$

(b) **Squares**

$$\cosh^2 z - \sinh^2 z = 1$$

$$\text{sech}^2 z = 1 - \tanh^2 z$$

$$\text{cosech}^2 z = \coth^2 z - 1$$

(c) **Negatives**

$$\sinh(-z) = -\sinh z$$

$$\cosh(-z) = \cosh z$$

$$\tanh(-z) = -\tanh z$$

(d) **Periodicity**

$$\sinh(z + 2\pi i) = \sinh z$$

$$\cosh(z + 2\pi i) = \cosh z$$

$$\tanh(z + \pi i) = \tanh z$$
The following example and exercise show that the hyperbolic functions play an important role in the determination of the real and imaginary parts of $\sin z$ and $\cos z$.

**Example 4.4**

Let $z = x + iy$. Prove the following identities.

(a) $\sin z = \sin x \cosh y + i \cos x \sinh y$  
(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

**Solution**

(a) We have

$$
\sin(x + iy) = \sin x \cosh iy + \cos x \sin iy \quad \text{(Theorem 4.3(a))}
$$

$$
= \sin x \cosh y + i \cos x \sinh y \quad \text{(Theorem 4.4)}.
$$

(b) We have

$$
|\sin(x + iy)|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \quad \text{(part (a))}
$$

$$
= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \quad \text{(Theorem 4.5(b))}
$$

$$
= \sin^2 x + \sinh^2 y (\sin^2 x + \cos^2 x)
$$

$$
= \sin^2 x + \sinh^2 y.
$$

**Exercise 4.7**

Let $z = x + iy$. Prove the following identities.

(a) $\cos z = \cos x \cosh y - i \sin x \sinh y$  
(b) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

**Further exercises**

**Exercise 4.8**

Express the following complex numbers in Cartesian form.

(a) $e^{3\pi i}$  
(b) $ee^{\pi i/2}$  
(c) $e^{2\pi i/3}$  
(d) $e^{-3\pi i/2}$  
(e) $e^{2+\pi i}$  
(f) $e^{3+\pi i/2}$  
(g) $e^{(\pi i)/6}$  
(h) $e^{(\cos \theta + i \sin \theta)}$

**Exercise 4.9**

(a) Express the following complex numbers in the polar form $re^{i\theta}$.

(i) $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$  
(ii) $-1 - i$  
(iii) $1 + \sqrt{3}i$
(b) Hence evaluate the following complex numbers, giving your answers in Cartesian form.

(i) \( \left( \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)^3 \)

(ii) \( (1 + \sqrt{3}i)^{-7} \)

**Exercise 4.10**

Express the following complex numbers in Cartesian form.

(a) \( \sin(\pi + 2i) \)  
(b) \( \cos(\pi/2 - i) \)  
(c) \( \tan i \)

In parts (a) and (b) you can work from the definitions of \( \sin \) and \( \cos \), or use identities established in the section.

**Exercise 4.11**

Prove the following identities.

(a) \( e^z = e^\bar{z} \)  
(b) \( \sin 2z = 2 \sin z \cos z \)  
(c) \( \sin \bar{z} = \sin z \)

(d) \( \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \)

(e) \( \cosh^2 z - \sinh^2 z = 1 \)

In parts (a), (b) and (c), work from the definitions of \( e^z \), \( \sin 2z \) and \( \sin z \); in parts (d) and (e), use identities established in this section.

5 Logarithms and powers

After working through this section, you should be able to:

• determine the principal logarithm \( \text{Log} \) of a non-zero complex number \( z \), and describe the geometric effect of the function \( z \mapsto \text{Log} z \)

• determine the principal power \( z^\alpha \), where \( \alpha \in \mathbb{C} \), of a non-zero complex number \( z \).

5.1 Logarithms of complex numbers

In real analysis the natural logarithm function \( \log \) (that is, the logarithm to base \( e \), sometimes denoted \( \ln \) or \( \log_e \)) is defined as the inverse function of the exponential function. Since \( x \mapsto e^x \) is a one-to-one function on \( \mathbb{R} \) with image set \( (0, \infty) \), the inverse function \( \log \) has domain \( (0, \infty) \), and is defined by the rule

\[ \log y = x, \quad \text{where } y = e^x. \]

The graph of the function \( \log \) can be obtained by reflecting the graph of \( y = e^x \) in the line \( y = x \) (Figure 5.1).

Now consider the complex exponential function \( f(z) = e^z \). In trying to define an inverse function \( f^{-1} \) for \( f \), we encounter the fact that \( f \) is not a one-to-one function. For example,

\[ \cdots = f(-2\pi i) = f(0) = f(2\pi i) = f(4\pi i) = \cdots = 1. \]
To get around this difficulty, we seek a set $A$ (preferably as large as possible) on which $f(z) = e^z$ is one-to-one. We then restrict $f$ to the set $A$, retaining the name $f$ for the restricted function with domain $A$, and define an inverse function $f^{-1}$ with domain $f(A)$. This approach was used in Subsection 1.5 with the function $f(z) = z^2$. We will find (just as we did for the function $f(z) = z^2$) that there are many choices for the set $A$. The choice in the following example is motivated by the periodicity property

$$e^{z+2\pi i} = e^z, \quad \text{for all } z \in \mathbb{C},$$

which suggests that $A$ should be chosen in such a way that it does not contain two points that differ by $2\pi i$.

Example 5.1

Let

$$A = \{x + iy : -\pi < y \leq \pi\},$$

which is a horizontal strip (Figure 5.2). Prove that the function

$$f(z) = e^z \quad (z \in A)$$

has an inverse function $f^{-1}$, and determine the domain and rule of $f^{-1}$.

Solution

First we determine the image set of $f$:

$$f(A) = \{e^z : z \in A\}$$

$$= \{w = e^{x+iy} : x \in \mathbb{R}, -\pi < y \leq \pi\}$$

$$= \{w = e^x e^{iy} : x \in \mathbb{R}, -\pi < y \leq \pi\}$$

$$= \{w = \rho e^{i\phi} : \rho > 0, -\pi < \phi \leq \pi\}$$

$$= \mathbb{C} - \{0\},$$

where $\rho = e^x$ and $\phi = y$. (In fact, we have already observed that $f(A) = \mathbb{C} - \{0\}$ in Figure 4.7.)

Now, for each $w \in \mathbb{C} - \{0\}$, we wish to solve the equation

$$w = e^z \quad (5.1)$$

to obtain a unique solution $z$ in $A$.

Each $w$ in $\mathbb{C} - \{0\}$ can be written in the form

$$w = \rho e^{i\phi}, \quad \text{where } \rho > 0 \text{ and } -\pi < \phi \leq \pi,$$

and equation (5.1) is then

$$\rho e^{i\phi} = e^z = e^x e^{iy}, \quad \text{where } z = x + iy.$$

Equating moduli on both sides of the equation $\rho e^{i\phi} = e^x e^{iy}$, we obtain $\rho = e^x$, and on dividing both sides by $\rho$ we obtain $e^{i\phi} = e^{iy}$. 


---

**Figure 5.2** A horizontal strip
Hence
\[ x = \log \rho \quad \text{and} \quad y = \phi + 2n\pi, \]
where \( n \in \mathbb{Z} \). For \( n = 0 \), the solution is
\[ z = x + iy = \log \rho + i\phi, \]
which lies in \( A \), since \( -\pi < \phi \leq \pi \), whereas the other solutions (with \( n \neq 0 \)) lie outside \( A \).
Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} - \{0\} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} - \{0\} \) and rule
\[ f^{-1}(w) = \log \rho + i\phi, \]
where \( w = \rho e^{i\phi} \) and \( \rho > 0, -\pi < \phi \leq \pi \).

**Remarks**

1. Since \( \phi = \text{Arg} \ w \) and \( \rho = |w| \) in this solution, the rule for \( f^{-1} \) can also be written in the form
\[ f^{-1}(w) = \log |w| + i \text{Arg} \ w \quad (w \neq 0). \]

2. Example 5.1 is important because it will be used shortly in defining logarithms of complex numbers.

**Exercise 5.1**

Let \( A = \{x + iy : 0 \leq y < 2\pi\} \). Prove that the function
\[ f(z) = e^z \quad (z \in A) \]
has an inverse function \( f^{-1} \), and determine the domain and rule of \( f^{-1} \).

The solution to Example 5.1 and Remark 1 that follows it show that if \( w \neq 0 \), then the equation \( e^z = w \) has infinitely many solutions of the form
\[ z = \log |w| + i(\text{Arg} \ w + 2n\pi), \quad n \in \mathbb{Z}. \]
Each of these solutions is called a **logarithm** of \( w \). The infinitely many logarithms of \( w \) correspond to the infinitely many arguments of \( w \), which are all of the form
\[ \text{Arg} \ w + 2n\pi, \quad n \in \mathbb{Z}. \]
Some texts use the notation ‘\( \log w \)’ to denote a particular logarithm of \( w \), determined by a particular choice of argument. However, to prevent confusion, we will avoid this ambiguous notation, and instead we will almost always use the logarithm of \( w \) that corresponds to the principal argument of \( w \) (that is, \( n = 0 \)). This solution,
\[ z = \log |w| + i \text{Arg} \ w, \]
of \( e^z = w \) is called the **principal logarithm** of \( w \), written \( \text{Log} \ w \). (Note the capital \( L \) in \( \text{Log} \), corresponding to the capital \( A \) in \( \text{Arg} \).)
Thus the inverse function $f^{-1}$ of Example 5.1 can be written as

$$f^{-1}(w) = \log w \quad (w \in \mathbb{C} - \{0\}).$$

**Definitions**

For $z \in \mathbb{C} - \{0\}$, the principal logarithm of $z$ is

$$\log z = \log |z| + i \arg z.$$

The corresponding principal logarithm function is called $\log$.

**Remarks**

1. If $z$ is real and positive (that is, $z = x + 0i$, where $x > 0$), then

$$\log z = \log x,$$

where $\log x$ denotes the usual real logarithm of $x$, as expected. Thus the restriction of $\log$ to $(0, \infty)$ is $\log$.

2. Note that the definition of $\log$ applies if $z$ is a negative real number. For example,

$$\log(-2) = \log |-2| + i \arg(-2) = \log 2 + i \pi.$$

3. Since $\log$ is the inverse function of the function

$$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \leq \pi\}),$$

we have the following two identities.

$$e^{\log z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$

$$\log(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \leq \pi\}.$$

The latter identity is false if $z$ lies outside the strip $\{x + iy : -\pi < y \leq \pi\}$.

For example, if $z = 2\pi i$, then

$$\log(e^{2\pi i}) = \log 1 = \log 1 = 0 \neq 2\pi i.$$

4. The function $\log$ has domain $\mathbb{C} - \{0\}$, and its image set, written in terms of $w$, is $\{w : -\pi < \text{Im} w \leq \pi\}$, as shown in Figure 5.3.

**Figure 5.3** $\log$ has domain $\mathbb{C} - \{0\}$ and image set $\{x + iy : -\pi < y \leq \pi\}$.
Example 5.2
Evaluate \( \log(-1 + i) \) in Cartesian form.

Solution
Since \(| -1 + i | = \sqrt{2} \) and \( \text{Arg}(-1 + i) = 3\pi/4 \), we have
\[
\log(-1 + i) = \log \sqrt{2} + i3\pi/4.
\]

Exercise 5.2
Evaluate the following complex numbers in Cartesian form.
(a) \( \log i \)  
(b) \( \log(\sqrt{3} - i) \)  
(c) \( \log(\frac{1}{2} + \frac{1}{2}i) \)

The real function \( \log \) satisfies various identities, such as
\[
\log(x_1 x_2) = \log x_1 + \log x_2, \quad \text{for } x_1, x_2 > 0,
\]
and
\[
\log(1/x) = -\log x, \quad \text{for } x > 0.
\]
It is natural to hope that similar identities will hold for the complex function \( \log \), and this is indeed the case, provided that suitable restrictions are placed on the variables involved.

Theorem 5.1 Logarithmic Identities
(a) **Multiplication**
\[
\log(z_1 z_2) = \log z_1 + \log z_2, \quad \text{if } \text{Arg} z_1, \text{Arg} z_2 \in (-\pi/2, \pi/2].
\]
(b) **Reciprocals**
\[
\log(1/z) = -\log z, \quad \text{if } \text{Arg} z \in (-\pi, \pi).
\]

Part (b) does not hold if \( \text{Arg} z = \pi \), as you can check by choosing \( z = -1 \).

Proof
(a) If \( \text{Arg} z_1, \text{Arg} z_2 \in (-\pi/2, \pi/2] \), then \( \text{Arg} z_1 + \text{Arg} z_2 \in (-\pi, \pi) \), so
\[
\text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2.
\]
(You met this property of Arg in Subsection 2.3 of Unit A1.) Hence
\[
\log(z_1 z_2) = \log |z_1 z_2| + i \text{Arg}(z_1 z_2)
= \log |z_1| + \log |z_2| + i (\text{Arg} z_1 + \text{Arg} z_2)
= (\log |z_1| + i \text{Arg} z_1) + (\log |z_2| + i \text{Arg} z_2)
= \log z_1 + \log z_2.
\]
(b) Since \(1/z = \bar{z}/|z|^2\) and \(\text{Arg } z \neq \pi\), it follows that
\[
\text{Arg}(1/z) = \text{Arg } \bar{z} = -\text{Arg } z.
\]

Hence
\[
\text{Log}(1/z) = \log |1/z| + i \text{Arg}(1/z)
= \log(1/|z|) + i \text{Arg}(1/z)
= -\log |z| - i \text{Arg } z
= -\text{Log } z.
\]

\[\blacksquare\]

**Remark**

Using properties of \(\text{Arg}\) described in Subsection 2.3 of Unit A1, it can be shown that the identity in Theorem 5.1(a) holds in the following form for any values in the domain \(\mathbb{C} - \{0\}\) of \(\text{Log}\):
\[
\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2 + 2n\pi i,
\]
where \(n\) is \(-1\), 0 or 1, depending on whether \(\text{Arg } z_1 + \text{Arg } z_2\) is greater than \(\pi\), lies in the interval \((-\pi, \pi]\), or is less than or equal to \(-\pi\).

For example, if \(z_1 = z_2 = -1\), then
\[
\text{Log}(z_1 z_2) = \text{Log } 1 = 0,
\]
\[
\text{Log } z_1 + \text{Log } z_2 = (\log 1 + i\pi) + (\log 1 + i\pi) = 2\pi i.
\]
Thus the identity (5.2) holds with \(n = -1\) in this case.

**The geometric effect of the function Log**

We now briefly discuss the geometric effect of the function \(\text{Log}\), drawing on our knowledge of the geometric effect of the exponential function \(z \mapsto e^z\) obtained in Subsection 4.1 and depicted in Figure 4.6. Since \(\text{Log}\) is the inverse function of the restriction of the exponential function to the horizontal strip \(\{z : -\pi < \text{Im } z \leq \pi\}\), we can understand the geometric effect of \(\text{Log}\) by ‘reversing’ the effect of \(\text{exp}\) observed in Figure 4.6. Thus \(\text{Log}\) maps circles with centre 0 and radius \(r\) onto line segments of the form \(u = \log r, -\pi < v \leq \pi\) (where \(w = u + iv\)), and it maps rays \(\text{Arg } z = \theta\) onto lines of the form \(v = \theta\). Figure 5.4 shows the image of this polar grid under the function \(\text{Log}\).

Notice that

- points lying outside the unit circle \(\{z : |z| = 1\}\) have images lying in the right half-strip, whereas non-zero points inside the unit circle have images lying in the left half-strip
- the \(\text{Log}\) function behaves in a rather strange manner near the negative real axis. For example, as \(z\) approaches the point \(-1\) on the negative real axis from above, the image point \(w = \text{Log } z\) approaches the point \(i\pi\), but if \(z\) approaches \(-1\) from below, then \(w = \text{Log } z\) approaches the point \(-i\pi\) (which is not in the image set of \(\text{Log}\)). This strange behaviour occurs near the negative real axis because of the particular definition of \(\text{Arg}\) that we have chosen (and the fact that \(\text{Log } z = \log |z| + i \text{Arg } z\)). We consider this behaviour further in Unit A3.
Figure 5.4 Image of a polar grid under $f(z) = \text{Log } z$

### Exercise 5.3

Classify the following statements as True or False.

(a) The image under the function $\text{Log}$ of the ellipse

$$4x^2 + 9y^2 = 1$$

lies in the right half-plane.

(b) The image under the function $\text{Log}$ of the ray $\theta = \pi/4$ lies in the right half-plane.

(c) There is a point $z \in \mathbb{C}$ such that

$$\text{Log } z = 1 + 4i.$$

(d) There is a point $z \in \mathbb{C}$ such that

$$\text{Log } z = 1 + \frac{1}{4}i.$$
Leonhard Euler and logarithms

The first person to recognise the many-valued nature of complex logarithms was the prolific Swiss mathematician and scientist Leonhard Euler (1707–1783). In a letter to another Swiss mathematician Gabriel Cramer (1704–1752) in 1746, Euler wrote:

I have finally discovered the true solution: in the same way that to one sine there correspond an infinite number of different angles I have found that it is the same with logarithms, and each number has an infinity of different logarithms, all of them imaginary unless the number is real and positive; there is only one logarithm that is real, and we regard it as its unique logarithm.

(Speziali, 1983, p. 428, cited in Bottazzini and Gray, 2013, p. 82)

In Euler’s time, ‘imaginary’ meant ‘complex and not real’.

As the quote demonstrates, Euler’s understanding of logarithms was remarkably advanced; his work on complex numbers led to considerable developments in the subject. He was the first to publish the equation $e^{ix} = \cos x + i \sin x$ (Euler’s Identity) which he obtained not by defining $e^{ix}$ from that formula as we have done, but by defining the exponential and trigonometric functions in terms of series, and then proceeding with a method similar to that presented at the start of Subsection 4.1.

5.2 Powers of complex numbers

In this subsection we define the expression $z^\alpha$, where $z$ is any non-zero complex number and $\alpha$ is any complex number. In Subsection 3.1 of Unit A1 you saw that any non-zero complex number $z$ has $n$ $n$th roots and that the expression $z^{1/n}$ is reserved for just one of these roots, called the principal $n$th root of $z$. It seems likely, therefore, that there is going to be some difficulty in defining the expression $z^\alpha$ in a unique way.

Recall first that if $a > 0$ and $x \in \mathbb{R}$, then $a^x$ satisfies the equation

$$a^x = e^{x \log a}.$$ 

It is tempting to define $z^\alpha$ by means of a similar formula, namely

$$z^\alpha = e^{\alpha \log z},$$

where ‘$\log z$’ is a logarithm of $z$.

The problem is, however, that any non-zero complex number $z$ has infinitely many logarithms, so the formula above would give rise to infinitely many possible values of $z^\alpha$. To avoid confusion, we will define $z^\alpha$ using $\text{Log } z$, the principal logarithm of $z$. 

Leonhard Euler
Definitions
For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the **principal $\alpha$th power** of $z$ is
$$
z^\alpha = \exp(\alpha \Log z).
$$
The function $z \mapsto z^\alpha$ is called the **principal $\alpha$th power function**.

Remarks
1. It can be shown that this definition agrees with the usual meaning of $z^\alpha$
   if $\alpha = n$ or $\alpha = 1/n$, where $n$ is a positive integer. For example, if $\alpha = n$
   and $z \neq 0$, then
   $$
e^n \Log z = e^{\Log z + \cdots + \Log z}
   = e^{\Log z} \times \cdots \times e^{\Log z}
   = z \times \cdots \times z = z^n.
$$
2. This definition assigns no value to $0^\alpha$. However, in Subsection 3.1 of
   Unit A1, we defined $0^n$ and $0^{1/n}$ to be 0, for $n = 1, 2, 3, \ldots$.
3. Some texts take a different approach and allow both ‘log $z$’ and $z^\alpha$ to
   represent infinitely many different values, and specify, when
   appropriate, which value is being considered at a given time.

Example 5.3
Express each of the following numbers in Cartesian form.
(a) $(-1)^{1/2}$ (b) $(1 + i)^i$ (c) $i^i$

Solution
(a) We have
$$(-1)^{1/2} = \exp(\frac{1}{2} \Log(-1))
= e^{i\pi/2} \quad \text{(since } \Log(-1) = i\pi)$$
$$= i,$$
as expected!
(b) Since $1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$, we have
$$(1 + i)^i = \exp(i \Log(1 + i))
= \exp(i(\log \sqrt{2} + i\pi/4))
= \exp(-\pi/4 + i \log \sqrt{2})
= e^{-\pi/4}(\cos(\log \sqrt{2}) + i \sin(\log \sqrt{2})).$$
(c) We have
$$i^i = \exp(i \Log i)
= \exp(i(i\pi/2)) \quad \text{(since } \Log i = i\pi/2)
= e^{-\pi/2},$$
a real number!
**Exercise 5.4**

Express each of the following numbers in Cartesian form.

(a) \((1 + i)^{2/3}\)  
(b) \(i^{1+i}\)

**Exercise 5.5**

Show that for \(\alpha = 1/n\), where \(n\) is a positive integer, the definition of \(z^\alpha\) given above agrees with the definition of \(z^{1/n}\) given in Subsection 3.1 of Unit A1 (where we used the notation \(\rho = |z|\) and \(\phi = \text{Arg} \ z\)).

**Exercise 5.6**

(a) Find non-zero complex numbers \(z_1, z_2\) and \(\alpha\) for which the equation \(z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha\) does not hold.

(b) Prove that \(z^\alpha z^\beta = z^{\alpha + \beta}\) for all \(z \in \mathbb{C} - \{0\}\) and \(\alpha, \beta \in \mathbb{C}\).

**Further exercises**

**Exercise 5.7**

Express each of the following complex numbers in Cartesian form.

(a) \(\log(-2)\)  
(b) \(\log(i^3)\)  
(c) \(\log(1 + i)\)  
(d) \(\log(\sqrt{3})\)  
(e) \(\log(i - \sqrt{3})\)  
(f) \(\log\left(\frac{1 - i}{\sqrt{2}}\right)\)

**Exercise 5.8**

Express each of the following complex numbers in Cartesian form.

(a) \(i^{-i}\)  
(b) \((-i)^i\)  
(c) \((1 - i)^i\)  
(d) \((-1)^i\)
Solutions to exercises

Solution to Exercise 1.1
(a) Domain \( \mathbb{C} \), codomain \( \mathbb{C} \).
(b) Domain \( \mathbb{C} \setminus \{-2\} \), codomain \( \mathbb{C} \).
(c) Domain \( \mathbb{C} \setminus \{0\} \), codomain \( \mathbb{C} \).
(d) Domain \( \mathbb{C} \setminus \{-i, i\} \), codomain \( \mathbb{C} \).

Solution to Exercise 1.2
(a) The domain of \( f \) is \( \mathbb{C} \). The image set of \( f \) is
\[
\begin{align*}
f(\mathbb{C}) &= \{3iz : z \in \mathbb{C}\} \\
&= \left\{ w : z = \frac{w}{3i} \in \mathbb{C}\right\} \\
&= \{ w : w \in \mathbb{C}\} \\
&= \mathbb{C}.
\end{align*}
\]
(b) The domain of \( f \) is \( \mathbb{C} \setminus \{-i\} \). The image set of \( f \) is
\[
\begin{align*}
f(\mathbb{C} \setminus \{-i\}) &= \left\{ \frac{3z + 1}{z + i} : z \in \mathbb{C} \setminus \{-i\}\right\} \\
&= \left\{ w : z = \frac{1 - iw}{w - 3} \neq -i \right\} \\
&= \{ w : w \neq 3\} \\
&= \mathbb{C} \setminus \{3\},
\end{align*}
\]
where we have used the fact that the equation
\[
\frac{1 - iw}{w - 3} = -i,
\]
or, equivalently, \( 1 - iw = -iw + 3i \), has no solutions.
(c) The domain of \( f \) is \( \mathbb{C} \). The image set of \( f \) is
\[
\begin{align*}
f(\mathbb{C}) &= \{ \text{Im} z : z \in \mathbb{C}\} \\
&= \{ y : y \in \mathbb{R}\} \quad (z = x + iy) \\
&= \mathbb{R}.
\end{align*}
\]

Solution to Exercise 1.3
(a) The domain of \( f \) is \( \mathbb{C} \). The image set is
\[
f(\mathbb{C}) = \{ x \in \mathbb{R} : x \geq 0\}.
\]
(b) The domain of \( f \) is \( \mathbb{C} \setminus \{0\} \). The image set is
\[
f(\mathbb{C} \setminus \{0\}) = (-\pi, \pi].
\]

Solution to Exercise 1.4
(a) \( f + g \) has domain \( \mathbb{C} \setminus \{0, 1\} \) and rule
\[
\begin{align*}
(f + g)(z) &= f(z) + g(z) \\
&= \frac{1}{z} + \frac{z + 3i}{z^2 - z} \\
&= \frac{2z - 1 + 3i}{z^2 - z}.
\end{align*}
\]
(b) \( fg \) has domain \( \mathbb{C} \setminus \{0, 1\} \) and rule
\[
\begin{align*}
(fg)(z) &= f(z)g(z) \\
&= \frac{1}{z} \times \frac{z + 3i}{z^2 - z} \\
&= \frac{z + 3i}{z^3 - z^2}.
\end{align*}
\]
(c) \( f/g \) has domain \( \mathbb{C} \setminus \{0, 1, -3i\} \) and rule
\[
\begin{align*}
(f/g)(z) &= \frac{f(z)}{g(z)} \\
&= \frac{1}{z} / \left( \frac{z + 3i}{z^2 - z} \right) \\
&= \frac{z - 1}{z + 3i}.
\end{align*}
\]
(Note that 0 and 1 are excluded from the domain of \( f/g \) even though \((z - 1)/(z + 3i)\) is defined at these points.)

Solution to Exercise 1.5
(a) The domain of \( g \circ f \) is
\[
\text{domain of } f - \left\{ z : \frac{1}{z} \notin \mathbb{C} \setminus \{0, 1\} \right\}
\]
\[
= (\mathbb{C} \setminus \{0\}) - \left\{ z : \frac{1}{z} \notin \{0, 1\} \right\}
\]
\[
= (\mathbb{C} \setminus \{0\}) - \{1\} = \mathbb{C} \setminus \{0, 1\}.
\]
The rule of \( g \circ f \) is
\[
g(f(z)) = \frac{(1/z) + 3i}{(1/z)^2 - (1/z)} = \frac{z + 3iz^2}{1 - z}.
\]
(b) The domain of \( f \circ g \) is
\[
\text{domain of } g - \left\{ z : \frac{z + 3i}{z^2 - z} \notin \mathbb{C} \setminus \{0\} \right\}
\]
\[
= (\mathbb{C} \setminus \{0, 1\}) - \left\{ z : \frac{z + 3i}{z^2 - z} \notin \{0\} \right\}
\]
\[
= (\mathbb{C} \setminus \{0, 1\}) - \{-3i\} = \mathbb{C} \setminus \{0, 1, -3i\}.
\]
The rule of $f \circ g$ is
\[
f(g(z)) = \frac{1}{\left(\frac{z + 3i}{z^2 - z}\right)} = \frac{z^2 - z}{z + 3i}.
\]

**Solution to Exercise 1.6**

First we determine the image set of $f$. This is $C - \{3\}$, from Exercise 1.2(b).

Now, for each $w \in C - \{3\}$ we wish to solve the equation
\[
w = \frac{3z + 1}{z + i}
\]
to obtain a unique solution $z$ in $C - \{-i\}$. This is achieved by the rearrangement
\[
z = \frac{1 - iw}{w - 3}.
\]
Thus $f$ is a one-to-one function, with image set $C - \{3\}$. Hence $f$ has an inverse function $f^{-1}$ with domain $C - \{3\}$ and rule
\[
f^{-1}(w) = \frac{1 - iw}{w - 3}.
\]

**Solution to Exercise 1.7**

Let us first determine the image set of $f$, which is
\[
f(A) = \{w = z^2 : z \in A\} = \{0\} \cup \{w = z^2 : 0 \leq \text{Arg} z < \pi\}.
\]
By writing $z = r(cos \theta + i sin \theta)$, we see that $f(A)$ is equal to the union of $\{0\}$ and
\[
\{w = r^2(cos 2\theta + i sin 2\theta) : r > 0, 0 \leq \theta < \pi\}.
\]
Let $\rho = r^2$ and $\phi = 2\theta$; then $f(A)$ is the union of $\{0\}$ and
\[
\{w = \rho(cos \phi + i sin \phi) : \rho > 0, 0 \leq \phi < 2\pi\},
\]
so $f(A) = C$.

Now, for each $w \in C$ we wish to solve the equation
\[
w = z^2 \quad \text{(S1)}
\]
to obtain a unique solution $z$ in $A$. If $w = 0$, then equation (S1) has the unique solution $z = 0$. If $w \neq 0$, then $w$ can be written in the form
\[
w = \rho(cos \phi + i sin \phi),
\]
where $\rho > 0$ and $0 \leq \phi < 2\pi$, and equation (S1) then has exactly two solutions
\[
z_0 = \rho^{1/2}(cos(\phi/2) + i sin(\phi/2)),
\]
\[
z_1 = \rho^{1/2}(cos(\phi/2 + \pi) + i sin(\phi/2 + \pi)),
\]
by Theorem 3.1 of Unit A1. Clearly, $z_0 \in A$, since $0 \leq \phi/2 < \pi$, whereas $z_1 \notin A$.

Thus $f$ is a one-to-one function, with image set $C$. Hence $f$ has an inverse function $f^{-1}$ with domain $C$ and rule given by $f^{-1}(0) = 0$ and
\[
f^{-1}(w) = \rho^{1/2}(cos(\phi/2) + i sin(\phi/2)),
\]
where $w = \rho(cos \phi + i sin \phi)$, $\rho > 0$, $0 \leq \phi < 2\pi$.

**Remark:** Using the definition of the principal square root $\sqrt{w}$ given in Subsection 3.1 of Unit A1, we see that
\[
f^{-1}(w) = \sqrt{w}
\]
for values of $w$ for which $0 \leq \phi \leq \pi$. However, $f^{-1}(w)$ and $\sqrt{w}$ differ for $\pi < \phi < 2\pi$. For example, the number $w = -i$ has polar form
\[
\cos(3\pi/2) + i \sin(3\pi/2),
\]
and $0 \leq 3\pi/2 < 2\pi$, so
\[
f^{-1}(-i) = \cos(3\pi/4) + i \sin(3\pi/4) = \frac{-1 + i}{\sqrt{2}}.
\]
However, $-i$ also has polar form
\[
\cos(-\pi/2) + i \sin(-\pi/2),
\]
and $-\pi < -\pi/2 \leq \pi$, so
\[
\sqrt{-i} = \cos(-\pi/4) + i \sin(-\pi/4) = \frac{1 - i}{\sqrt{2}}.
\]

**Solution to Exercise 1.8**

(a) $C$

(b) $C - \{1\}$

(c) $C - \{-i, i\}$

(d) $C - \{z : \text{Re} z = 0\} = \{z : \text{Re} z \neq 0\}$

(e) $C - \{z : \vert z \vert = 1\} = \{z : \vert z \vert \neq 1\}$

(f) $C - \{\frac{1}{2}(1 + \sqrt{3}i), -1, \frac{1}{2}(1 - \sqrt{3}i)\}$
Solution to Exercise 1.9
The image set of each $f$ is determined as follows.

(a) $\{2z + 1 : z \in \mathbb{C}\}$
\[
= \{ w : z = (w - 1)/2 \in \mathbb{C} \}
= \{ w : w \in \mathbb{C} \} = \mathbb{C}
\]

(b) $\{1/(z - 1) : z \in \mathbb{C} - \{1\}\}$
\[
= \{ w : z = (1 + w)/w \neq 1 \}
= \{ w : w \neq 0 \} = \mathbb{C} - \{0\}
\]

(c) $\{z/(z - 1) : z \in \mathbb{C} - \{1\}\}$
\[
= \{ w : z = w/(w - 1) \neq 1 \}
= \{ w : w \neq 1 \} = \mathbb{C} - \{1\}
\]

(d) $\{|z - 1| : z \in \mathbb{C}\} = \{ r : r \geq 0 \} = [0, \infty)$

(e) Writing $z = x + iy$, we see that
\[
\{ \text{Re}(z + i) : z \in \mathbb{C} \} = \{ x : x \in \mathbb{R} \} = \mathbb{R}.
\]

(f) $\{ |\text{Arg} z| : z \in \mathbb{C} - \{0\} \}$
\[
= \{ |\theta| : \theta \in (-\pi, \pi] \} = [0, \pi]
\]

Solution to Exercise 1.10
Observe that $f$ has domain $\mathbb{C} - \{0\}$ and $g$ has domain $\mathbb{C} - \{1\}$.

(a) $f + g$ has domain $\mathbb{C} - \{0, 1\}$ and rule
\[
(f + g)(z) = \frac{z - 1}{z} + \frac{z}{z - 1}
= 2z^2 - 2z + 1
= \frac{z(z - 1)}{z - 1}.
\]

(b) $3f$ has domain $\mathbb{C} - \{0\}$ and $2ig$ has domain $\mathbb{C} - \{1\}$; hence $3f - 2ig$ has domain $\mathbb{C} - \{0, 1\}$ and rule
\[
(3f - 2ig)(z) = \frac{3(z - 1) - 2iz}{z}
= \frac{3z^2 - 6z + 3 - 2iz^2}{z(z - 1)}
= \frac{(3 - 2i)z^2 - 6z + 3}{z(z - 1)}.
\]

(c) $fg$ has domain $\mathbb{C} - \{0, 1\}$ and rule
\[
(fg)(z) = \frac{z - 1}{z} \times \frac{z}{z - 1}
= 1.
\]

(d) $f/g$ has domain
\[
(\mathbb{C} - \{0, 1\}) - \{0\} = \mathbb{C} - \{0, 1\}
\]

and rule
\[
(f/g)(z) = \frac{z - 1}{z} \div \frac{z}{z - 1}
= \left(\frac{z - 1}{z}\right)^2.
\]

Solution to Exercise 1.11
(a) The domain of $f \circ g$ is
\[
\text{domain of } g = \left\{ z : \frac{z}{z - 1} \notin \mathbb{C} - \{0\} \right\}
= (\mathbb{C} - \{1\}) - \{ z : \frac{z}{z - 1} = 0 \}
= (\mathbb{C} - \{1\}) - \{0\} = \mathbb{C} - \{0\}.
\]
The rule of $f \circ g$ is
\[
f(g(z)) = \frac{\left(\frac{z}{z - 1}\right) - 1}{\frac{z}{z - 1} - 1}
= 1 - \left(\frac{z - 1}{z}\right)
= \frac{1}{z}.
\]

(b) The domain of $g \circ f$ is
\[
\text{domain of } f = \left\{ z : \frac{z - 1}{z} \notin \mathbb{C} - \{1\} \right\}
= (\mathbb{C} - \{0\}) - \{ z : \frac{z - 1}{z} = 1 \}
= (\mathbb{C} - \{0\}) - \{0\} = \mathbb{C} - \{0\}.
\]
The rule of $g \circ f$ is
\[
g(f(z)) = \frac{\left(\frac{z - 1}{z}\right)}{\left(\frac{z - 1}{z}\right) - 1}
= \frac{z - 1}{z - 1 - z}
= 1 - z.
\]

(c) The domain of $f \circ f$ is
\[
\text{domain of } f = \left\{ z : \frac{z - 1}{z} \notin \mathbb{C} - \{0\} \right\}
= (\mathbb{C} - \{0\}) - \{ z : z - 1 = 0 \}
= \mathbb{C} - \{0, 1\}.
\]
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The rule of \( f \circ f \) is

\[
f(f(z)) = \left( \frac{z - 1}{z} \right) - 1\]
\[
= 1 - \left( \frac{z}{z - 1} \right)\]
\[
= \frac{1}{1 - z}.
\]

**Solution to Exercise 1.12**

The functions in parts (a), (b) and (c) are one-to-one, as we now show. We already know their image sets from Exercise 1.9.

(a) For each \( w \in \mathbb{C} \), the image set of \( f \), we wish to solve the equation

\[
w = 2z + 1
\]

to obtain a unique solution \( z \) in \( \mathbb{C} \), the domain of \( f \). This is achieved by the rearrangement

\[
z = (w - 1)/2.
\]

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} \) and rule

\[
f^{-1}(w) = (w - 1)/2.
\]

(b) For each \( w \in \mathbb{C} - \{0\} \), the image set of \( f \), we wish to solve the equation

\[
w = 1/(z - 1)
\]

to obtain a unique solution \( z \) in \( \mathbb{C} - \{1\} \), the domain of \( f \). This is achieved by the rearrangement

\[
z = (1 + w)/w.
\]

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} - \{0\} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} - \{0\} \) and rule

\[
f^{-1}(w) = (1 + w)/w.
\]

(c) For each \( w \in \mathbb{C} - \{1\} \), the image set of \( f \), we wish to solve the equation

\[
w = z/(z - 1)
\]

to obtain a unique solution \( z \) in \( \mathbb{C} - \{1\} \), the domain of \( f \). This is achieved by the rearrangement

\[
z = w/(w - 1).
\]

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} - \{1\} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} - \{1\} \) and rule

\[
f^{-1}(w) = w/(w - 1).
\]

(Note that \( f^{-1} = f \) in this case.)

(d) This function is not one-to-one, because (for example)

\[
f(2) = |2 - 1| = 1\quad \text{and}
\]
\[
f(0) = |0 - 1| = 1.
\]

(e) This function is not one-to-one, because (for example)

\[
f(i) = \text{Re}(i + i) = 0\quad \text{and}
\]
\[
f(0) = \text{Re}(0 + i) = 0.
\]

(f) This function is not one-to-one, because (for example)

\[
f(i) = |\text{Arg} i| = |\pi/2| = \pi/2\quad \text{and}
\]
\[
f(-i) = |\text{Arg}(-i)| = |\pi/2| = \pi/2.
\]

**Solution to Exercise 1.13**

Let us first determine the image set of \( f \), which is

\[
f(A) = \{ z^3 : z \in A \} = \{ 0 \} \cup \{ w = z^3 : -\pi/3 < \text{Arg} z \leq \pi/3 \}.
\]

By writing \( z = r(\cos \theta + i \sin \theta) \), we see that \( f(A) \) is equal to the union of \( \{0\} \) and

\[
\{ w = r^3(\cos 3\theta + i \sin 3\theta) : r > 0,\quad -\pi/3 < \theta \leq \pi/3 \}.
\]

Let \( \rho = r^3 \) and \( \phi = 3\theta \); then \( f(A) \) is the union of \( \{0\} \) and

\[
\{ w = \rho(\cos \phi + i \sin \phi) : \rho > 0,\quad -\pi < \phi \leq \pi \},
\]

so \( f(A) = \mathbb{C} \).

Now, for each \( w \in \mathbb{C} \) we wish to solve the equation

\[
w = z^3\quad \text{(S2)}
\]

to obtain a unique solution \( z \) in \( A \). If \( w = 0 \), then equation (S2) has the unique solution \( z = 0 \). If \( w \neq 0 \), then \( w \) can be written in the form

\[
w = \rho(\cos \phi + i \sin \phi),
\]
where \( \rho > 0 \) and \(-\pi < \phi \leq \pi\), and equation (S2) then has exactly three solutions:
\[
\begin{align*}
  z_0 &= \rho^{1/3} \left( \cos \left( \frac{\phi}{3} \right) + i \sin \left( \frac{\phi}{3} \right) \right), \\
  z_1 &= \rho^{1/3} \left( \cos \left( \frac{\phi}{3} + \frac{2\pi}{3} \right) + i \sin \left( \frac{\phi}{3} + \frac{2\pi}{3} \right) \right), \\
  z_2 &= \rho^{1/3} \left( \cos \left( \frac{\phi}{3} + \frac{4\pi}{3} \right) + i \sin \left( \frac{\phi}{3} + \frac{4\pi}{3} \right) \right),
\end{align*}
\]
by Theorem 3.1 of Unit A1. Clearly, \( z_0 \in A \), since \(-\pi/3 < \phi/3 \leq \pi/3\), whereas \( z_1 \) and \( z_2 \) are not in \( A \). Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} \). Hence \( f \) has an inverse function \( f^{-1} \) with domain \( \mathbb{C} \) and rule given by \( f^{-1}(0) = 0 \) and
\[
  f^{-1}(w) = \rho^{1/3}(\cos(\phi/3) + i\sin(\phi/3)),
\]
where \( w = \rho(\cos \phi + i\sin \phi) \), \( \rho > 0 \), \(-\pi < \phi \leq \pi\).

(Observe that for \( w \neq 0 \), \(-\pi < \phi \leq \pi\), we have \( \phi = \text{Arg} \, w \), so \( z_0 \) is the principal cube root of \( w \), namely \( w^{1/3} \). Also, \( 0^{1/3} = 0 \), by definition, so
\[
  f^{-1}(w) = w^{1/3} \quad (w \in \mathbb{C}).
\]

**Solution to Exercise 2.1**

Observe that in each of the following sketches we have rotated the \( x \)- and \( y \)-axes about the \( s \)-axis to help illustrate the shape of the surface.

(a) The surface with equation \( s = \text{Re} \, z \) is the plane that contains the \( y \)-axis and any line given by \( s = x \) and \( y = c \), for some constant \( c \).

(b) The surface with equation \( s = \text{Im} \, z \) is the plane that contains the \( x \)-axis and any line given by \( s = y \) and \( x = c \), for some constant \( c \).

**Solution to Exercise 2.2**

For \( z \in \mathbb{C} \setminus \{0\} \),
\[
  f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.
\]
So
\[
  \text{Re} \, f : z \mapsto \frac{x}{x^2 + y^2} \quad (z \in \mathbb{C} \setminus \{0\})
\]
and
\[
  \text{Im} \, f : z \mapsto \frac{-y}{x^2 + y^2} \quad (z \in \mathbb{C} \setminus \{0\}).
\]

**Solution to Exercise 2.3**

(a) Since \( \gamma(t) = 1 + it \) \((t \in \mathbb{R})\), we have
\[
  x = 1, \quad y = t.
\]
Hence \( \Gamma \) is the line with equation \( x = 1 \), as shown.

(b) Since \( \gamma(t) = t^2 + it \) \((t \in [-1,1])\), we have
\[
  x = t^2, \quad y = t.
\]
A brief table of values is as follows.
### Complex functions

<table>
<thead>
<tr>
<th>$t$</th>
<th>$-1$</th>
<th>$-\frac{1}{2}$</th>
<th>$0$</th>
<th>$\frac{1}{2}$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$y$</td>
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<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Hence the path $\Gamma$ is as shown.

A brief table of values is as follows.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$0$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\pi$</th>
<th>$\frac{5\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{7\pi}{4}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$2$</td>
<td>$\sqrt{2}$</td>
<td>$0$</td>
<td>$-\sqrt{2}$</td>
<td>$-2$</td>
<td>$-\sqrt{2}$</td>
<td>$0$</td>
<td>$\sqrt{2}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$y$</td>
<td>$0$</td>
<td>$\frac{5}{\sqrt{2}}$</td>
<td>$5$</td>
<td>$\frac{5}{\sqrt{2}}$</td>
<td>$0$</td>
<td>$-5$</td>
<td>$-5$</td>
<td>$\frac{-5}{\sqrt{2}}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Hence the path $\Gamma$ is as shown.

### Solution to Exercise 2.4

In each case we use the table of standard parametrisations.

(a) $\gamma(t) = (1 - t)(-2) + ti$

$$= 2(t - 1) + it \quad (t \in \mathbb{R}).$$

Hence

$$x = 2(t - 1), \quad y = t.$$

(b) $\gamma(t) = (1 - t)(1) + t(1 + i)$

$$= 1 + ti \quad (t \in [0, 1]).$$

Hence

$$x = 1, \quad y = t.$$
(c) \( \gamma(t) = (1 + it) + 1(\cos t + i \sin t) \)
\[= 1 + \cos t + (1 + \sin t)i \quad (t \in [0, 2\pi]). \]
Hence
\[x = 1 + \cos t, \quad y = 1 + \sin t. \]

(d) The parabola is in standard form with \( a = \frac{1}{4} \), so the standard parametrisation is
\[\gamma(t) = \frac{1}{4}t^2 + \frac{1}{2}it \quad (t \in \mathbb{R}). \]
Hence
\[x = \frac{1}{4}t^2, \quad y = \frac{1}{2}t. \]
(Of course, you may feel that the parametrisation
\[\gamma(t) = t^2 + it \quad (t \in \mathbb{R}) \]
is simpler!)

**Solution to Exercise 2.5**

(a) Since \( f(z) = \overline{z} = x - iy \), we have
\[\text{Re } f: z \mapsto x \quad (z \in \mathbb{C}), \]
\[\text{Im } f: z \mapsto -y \quad (z \in \mathbb{C}). \]

(b) Since \( f(z) = iz = -y + ix \), we have
\[\text{Re } f: z \mapsto -y \quad (z \in \mathbb{C}), \]
\[\text{Im } f: z \mapsto x \quad (z \in \mathbb{C}). \]

(c) Since
\[f(z) = z^3 = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3), \]
we have
\[\text{Re } f: z \mapsto x^3 - 3xy^2 \quad (z \in \mathbb{C}), \]
\[\text{Im } f: z \mapsto 3x^2y - y^3 \quad (z \in \mathbb{C}). \]

(d) Since \( f(z) = |z| = \sqrt{x^2 + y^2} \), we have
\[\text{Re } f: z \mapsto \sqrt{x^2 + y^2} \quad (z \in \mathbb{C}), \]
\[\text{Im } f: z \mapsto 0 \quad (z \in \mathbb{C}). \]

**Solution to Exercise 2.6**

(a) \( \gamma(t) = 1 - it \quad (t \in \mathbb{R}) \)

(b) \( \gamma(t) = i + (1 - it) \quad (t \in [0, 1]) \)

(c) \( \gamma(t) = \cos t - i \sin t \)
\[= \cos(-t) + i\sin(-t) \quad (t \in [0, 2\pi]) \]

**Solution to Exercise 2.7**

(a) \( \gamma(t) = (1 - t)(1 + i) + ti \)
\[= 1 - t + i; \]
so the parametric equations are
\[x = 1 - t, \quad y = 1 \quad (t \in \mathbb{R}). \]
The path $\Gamma$ is the line $y = 1$.

(b) From the table of standard parametrisations,
$$\gamma(t) = 2 \cos t + 3i \sin t \quad (t \in [0, 2\pi])$$
is a parametrisation of the ellipse
$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

(c) Since $\gamma(t) = 1 + 2 \cos t - (1 - 2 \sin t)i$, the parametric equations are
$$x = 1 + 2 \cos t, \quad y = -1 + 2 \sin t,$$where $t \in [0, 2\pi]$. Hence
$$(x-1)^2 = 4 \cos^2 t \quad \text{and} \quad (y+1)^2 = 4 \sin^2 t,$$so
$$(x-1)^2 + (y+1)^2 = 4.$$This is the equation of the circle with radius 2 and centre $1 - i$.

Solution to Exercise 2.8
(a) $\gamma(t) = 1 - i + 3(\cos t + i \sin t) \quad (t \in [0, 2\pi]).$
(b) The equation $2x^2 + 3y^2 = 6$ is equivalent to
$$\frac{x^2}{3} + \frac{y^2}{2} = 1,$$for which the standard parametrisation is
$$\gamma(t) = \sqrt{3} \cos t + i \sqrt{2} \sin t \quad (t \in [0, 2\pi]).$$
(c) The equation $8y^2 = x$ is equivalent to
$$\frac{y^2}{\frac{1}{8}} = x,$$for which the standard parametrisation is
$$\gamma(t) = \frac{1}{\sqrt{2}} t^2 + \frac{1}{16} it \quad (t \in \mathbb{R}).$$

Solution to Exercise 2.9
We have
$$\gamma(t) = \frac{1}{2} (\cos t + i \sin t) - \frac{1}{4} (\cos 2t + i \sin 2t) = \frac{1}{2} \cos t - \frac{1}{4} \cos 2t + i(\frac{1}{2} \sin t - \frac{1}{4} \sin 2t),$$where $t \in [-\pi, \pi]$. Hence the table of values is as follows (where each non-zero value of $x$ and $y$ is given to two decimal places).

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>$\pm \frac{\pi}{4}$</th>
<th>$\pm \frac{\pi}{2}$</th>
<th>$\pm \frac{3\pi}{4}$</th>
<th>$\pm \pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.25</td>
<td>0.35</td>
<td>0.25</td>
<td>-0.35</td>
<td>-0.75</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>$\pm 0.10$</td>
<td>$\pm 0.50$</td>
<td>$\pm 0.60$</td>
<td>0</td>
</tr>
</tbody>
</table>

Plotting these points, we obtain the following rough sketch of the path (which is a curve called a cardioid).
Since
\[ x = \frac{1}{2} \cos t - \frac{1}{4} \cos 2t, \quad y = \frac{1}{2} \sin t - \frac{1}{4} \sin 2t, \]
we have
\[ x^2 + y^2 = (\frac{1}{4} \cos^2 t - \frac{1}{4} \cos t \cos 2t + \frac{1}{16} \cos^2 2t) + (\frac{1}{4} \sin^2 t - \frac{1}{4} \sin t \sin 2t + \frac{1}{16} \sin^2 2t) \]
\[ = \frac{1}{4} (\cos^2 t + \sin^2 t) - \frac{1}{4} (\cos t \cos 2t + \sin t \sin 2t) \]
\[ + \frac{1}{16} (\cos^2 2t + \sin^2 2t) \]
\[ = \frac{1}{4} - \frac{1}{4} \cos(2t - t) + \frac{1}{16} \]
\[ = \frac{5}{16} - \frac{1}{4} \cos t \]
\[ = \frac{1}{16} (5 - 4 \cos t). \]
Hence the image of the circle $|z| = 1$ is the semicircle shown below, with one endpoint missing.

**Solution to Exercise 2.10**

The principal square root function $f(z) = \sqrt{z}$ maps $z = r(\cos \theta + i \sin \theta)$, where $\theta = \text{Arg } z$, to $w = r^{1/2}(\cos \theta/2 + i \sin \theta/2)$.

(a) If $z$ is on the negative $x$-axis, then $\theta = \pi$, so its image $w$ is such that $\text{Arg } w = \pi/2$. Thus the negative real axis maps to the positive $v$-axis in the $w$-plane.

(b) The image of the point $z$ with modulus 1 and principal argument $\theta \in (-\pi, \pi]$ is the point $w$ with modulus 1 and principal argument $\theta/2 \in (-\pi/2, \pi/2]$.

**Solution to Exercise 3.1**

\[ 4(x^2 + y^2) - \frac{3}{2}(x^2 + y^2) + \frac{1}{4}x \]
\[ = \frac{1}{64}(5 - 4 \cos t)^2 - \frac{3}{32}(5 - 4 \cos t) \]
\[ + \frac{1}{8}(2 \cos t - \cos 2t) \]
\[ = (\frac{25}{64} - \frac{5}{8} \cos t + \frac{1}{2} \cos^2 t) - (\frac{15}{32} - \frac{3}{8} \cos t) \]
\[ + (\frac{1}{4} \cos t - \frac{1}{8}(2 \cos^2 t - 1)) \]
\[ = \frac{3}{64}, \]
as required.

**Solution to Exercise 3.2**

(a) $u = a - t, \quad v = a + t$;
adding these equations, we obtain
\[ u + v = 2a. \]
(b) $u = a^2 - t^2, \quad v = 2at$;

\[ t^2 = \left(\frac{v}{2a}\right)^2, \]
so
\[ u = a^2 - \frac{v^2}{4a^2}, \]
which gives
\[ v^2 = 4a^2(a^2 - u). \]
(c) \[ u = \frac{a}{a^2 + t^2}, \quad v = \frac{-t}{a^2 + t^2}; \]
squaring each of these expressions and adding the results, we obtain
\[ u^2 = \frac{a^2}{(a^2 + t^2)^2}, \quad v^2 = \frac{t^2}{(a^2 + t^2)^2}, \]
so
\[ u^2 + v^2 = \frac{a^2 + t^2}{(a^2 + t^2)^2} = \frac{1}{a^2 + t^2} = \frac{u}{a}. \]

**Solution to Exercise 3.3**

The line \( y = b \) has parametric equations
\[ x = t, \quad y = b \quad (t \in \mathbb{R}). \]
Substituting these in
\[ u = x - y, \quad v = x + y \]
(from equations (3.1)) gives the parametric equations of the image of the line \( y = b \) under the function \( f(z) = (1 + i)z \). Thus
\[ u = t - b, \quad v = t + b \quad (t \in \mathbb{R}). \]
Eliminating \( t \), we obtain
\[ v - u = 2b, \]
which is the equation of a line.

The images of the lines \( y = 1 \) and \( y = 0 \) are, respectively,
\[ v - u = 2 \quad \text{and} \quad v - u = 0. \]
They are shown below, as are the directions of increasing \( t \). (The directions were not asked for in the exercise, but it is illuminating to include them anyway.)

**Solution to Exercise 3.4**

The line \( y = b \) has parametric equations
\[ x = t, \quad y = b \quad (t \in \mathbb{R}). \]
Substituting these in
\[ u = x^2 - y^2, \quad v = 2xy \]
(from equations (3.2)) gives the parametric equations of the image of the line \( y = b \) under the function \( f(z) = z^2 \). Thus
\[ u = t^2 - b^2, \quad v = 2tb \quad (t \in \mathbb{R}). \]
Eliminating \( t \), we obtain
\[ v^2 = 4b^2(u + b^2), \quad b \neq 0, \]
which is the equation of a parabola.

When \( b = 0 \) we obtain the parametric equations
\[ u = t^2, \quad v = 0 \quad (t \in \mathbb{R}), \]
which are equations for the non-negative \( u \)-axis.

Therefore the images of the lines \( y = 1 \) and \( y = 0 \) are, respectively,
- the parabola \( v^2 = 4(u + 1) \),
- the non-negative \( u \)-axis, i.e. \( v = 0, \ u \geq 0 \).

They are shown below, as are the directions of increasing \( t \).
Solution to Exercise 3.5

The line \( y = b \) has parametric equations

\[
x = t, \quad y = b \quad (t \in \mathbb{R}).
\]

Substituting these in

\[
u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}
\]

(from equations (3.3)) gives the parametric equations of the image of the line \( y = b \) under the function \( f(z) = 1/z \). Thus

\[
u = \frac{t}{t^2 + b^2}, \quad v = \frac{-b}{t^2 + b^2} \quad (t \in \mathbb{R}).
\]

In the case \( b \neq 0 \), we obtain, on eliminating \( t \),

\[
u^2 + v^2 = -\frac{v}{b}
\]

which is the equation of a circle through the origin, though the origin itself is excluded from the image.

When \( b = 0 \) we obtain the parametric equations

\[
u = \frac{1}{t}, \quad v = 0 \quad (t \in \mathbb{R} - \{0\}),
\]

which are parametric equations for the real axis, excluding the origin.

Therefore the images of the lines \( y = 1 \) and \( y = 0 \) are, respectively, the circle \( u^2 + v^2 + v = 0 \), excluding the origin, and the line \( v = 0, u \neq 0 \). They are shown below, as are the directions of increasing \( t \).

The image of \( y = 1 \)

The image of \( y = 0 \)

Solution to Exercise 3.6

(a) The function \( f(z) = iz + 1 \) has the following geometric effect: it rotates the point \( z \) anticlockwise about the origin through \( \pi/2 \) and then translates the result to the right by one unit. Thus

- the image of the line \( y = 0 \) is the line \( u = 1 \)
- the image of the line \( x = 1 \) is the line \( v = 1 \)
- the image of the line \( y = 1 \) is the line \( u = 0 \)
- the image of the line \( x = 0 \) is the line \( v = 0 \);

and the image of \( S \) is \( S \) (in the \( w \)-plane), as shown in the figure.

(b) Using the geometrical interpretation above, we obtain the image of the polar grid as follows.

Solution to Exercise 3.7

If \( z \) has modulus \( r \) and argument \( \theta \), then

\( w = f(z) = z^3 \) has modulus \( r^3 \) and argument \( 3\theta \).

Thus the image of the polar grid (with the circle \( r = 3 \) omitted) is as shown in the following figure.
Solution to Exercise 3.8

If \( z \) has modulus \( r \) and principal argument \( \theta \), then \( w = f(z) = \sqrt{z} \) has modulus \( r^{1/2} \) and principal argument \( \theta/2 \). Thus

- the image of the ray \( \theta = b \), where \( b \) is a constant in the interval \( (-\pi, \pi] \), is the ray \( \theta = b/2 \)
- the image of the circle with radius \( r \) and centre the origin is the semicircle (with one endpoint missing) given by
  \[
  |w| = \sqrt{r}, \quad \theta \in (\pi/2, \pi/2].
  \]

Hence the image of the polar grid is as shown below.

Solution to Exercise 3.9

(a) The function \( f(z) = z + i \) translates the point \( z \) one unit in the \( y \)-direction. The images of the Cartesian grid and the polar grid are shown below.

(b) The function \( f(z) = 2z \) doubles the modulus of the point \( z \), but leaves its argument unchanged. The images of the Cartesian grid and the polar grid are as follows.

![Image of Cartesian grid](image1)

![Image of polar grid](image2)
Solutions to exercises

(c) The function \( f(z) = 2 - iz \) rotates the point \( z \) about the origin through \( \pi/2 \) clockwise and then translates it 2 units to the right. Thus the images of the Cartesian grid and the polar grid are as follows.

Image of Cartesian grid

Image of polar grid

Alternatively, for the Cartesian grid we can use the parametric approach, as follows.

The image of \( z \) is
\[
w = f(z) = 2 - iz = 2 + y - ix,
\]
so
\[
u = 2 + y, \quad v = -x.
\]

The line \( x = a \) has parametric equations
\[
x = a, \quad y = t \quad (t \in \mathbb{R}).
\]
Substituting these in equations (S3) gives the parametric equations of the image,

\[ u = 2 + t, \quad v = -a \quad (t \in \mathbb{R}), \]

which is the line \( v = -a \).

Similarly, the line \( y = b \) has parametric equations

\[ x = t, \quad y = b \quad (t \in \mathbb{R}). \]

Substituting these in equations (S3) gives the parametric equations of the image,

\[ u = 2 + b, \quad v = -t \quad (t \in \mathbb{R}), \]

which is the line \( u = 2 + b \).

(d) Since \( f(z) = iz^2 = i \times z^2 \) and multiplication by \( i \) corresponds to rotation about the origin through \( \pi/2 \) anticlockwise, the images are found by rotating those in Figures 3.7 and 3.8 through \( \pi/2 \) anticlockwise.

### Solution to Exercise 4.1

(a) \( e^{2\pi i} = e^0 (\cos 2\pi + i \sin 2\pi) = 1 \)

(b) \( e^{2+i\pi/3} = e^2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = \frac{e^2}{2} (1 + \sqrt{3}i) \)

(c) \( e^{-(1+i\pi)} = e^{-1} (\cos(-\pi) + i \sin(-\pi)) = -1/e \)

### Solution to Exercise 4.2

(a) \( e^{z+2\pi i} = e^z e^{2\pi i} \) (Theorem 4.1(a))

because \( e^{2\pi i} = 1 \) (see Exercise 4.1(a)).

(b) \( |e^z| = e^{\text{Re} z} \) (Theorem 4.1(b))

\[ \leq e^{|z|}, \]

because \( \text{Re} z \leq |z| \) and \( x \mapsto e^x \) is an increasing function.

(c) The (complex) function \( \exp \) is not one-to-one because \( e^0 = e^{2\pi i} = 1 \).

(d) (i) By equating moduli on each side of the equation \( e^{x+iy} = 1 \), we see that \( e^x = 1 \), and hence \( e^{iy} = 1 \) also. Therefore

\[ e^{x+iy} = 1 \iff e^x = 1 \text{ and } \cos y + i \sin y = 1 \iff x = 0 \text{ and } y = 2n\pi, \]

where \( n \in \mathbb{Z} \). It follows that

\[ \{ z : e^z = 1 \} = \{ 2n\pi i : n \in \mathbb{Z} \}. \]

(ii) By equating moduli on each side of the equation \( e^{x+iy} = -1 \), we see that \( e^x = 1 \), and hence \( e^{iy} = -1 \). Therefore

\[ e^{x+iy} = -1 \iff e^x = 1 \text{ and } \cos y + i \sin y = -1 \iff x = 0 \text{ and } y = (2n+1)\pi, \]

where \( n \in \mathbb{Z} \). It follows that

\[ \{ z : e^z = -1 \} = \{ (2n+1)\pi i : n \in \mathbb{Z} \}. \]

### Solution to Exercise 4.3

Since \( \sqrt{3} + i = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = 2e^{i\pi/6} \), we have

\[ (\sqrt{3} + i)^{-6} = (2e^{i\pi/6})^{-6} = 2^{-6} (e^{i\pi/6})^{-6} = 2^{-6} e^{-6 \times i\pi/6} = 2^{-6} e^{-i\pi} = -1/64. \]
Solution to Exercise 4.4

(a) \[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \]

(b) \[ |w| = 1/e \]

(c) \[ \cos(-\pi/4) + i \sin(-\pi/4) \]

Solution to Exercise 4.5

(a) \[ \sin(\pi/2 + i) = \frac{1}{2i} (e^{i(\pi/2+i)} - e^{-i(\pi/2+i)}) \]
\[ = \frac{1}{2i} (e^{-1+ix/2} - e^{1-ix/2}) \]
\[ = \frac{1}{2i} (e^{-1}i - e(-i)) \]
\[ = \frac{1}{2} (e + e^{-1}) \]

(b) \[ \cos i = \frac{1}{2} (e^{ix} + e^{-ix}) \]
\[ = \frac{1}{2} (e + e^{-1}) \]

(which equals \(\sin(\pi/2 + i)\), in fact).

Solution to Exercise 4.6

(a) (i) \[ \sin(-z) = \frac{1}{2i} (e^{i(-z)} - e^{-i(-z)}) \]
\[ = \frac{1}{2i} (e^{-iz} - e^{iz}) \]
\[ = \frac{1}{2i} (e^{iz} - e^{-iz}) \]
\[ = - \sin z \]

(ii) \[ \cos(z + 2\pi) = \frac{1}{2} (e^{i(z+2\pi)} + e^{-i(z+2\pi)}) \]
\[ = \frac{1}{2} (e^{iz} e^{2\pi i} + e^{-iz} e^{-2\pi i}) \]
\[ = \frac{1}{2} (e^{iz} + e^{-iz}) \]
\[ = \cos z \]

(b) (i) \[ \cos 2z = \cos(z + z) \]
\[ = \cos z \cos z - \sin z \sin z \]
\[ = 2 \cos^2 z - 1 \] (Theorem 4.3(a))

(ii) \[ \tan(z_1 - z_2) \]
\[ = \tan(z_1 + (-z_2)) \]
\[ = \frac{\tan z_1 + \tan(-z_2)}{1 - \tan z_1 \tan(-z_2)} \] (Theorem 4.3(a))
\[ = \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2} \] (Theorem 4.3(c))
Solution to Exercise 4.7

(a) By Theorem 4.3(a),
\[
\cos z = \cos(x + iy) \\
= \cos x \cos(iy) - \sin x \sin(iy) \\
= \cos x \cosh y - \sin x(i \sinh y) \\
= \cos x \cosh y - i \sin x \sinh y.
\]

(b) Using part (a),
\[
|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
= \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\
= \cos^2 x + \sinh^2 y(\cos^2 x + \sin^2 x) \\
= \cos^2 x + \sinh^2 y.
\]

Solution to Exercise 4.8

(a) \(e^{3\pi i} = e^0(\cos 3\pi + i \sin 3\pi) = -1\)

(b) \(e^{\pi i/2} = e(\cos \pi/2 + i \sin \pi/2) = ei\)

(c) \(e^{2\pi i/3} = e^0(\cos 2\pi/3 + i \sin 2\pi/3) = -1/2 + \sqrt{3}/2i\)

(d) \(e^{-3\pi i/2} = e^0(\cos(-3\pi/2) + i \sin(-3\pi/2)) = i\)

(e) \(e^{2+\pi i} = e^2(\cos \pi + i \sin \pi) = -e^2\)

(f) \(e^{3+\pi i/2} = e^3(\cos \pi/2 + i \sin \pi/2) = e^3i\)

(g) \(e^{(\pi i/6)-1} = e^{-1}(\cos \pi/6 + i \sin \pi/6) = \sqrt{3}/2e + 1/2i\)

(h) \(e^{(\cos \theta + i \sin \theta)} = e^{\cos \theta}(\cos(\sin \theta) + i \sin(\sin \theta))\)

Solution to Exercise 4.9

(a) (i) We have
\[
\left| \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right| = 1 \text{ and } \text{Arg} \left( \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = -\pi/4,
\]
so
\[
\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = e^{-\pi i/4} = e^{-\pi i/4}.
\]

(ii) We have
\[
|-1 - i| = \sqrt{2} \text{ and } \text{Arg}(-1 - i) = -3\pi/4,
\]
so
\[
-1 - i = \sqrt{2}e^{-3\pi i/4}.
\]

(iii) We have
\[
|1 + \sqrt{3}i| = 2 \text{ and } \text{Arg}(1 + \sqrt{3}i) = \pi/3,
\]
so
\[
1 + \sqrt{3}i = 2e^{\pi i/3}.
\]

Solution to Exercise 4.10

(a) \(\sin(\pi + 2i) = \frac{1}{2i}(e^{i(\pi+2i)} - e^{-i(\pi+2i)}) = \frac{1}{2i}(e^{-2+i\pi} - e^{2-i\pi}) = \frac{1}{2i}(e^{-2}e^{2i\pi} - e^{2}e^{-2i\pi}) = \frac{1}{2i}(e^{-2}e^{2} - e^{2}e^{-2}) = -i\left(\frac{e^{2} - e^{2}}{2}\right) = -i \sinh 2.\)

Alternatively, using Theorems 4.3(a) and 4.4, we have
\[
\sin(\pi + 2i) = \sin \pi \cos 2i + \cos \pi \sin 2i = 0 + (-1) \times (i \sinh 2) = -i \sinh 2.
\]
(b) \[ \cos(\pi/2 - i) = \frac{1}{2}(e^{i(\pi/2 - i)} + e^{-i(\pi/2 - i)}) \]
\[ = \frac{1}{2}(e^{1+i\pi/2} + e^{-1-i\pi/2}) \]
\[ = \frac{1}{2}(ee^{i\pi/2} + e^{-1}e^{-i\pi/2}) \]
\[ = \frac{1}{2}(ei - e^{-1}i) \]
\[ = \left(e - e^{-1}\right)i \]
\[ = i \sinh 1. \]
Alternatively, using Theorems 4.3(a) and 4.4, we have
\[ \cos(\pi/2 - i) = \cos \pi/2 \cos i + \sin \pi/2 \sin i \]
\[ = 0 + 1 \times i \sinh 1 \]
\[ = i \sinh 1. \]

(c) \[ \tan i = \frac{\sin i}{\cos i} \]
\[ = \frac{i \sinh 1}{\cosh 1} \]
\[ = i \tanh 1 \]

Solution to Exercise 4.11

(a) Writing \( z = x + iy \), and observing that \( e^x \) is real, we obtain
\[ e^z = e^{x+iy} \]
\[ = e^x e^{iy} \]
\[ = e^x (\cos y - i \sin y). \]
Also, \( e^\bar{z} = e^{x-iy} \]
\[ = e^x e^{-iy} \]
\[ = e^x (\cos(-y) + i \sin(-y)) \]
\[ = e^x (\cos y - i \sin y). \]
Hence \( e^z = e^\bar{z} \).

(b) We have
\[ \sin 2z = \frac{1}{2i}(e^{2iz} - e^{-2iz}) \]
and
\[ 2 \sin z \cos z = 2 \times \left( \frac{e^{iz} - e^{-iz}}{2i} \right) \times \left( \frac{e^{iz} + e^{-iz}}{2} \right) \]
\[ = \frac{1}{2i}(e^{2iz} - e^{-2iz}). \]
Hence \( \sin 2z = 2 \sin z \cos z \).

(c) Using part (a), we see that
\[ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \]
\[ = \frac{1}{2i}(e^{iz} - e^{-iz}) \]
\[ = \frac{1}{2(-i)}(e^{iz} - e^{-iz}) \]
\[ = -\frac{1}{2i}(e^{-iz} - e^{iz}) \]
\[ = \frac{1}{2i}(e^{iz} - e^{-iz}) \]
\[ = \sin z. \]

(d) We have
\[ \cosh(z_1 + z_2) \]
\[ = \cos(i(z_1 + z_2)) \] (Theorem 4.4)
\[ = \cos(iz_1 + iz_2) \]
\[ = \cos(iz_1) \cos(iz_2) - \sin(iz_1) \sin(iz_2) \] (Theorem 4.3(a))
\[ = \cosh z_1 \cosh z_2 - (i \sinh z_1)(i \sinh z_2) \] (Theorem 4.4)
\[ = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2. \]

(e) Using Theorem 4.4, we see that
\[ \cosh^2 z - \sinh^2 z = \cos^2(iz) - i^2 \sin^2(iz) \]
\[ = \cos^2(iz) + \sin^2(iz) = 1, \]
by Theorem 4.3(b).

Solution to Exercise 5.1

First we determine the image set of \( f \):
\[ f(A) = \{ e^z : z \in A \} \]
\[ = \{ w = e^{x+iy} : x \in \mathbb{R}, \ 0 \leq y < 2\pi \} \]
\[ = \{ w = e^x e^{iy} : x \in \mathbb{R}, \ 0 \leq y < 2\pi \} \]
\[ = \{ w = pe^{i\phi} : \rho > 0, \ 0 \leq \phi < 2\pi \} \]
\[ = \mathbb{C} - \{0\}, \]

where \( \rho = e^x \) and \( \phi = y \).

Now, for each \( w \in \mathbb{C} - \{0\} \), we wish to solve the equation
\[ w = e^z \]
to obtain a unique solution \( z \) in \( A \). Each \( w \) in \( \mathbb{C} - \{0\} \) can be written in the form
\[ w = \rho e^{i\phi}, \] where \( \rho > 0 \) and \( 0 \leq \phi < 2\pi \),
and the equation \( w = e^z \) is then
\[
\rho e^{i\phi} = e^z = e^{x + iy}, \quad \text{where } z = x + iy.
\]
Thus, by equating moduli in the equation above, we see that \( x \) and \( y \) must satisfy
\[
\rho = e^x \quad \text{and} \quad e^{i\phi} = e^{iy};
\]
that is,
\[
x = \log \rho \quad \text{and} \quad y = \phi + 2n\pi,
\]
where \( n \in \mathbb{Z} \). For \( n = 0 \), the solution is
\[
z = x + iy = \log \rho + i\phi,
\]
which lies in \( A \), since \( 0 \leq \phi < 2\pi \), whereas the other solutions (with \( n \neq 0 \)) lie outside \( A \).

Thus \( f \) is a one-to-one function, with image set \( \mathbb{C} - \{0\} \). Hence \( f \) has inverse function \( f^{-1} \) with domain \( \mathbb{C} - \{0\} \) and rule
\[
f^{-1}(w) = \log \rho + i\phi,
\]
where \( w = \rho e^{i\phi}, \rho > 0, 0 \leq \phi < 2\pi \).

**Solution to Exercise 5.2**

(a) \( \log i = \log|i| + i\arg i \)
\[
= \log 1 + i\frac{\pi}{2} = i\frac{\pi}{2}
\]
(b) \( \log(\sqrt{3} - i) = \log|\sqrt{3} - i| + i\arg(\sqrt{3} - i) \)
\[
= \log 2 - i\frac{\pi}{6}
\]
(c) \( \log\left(\frac{1}{2} + \frac{1}{2}i\right) = \log\left|\frac{1}{2} + \frac{1}{2}i\right| + i\arg\left(\frac{1}{2} + \frac{1}{2}i\right) \)
\[
= \log\frac{\sqrt{2}}{2} + i\frac{\pi}{4}
\]
\[
= -\log\sqrt{2} + i\frac{\pi}{4}
\]

**Solution to Exercise 5.3**

(a) False. (The ellipse \( 4x^2 + 9y^2 = 1 \) lies entirely inside the unit circle \(|z| = 1\), so its image lies in the left half-plane.)

(b) False. (The ray \( \theta = \pi/4 \) lies inside, on and outside the unit circle \(|z| = 1\), so its image is not confined to the right half-plane.)

(c) False. (Since \( 1 + 4i \) does not lie in the strip \( \{w : -\pi < \text{Im} w \leq \pi\} \), which is the image set of the function \( \log \), there is no \( z \in \mathbb{C} \) such that \( \log z = 1 + 4i \).)

(d) True. (Since
\[
1 + \frac{1}{2}i \in \{w : -\pi < \text{Im} w \leq \pi\},
\]
there is a \( z \in \mathbb{C} \) such that \( \log z = 1 + \frac{1}{2}i \).)

**Solution to Exercise 5.4**

(a) \( (1 + i)^{2/3} = \exp\left(\frac{2}{3}\log(1 + i)\right) \)
\[
= \exp\left(\frac{2}{3}(\log \sqrt{2} + i\pi/4)\right) \]
\[
= \exp\left(\frac{2}{3}\log 2 + i\pi/6\right) \]
\[
= 2^{1/3}e^{i\pi/6} \]
\[
= 2^{1/3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \]
\[
= 2^{-2/3}(\sqrt{3} + i) \]

(b) \( i^{1+i} = \exp((1 + i) \log i) \)
\[
= \exp\left(1 + i\frac{\pi}{2}\right) \quad \text{(Exercise 5.2(a))} \]
\[
= e^{-\pi/2}e^{i\pi/2} \]
\[
= e^{-\pi/2}i \]

**Solution to Exercise 5.5**

Since \( z^\alpha = \exp(\alpha \log z) \), we have
\[
z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \]
\[
= \exp\left(\frac{1}{n}(\log |z| + i\arg z)\right) \]
\[
= \exp\left(\frac{1}{n}(\log \rho + i\phi)\right) \]
\[
= \exp\left(\log \rho^{1/n} + i\frac{\phi}{n}\right) \]
\[
= \rho^{1/n}e^{i\phi/n} \]
\[
= \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right),
\]
which is the principal \( n \)th root of \( z = \rho(\cos \phi + i\sin \phi) \) because \( \phi \) is the principal argument of \( z \) (see Subsection 3.1 of Unit A1).
Solution to Exercise 5.6

(a) Consider \( z_1 = -1 = z_2 \) and \( \alpha = \frac{1}{2} \). Then
\[
z_1^\alpha z_2^\alpha = (-1)^{1/2}(-1)^{1/2} = i \times i \quad \text{(Example 5.3(a))}
\]

However,
\[
( z_1 z_2 )^\alpha = ((-1) \times (-1))^{1/2} = 1,
\]

so \( z_1^\alpha z_2^\alpha \neq (z_1 z_2)^\alpha \).

(b) By definition,
\[
z^\alpha = \exp(\alpha (\log |z| + i \text{Arg } z)),
\]
\[
z^\beta = \exp(\beta (\log |z| + i \text{Arg } z)),
\]

so, using Theorem 4.1(a),
\[
z^\alpha z^\beta = e^{\alpha (\log |z|+i \text{Arg } z)} e^{\beta (\log |z|+i \text{Arg } z)}
\]
\[
= e^{\alpha (\log |z|+i \text{Arg } z)+\beta (\log |z|+i \text{Arg } z)}
\]
\[
= e^{(\alpha+\beta)(\log |z|+i \text{Arg } z)}
\]
\[
= z^{\alpha+\beta},
\]

by definition.

Solution to Exercise 5.7

(a) \( \log(-2) = \log|-2| + i \text{Arg}(-2) \)
\[
= \log 2 + i \pi
\]

(b) \( \log(i^3) = \log(-i) \)
\[
= \log|-i| + i \text{Arg}(-i)
\]
\[
= -i \pi/2
\]

(c) \( \log(1+i) = \log|1+i| + i \text{Arg}(1+i) \)
\[
= \log \sqrt{2} + i \pi/4
\]

(d) \( \log \sqrt{3} = \log \sqrt{3} + i \text{Arg } \sqrt{3} \)
\[
= \log \sqrt{3}
\]

(e) \( \log(i - \sqrt{3}) = \log|i - \sqrt{3}| + i \text{Arg}(i - \sqrt{3}) \)
\[
= \log 2 + i5 \pi/6
\]

(f) \( \log \left( \frac{1-i}{\sqrt{2}} \right) = \log \left| \frac{1-i}{\sqrt{2}} \right| + i \text{Arg} \left( \frac{1-i}{\sqrt{2}} \right) \)
\[
= \log 1 + i(-\pi/4)
\]
\[
= -i \pi/4
\]

Solution to Exercise 5.8

(a) \( i^{-i} = \exp(-i \log i) \)
\[
= \exp(-i(0 + i\pi/2))
\]
\[
= e^{\pi/2}
\]

(b) \( (-i)^i = \exp(i \log(-i)) \)
\[
= \exp(i(0 - i\pi/2))
\]
\[
= e^{\pi/2}
\]

(c) \( (1-i)^i = \exp(i \log(1-i)) \)
\[
= \exp(i(\log |1-i| + i \text{Arg}(1-i)))
\]
\[
= \exp(i(\log \sqrt{2} + i(-\pi/4)))
\]
\[
= \exp(i \log \sqrt{2} + \pi/4)
\]
\[
= e^{\pi/4}(\cos(\log \sqrt{2}) + i \sin(\log \sqrt{2}))
\]

(d) \( (-1)^i = \exp(i \log(-1)) \)
\[
= \exp(i(0 + i\pi))
\]
\[
= e^{-\pi}
\]