Unit A1
Complex numbers
Introduction

Historical survey

Before describing the contents of this unit, we provide a brief historical account of the development of complex numbers.

Many mathematical problems that have been studied since ancient times lead to quadratic equations of the form

\[ ax^2 + bx + c = 0, \]  

where \( a, b, c \) are real numbers, with \( a \neq 0 \), and \( x \) is an unknown number.

The formula

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

for the solutions of equation (0.1) gives

- two distinct real solutions if \( b^2 - 4ac > 0 \)
- one (repeated) real solution if \( b^2 - 4ac = 0 \)
- no real solutions if \( b^2 - 4ac < 0 \).

When \( b^2 - 4ac < 0 \), equation (0.1) has no real solutions because a negative number cannot have a real square root.

However, as long ago as the sixteenth century, Italian scholars such as Gerolamo Cardano (1501–1576) began to experiment with the manipulation of symbols such as \( \sqrt{-1} \), using the ordinary rules for real numbers. For example, he considered the problem of finding numbers \( x \) and \( y \) such that

\[ x + y = 10 \quad \text{and} \quad xy = 40. \]  

(0.2)

This problem has no real solutions, since if we substitute \( y = 40/x \) into \( x + y = 10 \), then we obtain the quadratic equation

\[ x^2 - 10x + 40 = 0, \]

which has no real solutions (because \( b^2 - 4ac = -60 < 0 \)).

Cardano pointed out, however, that if \( \sqrt{-15} \) is manipulated using the ordinary rules for real numbers, then

\[ x = 5 + \sqrt{-15} \quad \text{and} \quad y = 5 - \sqrt{-15} \]

do satisfy equations (0.2). As he wrote:

Putting aside the mental tortures involved, multiply 5 + \( \sqrt{-15} \) by 
5 − \( \sqrt{-15} \), making 25 − (−15), whence the product is 40.

(Burton, 2002, p. 297)

At the time there was little enthusiasm for such a solution because of doubts about the existence of entities such as \( \sqrt{-15} \), doubts which Cardano himself harboured, but soon the idea of taking square roots of negative numbers was to prove its worth in a significant way, which we now describe.
The general cubic equation
\[ ax^3 + bx^2 + cx + d = 0, \]  
(0.3)
where \( a, b, c, d \) are real numbers, with \( a \neq 0 \), is much more difficult to solve algebraically than is the quadratic equation. One of the first to find real-number solutions to general cubic equations was the Persian mathematician (and poet) Omar Khayyām (1048–1131), who made ingenious use of conic sections. However, only after the work of Cardano and his contemporaries in the sixteenth century did mathematicians begin to appreciate that there are other, non-real solutions of cubic equations. Their breakthrough is described by the following remarkable method, which is usually attributed to Scipioned del Ferro (1465–1526) and Niccolò Fontana Tartaglia (c.1500–1557).

First, equation (0.3) is reduced to the form
\[ x^3 + px + q = 0 \]  
(0.4)
by substituting \( x = \frac{1}{3}(b/a) \) in place of \( x \) and dividing through by \( a \). We will now solve this equation; there is no need to follow the details if you do not wish to. Substitute
\[ x = u + v, \quad \text{where} \quad uv = -p/3, \]
assuming that such real numbers \( u \) and \( v \) exist. This transforms equation (0.4) into
\[ (u + v)^3 + p(u + v) + q = 0, \]
which, on expanding the cubic term, gives
\[ u^3 + 3uv(u + v) + v^3 + p(u + v) + q = 0. \]
Since \( uv = -p/3 \), this reduces to
\[ u^3 + v^3 + q = 0, \]  
(0.5)
and multiplying through by \( u^3 \), we obtain
\[ u^6 + qa^3 - (p/3)^3 = 0. \]
This is a quadratic equation in \( u^3 \) with solutions
\[ u^3 = -q \pm \sqrt{q^2 + 4(p/3)^3}. \]
Then, by equation (0.5),
\[ v^3 = -q \mp \sqrt{q^2 + 4(p/3)^3}, \]
so one of the solutions of equation (0.4) is
\[ x = \sqrt[3]{-q + \sqrt{q^2 + 4(p/3)^3}} + \sqrt[3]{-q - \sqrt{q^2 + 4(p/3)^3}}. \]  
(0.6)
Formula (0.6) works extremely well in some cases. For example, the equation
\[ x^3 + 3x - 4 = 0 \]
has \( p = 3 \) and \( q = -4 \), so it follows from equation (0.6) that
\[
x = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}.
\]
Now, one can check that
\[
\left(\frac{1}{2}(1 + \sqrt{5})\right)^3 = 2 + \sqrt{5} \quad \text{and} \quad \left(\frac{1}{2}(1 - \sqrt{5})\right)^3 = 2 - \sqrt{5},
\]
so it follows that
\[
x = \frac{1}{2}(1 + \sqrt{5}) + \frac{1}{2}(1 - \sqrt{5}) = 1,
\]
which is indeed a solution of \( x^3 + 3x - 4 = 0 \).

For some values of \( p \) and \( q \), though, there is a difficulty. The Italian mathematician Rafael Bombelli (1526–1572), who was probably the first mathematician bold enough to accept the existence of square roots of negative numbers, considered the equation
\[
x^3 - 15x - 4 = 0. \tag{0.7}
\]
This has \( p = -15 \) and \( q = -4 \), so it follows from equation (0.6) that
\[
x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.
\]
This solution involves the expression \( \sqrt{-121} \), which suggests that, for this cubic equation, formula (0.6) does not give a solution. However, following Bombelli and treating \( \sqrt{-1} \) in the same way as a real number, we see that
\[
(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -\sqrt{-1},
\]
and hence
\[
(2 + \sqrt{-1})^3 = 2^3 + (3 \times 2^2 \times \sqrt{-1}) + (3 \times 2 \times (\sqrt{-1})^2) + (\sqrt{-1})^3
\]
\[
= 8 + 12\sqrt{-1} - 6 - \sqrt{-1}
\]
\[
= 2 + 11\sqrt{-1}
\]
\[
= 2 + \sqrt{-121}.
\]
Similarly,
\[
(2 - \sqrt{-1})^3 = 2 - \sqrt{-121}.
\]
Hence
\[
x = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4,
\]
which is indeed a solution of equation (0.7). Thus, by allowing the use of symbols that seemingly have no meaning, we can produce one correct solution to the original problem.

This method of solution did not immediately lead to acceptance of such ‘imaginary numbers’ (as they were called), which continued to be regarded with great suspicion. A century later, for example, the English mathematician and scientist Isaac Newton (1642–1727) stated that if the solution to a problem involved imaginary numbers, then the problem did not have a ‘genuine’ solution.
By the eighteenth century, however, mathematicians such as Johann Bernoulli (1667–1748), Gottfried Wilhelm Leibniz (1646–1716) and Leonhard Euler (1707–1783) were using imaginary numbers in applications to integration.

The symbol $i$ for $\sqrt{-1}$ was introduced by Euler, and the name complex number for an expression of the form $z = x + iy$, where $x$ and $y$ are real, was introduced by the German mathematician Carl Friedrich Gauss (1777–1855) at the end of the eighteenth century to replace the old phrase ‘imaginary number’. Gauss also advocated the geometric interpretation of a complex number $x + iy$ as a point with rectangular coordinates $(x, y)$ in a plane (see Figure 0.1).

The importance of complex numbers was underlined by the so-called Fundamental Theorem of Algebra, which was established in the late eighteenth and early nineteenth centuries. This theorem states that every polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

(0.8)

where $a_0, a_1, \ldots, a_n$ are complex numbers, with $a_n \neq 0$, and $z$ is an unknown complex number, has at least one complex solution (and hence $n$ complex solutions, some of which may be repeated). Thus, although it was necessary to introduce the new ‘complex’ numbers in order to solve quadratic equations, there was no need to introduce any further new numbers in order to solve all polynomial equations with complex coefficients. We should hasten to add that the mathematicians of the eighteenth and nineteenth centuries did not find a general formula for solving equation (0.8) for all values of $n$; indeed, in 1824 the Norwegian mathematician Niels Henrik Abel (1802–1829) proved that for $n \geq 5$ no formula that uses the usual operations of arithmetic exists. Nonetheless, the Fundamental Theorem of Algebra asserts the existence of a solution.

Any lingering doubts about the validity of complex numbers were laid to rest in the 1830s when the Irish mathematician William Rowan Hamilton (1805–1865) gave a definition of complex numbers as ordered pairs of real numbers, writing $(a, b)$ in place of $a + ib$, subject to the rules of manipulation

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b) \times (c, d) = (ac – bd, ad + bc).$$

This had the effect of placing complex numbers on a sound algebraic basis.

The groundwork had now been laid for the study of complex numbers to flourish. This development, associated with the names of Cauchy, Riemann and Weierstrass, went on throughout the nineteenth century.

Although most of this module deals with the classical theory of complex numbers and complex functions, at the end of the module you will see that even today there are still new and exciting developments in the subject.
Applications, supplementary topics and history

Scattered throughout the module you will find boxes like this one that contain additional information about complex analysis, to supplement the text. *The content of these boxes is not assessed.* Some of the boxes are about applications of complex analysis to science and engineering, and others describe further branches of mathematics related to complex analysis. You will also find out about the history of some of the mathematicians who have shaped the subject, some of whom you have encountered in this Introduction already.

Introduction to this unit

The first book of the module is an introduction to complex functions, and it is designed to familiarise you with their most basic properties. This unit is devoted solely to complex numbers themselves.

In Section 1 we define complex numbers and show you how to manipulate them, stressing the similarities with the manipulation of real numbers.

Section 2 is about the geometric representation of complex numbers. You will find that this is useful in understanding the arithmetic properties introduced in the first section.

In Section 3 we discuss methods of finding $n$th roots of complex numbers and the solutions of simple polynomial equations.

The final two sections deal with inequalities between real-valued expressions involving complex numbers. First we use inequalities in Section 4 to describe various subsets of the complex plane. Then in Section 5 we introduce the Triangle Inequality, which is a useful tool for manipulating inequalities.

Each section ends with a number of further exercises for additional practice. (This is the case for most sections throughout the module.)

Unit guide

After studying this unit, you should be able to perform basic algebraic manipulations with complex numbers and understand their geometric interpretation. Before you tackle later units, it is important that you become confident with these basic manipulations, and you should attempt as many of the exercises as you have time for.
1 Complex numbers and their properties

After working through this section, you should be able to:

• determine the real part, the imaginary part and the complex conjugate of a given complex number
• perform addition, subtraction, multiplication and division of complex numbers
• use the Binomial Theorem and the Geometric Series Identity to simplify complex expressions.

1.1 Defining complex numbers

We assume that you are already familiar with various different types of numbers, such as the natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \), the integers \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \), the rational numbers (or fractions) \( \mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\} \), and the real numbers \( \mathbb{R} \), which can be represented by decimals, either terminating (such as \( \frac{1}{2} = 0.5 \)) or non-terminating (such as \( \pi = 3.1415 \ldots \)). We assume also that you are familiar with the usual arithmetic operations of addition, subtraction, multiplication and division of real numbers.

We are now going to introduce the idea of a complex number, and we begin with some definitions.

**Definitions**

A complex number \( z \) is an expression of the form \( x + iy \), where \( x \) and \( y \) are real numbers and \( i \) is a symbol with the property that \( i^2 = -1 \). We write

\[
z = x + iy \quad \text{or, equivalently,} \quad z = x + yi,
\]

and say that \( z \) is expressed in **Cartesian form**. The real number \( x \) is the **real part** of \( z \) (written \( x = \text{Re } z \)) and the real number \( y \) is the **imaginary part** of \( z \) (written \( y = \text{Im } z \)).

Two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal.

The set of all complex numbers is denoted by \( \mathbb{C} \).

In this module, the mathematical symbol \( z \) always denotes a complex number, unless specified otherwise.

The word ‘Cartesian’ is derived from the surname of the French mathematician and philosopher René Descartes (1596–1650), who was a pioneer in relating algebra to geometry by representing pairs of real variables \( x \) and \( y \) by points in a plane.
The following table gives some examples of complex numbers \( z \) that correspond to given real numbers \( x \) and \( y \).

<table>
<thead>
<tr>
<th>( z = x + iy )</th>
<th>1 + 2i</th>
<th>( \sqrt{2} + i\pi )</th>
<th>3i</th>
<th>1</th>
<th>1 + i</th>
<th>0</th>
<th>1 - 2i</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Re } z ) = ( x )</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \text{Im } z ) = ( y )</td>
<td>2</td>
<td>( \pi )</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0 - 2</td>
</tr>
</tbody>
</table>

When working with complex numbers we use the following conventions, some of which you can see in the table.

- Any real number \( x \) can be thought of as a complex number whose imaginary part is zero (thus \( \mathbb{R} \) is a subset of \( \mathbb{C} \)). We write, for example, \( 1 + 0i = 1 \).
- If the real part of a complex number is 0, but the imaginary part is non-zero, then we omit the real part when writing the complex number; for example, \( 0 + 3i = 3i \).
- The complex number \( 0 + 0i \) is written 0, the zero complex number.
- We usually abbreviate \( 1i \) to \( i \) and \( -1i \) to \( -i \).
- If \( y \) is negative, then we usually write \( z \) as \( x - |y|i \); for example, \( 1 + (-2)i = 1 - 2i \).

In some texts, a complex number with imaginary part zero is called purely real, and a complex number with real part zero is called purely imaginary. However, those terms will not be used in this module. We also remark that in certain contexts such as electrical engineering (where \( i \) is used for current) it is common practice to write \( j \) instead of \( i \).

### 1.2 Arithmetic with complex numbers

The definition of a complex number contains the symbol ‘+’ and refers to the ‘square’ of \( i \). This suggests that arithmetic operations can be performed with complex numbers; the following definitions are made.

**Definitions**

The binary operations of **addition**, **subtraction** and **multiplication** of complex numbers are denoted by the same symbols as for real numbers and are performed by the usual procedure – that is, treating complex numbers as real expressions together with an algebraic symbol \( i \) with the property that \( i^2 = -1 \).

The following example shows some arithmetic operations involving complex numbers.
Example 1.1
Express each of the following numbers in Cartesian form.

(a) \((1 + 2i) + \left(\frac{1}{2} + \pi i\right)\)
(b) \((1 + 2i)\left(\frac{1}{2} + \pi i\right)\)
(c) \(2(1 + 2i) - 2i\left(\frac{1}{2} + \pi i\right)\)
(d) \((1 + 2i)(1 - 2i)\)

Solution

(a) By the usual procedure,
\[
(1 + 2i) + \left(\frac{1}{2} + \pi i\right) = 1 + 2i + \frac{1}{2} + \pi i = \frac{3}{2} + (2 + \pi)i.
\]

(b) By the usual procedure,
\[
(1 + 2i)\left(\frac{1}{2} + \pi i\right) = \frac{1}{2} + \pi i + i + 2\pi i^2.
\]
Applying the extra property that \(i^2 = -1\), we obtain
\[
(1 + 2i)\left(\frac{1}{2} + \pi i\right) = \left(\frac{1}{2} - 2\pi\right) + (\pi + 1)i.
\]

(c) By the usual procedure and the property that \(i^2 = -1\),
\[
2(1 + 2i) - 2i\left(\frac{1}{2} + \pi i\right) = 2 + 4i - i - 2\pi i^2 = (2 + 2\pi) + 3i.
\]

(d) By the usual procedure and the property that \(i^2 = -1\),
\[
(1 + 2i)(1 - 2i) = 1 - 2i + 2i - 4i^2 = 1 + 4 = 5.
\]

The following exercises provide practice at manipulating complex numbers.

Exercise 1.1

(a) Express each of the following numbers in Cartesian form.

(i) \((2 + i) + 3i(-1 + 3i)\)
(ii) \((2 + i)(-1 + 3i)\)
(iii) \((-1 + 3i)(-1 - 3i)\)

(b) Write down the real and imaginary parts of \(z = (2 + i) + 3i(-1 + 3i)\).

In the next exercise the letters \(x\) and \(y\) appear in the specification of complex numbers (with and without subscripts). You should assume in these circumstances – here and elsewhere in the module – that \(x\) and \(y\) are both real numbers.
Exercise 1.2

Express each of the following numbers in Cartesian form.

(a) \((x_1 + iy_1) + (x_2 + iy_2)\)  
(b) \((x_1 + iy_1) - (x_2 + iy_2)\)  
(c) \((x_1 + iy_1)(x_2 + iy_2)\)  
(d) \((x + iy)(x - iy)\)

As with real numbers, the negative \(-z\) of a complex number \(z\) is defined in such a way that \(z + (-z) = 0\).

**Definition**

The **negative** \(-z\) of a complex number \(z = x + iy\) is

\[-z = (-x) + i(-y),\]

usually written \(-z = -x - iy\).

For example, \(-(1 + i) = -1 - i\).

Next we discuss division of complex numbers. As with real numbers, the reciprocal \(1/z\) of a non-zero complex number \(z\) is defined in such a way that \(z(1/z) = 1\).

**Definitions**

The **reciprocal** \(1/z\) of a non-zero complex number \(z = x + iy\) is

\[\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.\]

The alternative notation \(z^{-1}\) is also used for the reciprocal.

The **quotient** \(z_1/z_2\) of a complex number \(z_1\) by a non-zero complex number \(z_2\) is

\[\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right).\]

This definition of \(1/z\) works because, as you saw in Exercise 1.2(d),

\[(x + iy)(x - iy) = x^2 + y^2,\]

which is strictly positive because \(z\) is non-zero, so

\[z\left(\frac{1}{z}\right) = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \frac{(x + iy)(x - iy)}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1.\]
The definition of quotient suggests that in order to evaluate $z_1/z_2$, we must first evaluate $1/z_2$ and then multiply by $z_1$. In practice, it is easier to do both operations at once using the following strategy.

**Strategy for obtaining a quotient**

To obtain the quotient

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \text{ where } y_2 \neq 0,$$

in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

**Example 1.2**

Express the following numbers in Cartesian form.

(a) $\frac{1}{1 + 2i}$  (b) $\frac{3 + 4i}{1 + 2i}$

**Solution**

(a) By the strategy for obtaining a quotient,

$$\frac{1}{1 + 2i} = \frac{1 - 2i}{(1 + 2i)(1 - 2i)} = \frac{1 - 2i}{1 + 4} = \frac{1}{5} - \frac{2}{5}i.$$

(b) By the strategy for obtaining a quotient,

$$\frac{3 + 4i}{1 + 2i} = \frac{(3 + 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{3 - 6i + 4i - 8i^2}{1 + 4} = \frac{11 - 2i}{5} = \frac{11}{5} - \frac{2}{5}i.$$

It would be acceptable to leave the solution to part (b) in the form $(11 - 2i)/5$, since this can readily be reduced to Cartesian form.

**Exercise 1.3**

(a) Express the following numbers in Cartesian form.

(i) $\frac{1}{i}$  (ii) $\frac{1}{1 + i}$  (iii) $\frac{1 + 2i}{2 + 3i}$

(b) Express the following quotient in Cartesian form:

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \text{ where } y_2 \neq 0.$$
The process of changing the sign of the imaginary part of a complex number, used in the strategy above, is often carried out, so the following terminology and notation is helpful.

**Definition**

The **complex conjugate** $\overline{z}$ of a complex number $z = x + iy$ is

$$\overline{z} = x - iy.$$ 

Some texts use the notation $z^*$ in place of $\overline{z}$. In particular, complex numbers feature considerably in the subject of quantum mechanics, and most texts on that subject prefer this alternative notation.

The complex conjugate of $z$ satisfies the simple identities

$$\text{Re} \overline{z} = \text{Re} z \quad \text{and} \quad \text{Im} \overline{z} = -\text{Im} z.$$ 

Several more identities involving complex conjugates are given in the following result.

**Theorem 1.1 Properties of the complex conjugate**

(a) If $z$ is a complex number, then

(i) $z + \overline{z} = 2 \text{Re} z$

(ii) $z - \overline{z} = 2i \text{Im} z$

(iii) $(\overline{z}) = z$.

(b) If $z_1$ and $z_2$ are complex numbers, then

(i) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

(ii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$

(iii) $\overline{z_1z_2} = \overline{z}_1 \overline{z}_2$

(iv) $z_1/\overline{z}_2 = \overline{z}_1/\overline{z}_2$, where $\overline{z}_2 \neq 0$.

Part (b)(i) says that ‘the conjugate of a sum is the sum of the conjugates’, and other parts can be described in similar terms. Note the use of the long conjugate bar over expressions involving several symbols.

**Proof**

(a) If $z = x + iy$, then $\overline{z} = x - iy$, so

(i) $z + \overline{z} = (x + iy) + (x - iy) = 2x = 2 \text{Re} z$

(ii) $z - \overline{z} = (x + iy) - (x - iy) = 2iy = 2i \text{Im} z$

(iii) $(\overline{z}) = (x - iy) = x + iy = z$.

(b) The proofs of these identities all follow from the results of Exercises 1.2 and 1.3(b). To illustrate the method, we prove part (iii).

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, by Exercise 1.2(c),

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \quad (1.1)$$
Now, \( z_1 = x_1 - iy_1 \) and \( z_2 = x_2 - iy_2 \), so if we replace \( y_1, y_2 \) by \(-y_1, -y_2\) in equation (1.1), then we see that

\[
\begin{align*}
\overline{z_1 z_2} &= (x_1 x_2 - (-y_1)(-y_2)) + i(x_1(-y_2) + (-y_1)x_2) \\
&= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2) \\
&= \overline{z_1} \overline{z_2},
\end{align*}
\]

as required. ■

Exercise 1.4

Prove the identities stated in Theorem 1.1(b), parts (i) and (iv).

Now that you have seen how to perform the usual arithmetic operations with complex numbers, it is natural to ask the following question. Do these operations have the usual properties that are known to hold for real numbers? It is a straightforward matter to check that, for example, addition of complex numbers is associative; that is, for all \( z_1, z_2, z_3 \) in \( \mathbb{C} \),

\[
(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).
\]

It is also straightforward, but more tedious, to show that multiplication of complex numbers is associative; that is, for all \( z_1, z_2, z_3 \) in \( \mathbb{C} \),

\[
(z_1 z_2)z_3 = z_1(z_2 z_3).
\]

In fact, it turns out that all the usual arithmetic properties do hold for complex numbers. These are summarised in the following table.

<table>
<thead>
<tr>
<th>Arithmetic in ( \mathbb{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Property</strong></td>
</tr>
<tr>
<td>Closure</td>
</tr>
<tr>
<td>Identity</td>
</tr>
<tr>
<td>Inverse</td>
</tr>
<tr>
<td>Associative</td>
</tr>
<tr>
<td>Commutative</td>
</tr>
<tr>
<td>Distributive</td>
</tr>
</tbody>
</table>
Once all these properties have been proved (and we will not give the details here), then the contents of the table can be described in algebraic terms as follows.

The complex numbers $\mathbb{C}$ form an abelian group under the operation of addition, with identity 0 (properties A1–A5).

The set of non-zero complex numbers is an abelian group under the operation of multiplication (noting in M1 that if $z_1, z_2 \neq 0$, then $z_1 z_2 \neq 0$), with identity 1 (properties M1–M5).

These two structures are linked by the distributive property (D).

Because $\mathbb{C}$ has all these properties, it is called a field. The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are also fields.

Notice that in property M3 we have used $z^{-1}$ to denote the reciprocal $1/z$. It is also standard practice to use the notation $z^n$, where $n \in \mathbb{Z}$, for integral powers of a non-zero complex number $z$. For example,

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1,$$
$$i^{-1} = -i, \quad i^{-2} = -1, \quad i^{-3} = i, \quad i^{-4} = 1.$$

By convention, $z^0 = 1$, for all non-zero $z$, and $0^0$ is not defined, except in special cases such as formulas in which it is convenient to assign a value to $0^0$ (for example, see binomial coefficients in the next subsection). The zero complex number has powers $0^n = 0$ for $n = 1, 2, 3, \ldots$ We will discuss the meaning of fractional powers, such as $z^{1/2} = \sqrt{z}$, in Section 3, and also in Unit A2.

### Rafael Bombelli

The Italian mathematician Rafael Bombelli, whom we met in the Introduction when discussing his contributions to solving cubic equations, is recognised as the first person to work with complex numbers in a systematic way. In his celebrated text *Algebra*, published in 1572, Bombelli lays out many of the rules for manipulating complex numbers that we have covered in this subsection, but with different terminology and notation. That text also includes an exposition of the solution of cubic and quartic equations, making use of complex numbers.

### 1.3 Identities with complex numbers

Because complex numbers satisfy the usual arithmetic properties, we can prove and then use all the usual algebraic identities. For example, if $z_1$ and $z_2$ are any complex numbers, then

$$(z_1 + z_2)^2 = z_1^2 + 2z_1 z_2 + z_2^2 \quad \text{and} \quad z_1^2 - z_2^2 = (z_1 - z_2)(z_1 + z_2).$$

Thus, for example, if $z^2 + 9 = 0$, then

$$z^2 + 9 = (z - 3i)(z + 3i) = 0,$$

so $z = 3i$ or $z = -3i$. 


Exercise 1.5

Prove the following identities.

(a) \((z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3\)

(b) \(z_1^3 - z_2^3 = (z_1 - z_2)(z_1^2 + z_1z_2 + z_2^2)\)

(c) \(z_1^3 + z_2^3 = (z_1 + z_2)(z_1^2 - z_1z_2 + z_2^2)\)

The identities in Exercise 1.5 are, in fact, special cases of two important general identities which will often be used in the module. The first of these is the Binomial Theorem, which we state in two forms. This theorem uses the binomial coefficient

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!},
\]

sometimes written as \(nC_k\). Note that, by convention, \(0! = 1\) and \(0^0 = 1\) in the formulas in these identities.

The proof of the Binomial Theorem is the same as in the real case, so it is omitted.

Theorem 1.2 Binomial Theorem

(a) If \(z \in \mathbb{C}\) and \(n \in \mathbb{N}\), then

\[
(1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + z^n.
\]

(b) If \(z_1, z_2 \in \mathbb{C}\) and \(n \in \mathbb{N}\), then

\[
(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k}z_2^k = z_1^n + n z_1^{n-1}z_2 + \frac{n(n-1)}{2!}z_1^{n-2}z_2^2 + \cdots + z_2^n.
\]

Remarks

1. Exercise 1.5(a) is the special case of part (b) of the Binomial Theorem with \(n = 3\).

2. You may have noticed that the formula

\[
(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + z^n \quad (1.2)
\]

from part (a) of the Binomial Theorem is a little misleading if \(n = 1\) or \(n = 2\). For example, if \(n = 1\), then there are only two terms in the expansion of \((1 + z)^1\), not four (or more). In this module, we adopt the convention that formulas presented in the manner of equation (1.2) are
valid for all sufficiently large values of $n$ for which they make sense (typically $n > 2$ should suffice), and for small values of $n$ the formula should be interpreted in the sensible way by omitting some terms.

3. It is worth remembering that the coefficients of powers of $z$ that appear in the Binomial Theorem can be arranged in the form of Pascal’s triangle. As you may recall, each entry in Pascal’s triangle (aside from the 1s that form the sides of the triangle) is the sum of the entries above left and above right. The first few rows of the triangle are shown below.

$$
\begin{array}{cccccc}
(1 + z)^0 & 1 \\
(1 + z)^1 & 1 & 1 \\
(1 + z)^2 & 1 & 2 & 1 \\
(1 + z)^3 & 1 & 3 & 3 & 1 \\
(1 + z)^4 & 1 & 4 & 6 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

**Exercise 1.6**

Use the Binomial Theorem to simplify the following expressions.

(a) $(1 + i)^4$  
(b) $(3 + 2i)^3$

Next we state two forms of an identity that can be used to sum a finite geometric series. The proof is the same as in the real case, so again it is omitted.

**Theorem 1.3  Geometric Series Identity**

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1}).$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$z_1^n - z_2^n = (z_1 - z_2)(z_1^{n-1} + z_1^{n-2}z_2 + z_1^{n-3}z_2^2 + \cdots + z_2^{n-1}).$$

**Remarks**

1. Exercise 1.5(b) is the special case of part (b) of the Geometric Series Identity with $n = 3$.

2. On replacing $n$ with $n + 1$, the first of these two identities can be written as

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \text{ for } z \neq 1.$$

This is the familiar formula for summing a finite geometric series.
Exercise 1.7

(a) Use the Geometric Series Identity to simplify the expression
\[1 + (1 + i) + (1 + i)^2 + (1 + i)^3.\]
(b) Use the Geometric Series Identity to find a factor of
\[z^5 - i\]
of the form \(z - a\), for some complex number \(a\).

(Hint: \(i^5 = i\).)

Further exercises

Exercise 1.8

Complete the following table.

<table>
<thead>
<tr>
<th>(z)</th>
<th>(\text{Re } z)</th>
<th>(\text{Im } z)</th>
<th>(-z)</th>
<th>(\overline{z})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 3i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3 - i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exercise 1.9

Express each of the following complex numbers in Cartesian form.

(a) \(i^3\) (b) \(i^4\) (c) \((1 + i)^2\) (d) \((1 - i)^2\) (e) \(\frac{1}{1 - i}\)

(f) \(\frac{1 + i}{1 - i}\) (g) \((1 + i)^3\) (h) \((2 + i)^2 - (2 - i)^2\) (i) \(\frac{3 + 5i}{2 - 3i}\)

(j) \(\frac{3 + 2i}{1 + 4i}\) (k) \((3 + 4i)^4 - (3 - 4i)^4\) (l) \(1 + i + i^2 + \cdots + i^{10}\)

Exercise 1.10

Write down the real part, imaginary part and complex conjugate of each of the complex numbers in parts (a), (e), (g) of Exercise 1.9.

Exercise 1.11

Prove that \(\text{Im } \overline{z} = -\text{Im } z\).
2 The complex plane

After working through this section, you should be able to:

- determine the **modulus** of a given complex number
- determine the **principal argument** and other arguments of a given non-zero complex number
- convert a complex number in Cartesian form to **polar form**, and vice versa
- interpret geometrically the sum, product and quotient of two complex numbers
- state De Moivre’s Theorem, and use it to evaluate powers of complex numbers.

2.1 Cartesian coordinates

In this section we describe a geometric interpretation of complex numbers, and we see how this interpretation gives us insight into the properties of complex numbers.

You are probably familiar with the idea of representing an ordered pair \((x, y)\) from \(\mathbb{R}^2\) by a point on a plane, called a Cartesian plane, with horizontal coordinate \(x\) and vertical coordinate \(y\) (the Cartesian coordinates of the point). It is common to refer to ‘the point \((x, y)\)’.

Complex numbers can likewise be represented by points on a Cartesian plane – the complex number \(z = x + iy\) is represented by the point \((x, y)\).

For example, the number \(4 + 3i\) is represented by the point \((4, 3)\), and in Figure 2.1 this point is labelled \(4 + 3i\).

Thus we often speak of ‘the point \(z = x + iy\)’ and, with this interpretation, refer to the plane as the **complex plane**.

**Definitions**

The **complex plane** or **z-plane** is a Cartesian plane used to represent the set of all complex numbers in which the complex number \(z = x + iy\) is represented by the point \((x, y)\).

The horizontal axis of the complex plane is called the **real axis** and the vertical axis is called the **imaginary axis**.

Since the complex plane represents the set of all complex numbers, we denote it by the symbol \(\mathbb{C}\).

In drawing the complex plane, we do not usually label the axes \(x\) and \(y\) unless it is helpful to do so (as in Unit A2, for example).

The four infinite regions of the complex plane separated off by (and not including) the axes are called **quadrants**. We label them upper-right, upper-left, lower-left and lower-right quadrants, as shown in Figure 2.2.
The various operations on complex numbers described in Section 1 can all be given geometric interpretations in the complex plane. For example, if \( z \) is a complex number, then, as shown in Figures 2.3(a) and 2.3(b),

\[-z \text{ is obtained by rotating } z \text{ through the angle } \pi \text{ about the origin}
\]
\[\overline{z} \text{ is obtained by reflecting } z \text{ in the real axis.}\]

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig2.3}
\caption{Transforming \( z \) by (a) rotating through the angle \( \pi \) about the origin, (b) reflecting in the real axis}
\end{figure}

Complex numbers can also be thought of as vectors, with the complex number \( x + iy \) corresponding to the vector from the point \((0,0)\) to the point \((x,y)\). It follows that the sum of two complex numbers, and also their difference, satisfy the parallelogram law for vectors, as shown in Figure 2.4. In that figure, the point \( z_1 - z_2 \) is obtained by observing that \( z_1 - z_2 = z_1 + (-z_2) \) and then applying the additive version of the parallelogram law to \( z_1 \) and \(-z_2\).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig2.4}
\caption{Using the parallelogram law to (a) add \( z_1 \) and \( z_2 \), (b) subtract \( z_2 \) from \( z_1 \)}
\end{figure}

**Exercise 2.1**

With \( z_1 = 3 + i \) and \( z_2 = -1 + 2i \), plot the following numbers (on two separate diagrams).

(a) \( z_1, z_2, -z_1, -z_2, z_1 + z_2, z_1 - z_2 \)
(b) \( z_1, z_2, \overline{z_1}, \overline{z_2}, z_1 + z_2, \overline{z_1 + z_2} \)
History of the complex plane

The complex plane is often called the Argand diagram, after a French mathematician with the surname Argand, who in 1806 wrote an essay on representing complex numbers by directed line segments in a plane. It is sometimes suggested that this mathematician is the Swiss-born man Jean-Robert Argand (1768–1822), but there is no evidence that this is so.

In fact, the idea of representing complex numbers in a plane had been proposed before, by the Norwegian–Danish surveyor and cartographer Caspar Wessel (1745–1818). Wessel’s work, published in a little-known Danish journal in 1799, went largely unnoticed for the next century, and Wessel published no other papers in mathematics.

The value of the complex plane in representing complex numbers gained general acceptance with the work of the eminent German mathematician Carl Friedrich Gauss (1777–1855). There is evidence that he understood this geometric representation of complex numbers in his doctoral dissertation of 1799, and the idea appears explicitly in his letters to colleagues in the years to follow. Gauss was the first to consider complex numbers as points in the plane rather than just as directed line segments (as considered by Argand and Wessel). He introduced the terminology ‘complex numbers’ in place of ‘imaginary numbers’, because he thought that the old terminology was unhelpful, ascribing mystery to complex numbers and obscuring their value.

Multiplication and division of complex numbers also have useful geometric interpretations. Before describing these, however, we need to introduce some other geometric concepts.

2.2 Polar form

The modulus, or absolute value, of a real number $x$ is defined as

$$|x| = \begin{cases} 
  x, & x \geq 0, \\
  -x, & x < 0.
\end{cases}$$

Equivalently, $|x|$ is the distance along the real line from 0 to $x$. The modulus of a complex number $z$ is similarly defined to be the distance from 0 to $z$ in the complex plane.
Definition

The **modulus**, or **absolute value**, of a complex number \( z = x + iy \) is the distance from 0 to \( z \); it is denoted by \(|z|\). Thus

\[
|z| = |x + iy| = \sqrt{x^2 + y^2}.
\]

Remarks

1. In this definition we use the standard convention that if \( a \geq 0 \), then \( \sqrt{a} \) denotes the *non-negative* square root of \( a \) (non-negative means positive or zero).

2. The plural of modulus is *moduli*.

For examples of moduli, observe that

\[
|3 + 4i| = \sqrt{3^2 + 4^2} = 5,
\]

\[
|-3| = \sqrt{(-3)^2} = 3,
\]

\[
|-2i| = \sqrt{(-2)^2} = 2.
\]

These moduli are shown as distances in Figure 2.5.

![Figure 2.5](image)

**Figure 2.5** Moduli of three different complex numbers

**Exercise 2.2**

(a) Evaluate the following moduli.

(i) \(|1 + i|\)  \quad (ii) \(|2 - 4i|\)  \quad (iii) \(|i|\)  \quad (iv) \(|-5 + 12i|\)

(b) Prove that \(|\overline{z}| = |z|\) and \(|-z| = |z|\).

If \( z_1, z_2 \) are any two complex numbers, then, by definition, \(|z_1 - z_2|\) is the distance from 0 to \( z_1 - z_2 \). Using the parallelogram law to add \( z_2 \) and \( z_1 - z_2 \) (see Figure 2.6), we deduce the following observation.

\[
|z_1 - z_2| \text{ is the distance from } z_2 \text{ to } z_1.
\]

Because \( z_1 + z_2 = z_1 - (-z_2) \), there is a similar geometric interpretation for \(|z_1 + z_2|\).
The complex plane

$|z_1 + z_2|$ is the distance from $-z_2$ to $z_1$.

Exercise 2.3

With $z_1 = 3 + i$ and $z_2 = -1 + 2i$, determine

(a) $|z_1 - z_2|$
(b) $|z_1 + z_2|$
(c) the distance from $z_2$ to $-z_1$.

We now collect together various basic properties of the modulus.

Theorem 2.1 Properties of the modulus

(a) $|z| \geq 0$, with equality if and only if $z = 0$.
(b) $|\overline{z}| = |z|$ and $|-z| = |z|$.
(c) $|z|^2 = z\overline{z}$.
(d) $|z_1 - z_2| = |z_2 - z_1|$.
(e) $|z_1 z_2| = |z_1||z_2|$, and $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$.

Property (d) says that the distance from $z_2$ to $z_1$ is the same as the distance from $z_1$ to $z_2$.

Proof Property (a) follows from the fact that $|z| = \sqrt{x^2 + y^2}$ (where $z = x + iy$), and property (b) was proved in Exercise 2.2(b). To prove property (c), note that if $z = x + iy$, then

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$ 

Property (d) follows from property (b), since $z_2 - z_1 = -(z_1 - z_2)$.

Each of the identities in property (e) can be proved by writing $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and then calculating both sides. However, it is neater to use property (c) and Theorem 1.1, as follows:

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) \quad \text{(property (c))}$$
$$= (z_1 z_2)(\overline{z_1} \overline{z_2}) \quad \text{(Theorem 1.1(b)(iii))}$$
$$= (z_1 \overline{z_1})(z_2 \overline{z_2}) \quad \text{(associativity and commutativity)}$$
$$= |z_1|^2 |z_2|^2 \quad \text{(property (c))},$$

so $|z_1 z_2| = |z_1||z_2|$. Similarly, if $z_2 \neq 0$ (so $\overline{z_2} \neq 0$, and $|z_2| > 0$ using property (a)), then

$$|z_1/z_2|^2 = (z_1/z_2)(\overline{z_1/z_2}) = (z_1/z_2)(\overline{z_1} \overline{z_2})$$
$$= (z_1 \overline{z_1})/(z_2 \overline{z_2}) = |z_1|^2/|z_2|^2,$$

so $|z_1/z_2| = |z_1|/|z_2|$.
If the modulus $|z|$ of a complex number $z$ is equal to 0, then $z$ itself must equal 0 (and vice versa). However, the modulus of a non-zero complex number does not determine the number completely; all the points that lie on the circle of radius $r$ centred at the origin have the same modulus, namely $r$. We can determine the non-zero complex number $z$ completely by giving its modulus $|z| = r$ together with the angle $\theta$ that the line from the origin to $z$ makes with the positive real axis.

Angles used to determine position in this way are conventionally taken to be positive when measured in an anticlockwise direction from the positive real axis, and negative when measured in a clockwise direction.

For example, $1 + i$ has modulus $\sqrt{2}$, and the (positive) angle that the line from the origin to $1 + i$ makes with the positive real axis is $\pi/4$ (see Figure 2.7). Of course, $\pi/4$ is not the only angle that, along with the modulus $\sqrt{2}$, specifies $1 + i$; any one of the angles

$$\ldots, \quad \frac{\pi}{4} - 2\pi, \quad \frac{\pi}{4}, \quad \frac{\pi}{4} + 2\pi, \quad \frac{\pi}{4} + 4\pi, \quad \ldots$$

would do just as well – for example, the (negative) angle $\pi/4 - 2\pi = -7\pi/4$ (see Figure 2.8). This feature is reflected in the following definition using the sine and cosine functions. Figure 2.9 illustrates the definition, showing one argument of $z = x + iy$.

### Definition

An **argument** of a non-zero complex number $z = x + iy$ with $|z| = r$ is an angle $\theta$ (measured in radians) such that

$$\cos \theta = \frac{x}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}.$$

### Remarks

1. No argument is assigned to the number 0.
2. Each non-zero complex number has infinitely many arguments, all differing by integer multiples of $2\pi$. For example, the arguments of $1 + i$ can be written collectively as

$$\frac{\pi}{4} + 2k\pi, \quad \text{where } k \in \mathbb{Z}.$$

3. As you can see in Figures 2.7, 2.8 and 2.9, arguments are represented in figures by directed arcs: anticlockwise arcs for positive arguments and clockwise arcs for negative arguments. We use directed arcs in this way to communicate whether an angle is positive or negative. In contrast, later figures (such as Figures 2.14, 2.15 and 2.16) use undirected arcs; these arcs always represent positive angles – the anticlockwise arrow is omitted for convenience.

4. For some complex numbers, arguments are easily obtained by plotting the point. For example, Figure 2.10 shows that $3\pi/4$ is an argument...
of \(-1 + i\), \(\pi/2\) is an argument of \(i\) and \(-\pi/4\) is an argument of \(1 - i\). The calculation of arguments is dealt with later in the section.

\[
-1 + i
\]

\[
\pi/2
\]

\[
1 - i
\]

**Figure 2.10** Arguments of three complex numbers

Since any non-zero complex number is completely determined by its modulus and any one of its arguments, these two quantities can be used to define an alternative coordinate system for non-zero complex numbers.

**Definitions**

The ordered pair \((r, \theta)\), where \(r\) is the modulus of a non-zero complex number \(z\) and \(\theta\) is an argument of \(z\), is called the **polar coordinates** of \(z\). The expression

\[
z = r(\cos \theta + i \sin \theta)
\]

is said to be a representation of \(z\) in **polar form**.

**Remarks**

1. We will rarely use polar coordinates, preferring almost always to use polar form.
2. It follows from the definition of polar form that if \(z = x + iy\), then

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

**Example 2.1**

Represent \(-1 - i\) in polar form.

**Solution**

Here

\[
r = |-1 - i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2},
\]

and, from Figure 2.11, one choice for \(\theta\) is \(5\pi/4\). Thus

\[
-1 - i = \sqrt{2}(\cos 5\pi/4 + i \sin 5\pi/4)
\]

is in polar form.

Another polar form for \(-1 - i\) is \(\sqrt{2}(\cos(-3\pi/4) + i \sin(-3\pi/4))\).
Exercise 2.4

(a) Represent the complex number \( i \) in polar form.
(b) Represent each of the following complex numbers in Cartesian form.
   
   (i) \( 2(\cos \pi/3 + i \sin \pi/3) \)
   
   (ii) \( 3(\cos(-\pi/4) + i \sin(-\pi/4)) \)

The terminology ‘arg \( z \)’ is often used in other texts to denote an argument of a non-zero complex number \( z \). Without further information, however, the expression arg \( z \) is ambiguous, since \( z \) has infinitely many arguments, so we will avoid using it. Instead we select one argument for special attention and call this the principal argument (a shortened version of the more conventional ‘principal value of the argument’).

Definition

The principal argument of a non-zero complex number \( z \) is the unique argument \( \theta \) of \( z \) satisfying \(-\pi < \theta \leq \pi\); it is denoted by

\[ \theta = \text{Arg} z. \]

(Note the capital A in Arg.)

Since the arguments of a non-zero complex number differ by multiples of \( 2\pi \), exactly one of them satisfies \(-\pi < \theta \leq \pi\).

For an example of a principal argument, the arguments of \( 1 + i \) are

\[ \ldots, -7\pi/4, \pi/4, 9\pi/4, 17\pi/4, \ldots, \]

hence \( \text{Arg}(1 + i) = \pi/4 \) because \(-\pi < \pi/4 \leq \pi\).

For complex numbers \( z \) such as \( 1 + i \), it is easy to determine Arg \( z \) by inspection. In general, the following strategy can be applied. (There are other equally valid strategies.) The figures to illustrate the strategy each have a small circle at the origin to indicate that the strategy does not apply to \( z = 0 \).

Strategy for determining principal arguments

To determine the principal argument \( \theta \) of a non-zero complex number \( z = x + iy \), apply the relevant case below.

**Case 1** If \( z \) lies on one of the axes, then \( \theta \) is evident (see Figure 2.12).

**Case 2** If \( z \) does not lie on one of the axes, then carry out the following two steps.

---

Figure 2.12 Principal arguments on the axes
The complex plane

(i) Decide in which quadrant \( z \) lies (by plotting \( z \) if necessary), and then calculate the acute angle

\[
\phi = \tan^{-1}\left(\frac{|y|}{|x|}\right)
\]

in radians (see Figure 2.13(a)).

(ii) Obtain \( \theta \) in terms of \( \phi \) by using the appropriate formula in Figure 2.13(b).

![Figure 2.13](https://example.com/image.png)

**Figure 2.13** Formulas for principal arguments in the four quadrants

Remarks

1. Having found Arg \( z \), the other arguments of \( z \) can be obtained by adding integer multiples of 2\( \pi \) to Arg \( z \).

2. Some texts define the principal argument Arg \( z \) to be the argument of \( z \) that lies in the interval \([0, 2\pi)\) rather than in the interval \((-\pi, \pi]\) that we use.

Example 2.2

Find the principal argument of each of the following complex numbers.

(a) \( 1 + 2i \)  
(b) \( -1 - \sqrt{3}i \)  
(c) \( -1 + \sqrt{3}i \)

**Solution**

We apply the strategy for determining principal arguments (case 2 each time).

(a) \( 1 + 2i \) lies in the upper-right quadrant (Figure 2.14), and

\[
\phi = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1} 2;
\]

thus the principal argument \( \theta \) is

\[
\theta = \phi \quad \text{(Figure 2.13(b))}
\]

\[
= \tan^{-1} 2 \quad \text{(approximately 1.11 radians)}.
\]

![Figure 2.14](https://example.com/image.png)

**Figure 2.14** Angle \( \phi \) for \( 1 + 2i \)
(b) \(-1 - \sqrt{3}i\) lies in the lower-left quadrant (Figure 2.15), and
\[
\phi = \tan^{-1}(|-\sqrt{3}|/|-1|) = \tan^{-1} \sqrt{3} = \pi/3;
\]
thus the principal argument \(\theta\) is
\[
\theta = -(\pi - \phi) \quad \text{(Figure 2.13(b))}
\]
\[
= -2\pi/3.
\]
(c) \(-1 + \sqrt{3}i\) lies in the upper-left quadrant (Figure 2.16), and
\[
\phi = \tan^{-1}(\sqrt{3}/|-1|) = \tan^{-1} \sqrt{3} = \pi/3;
\]
thus the principal argument \(\theta\) is
\[
\theta = \pi - \phi \quad \text{(Figure 2.13(b))}
\]
\[
= 2\pi/3.
\]
(We will see in the next subsection that \(\text{Arg}\overline{z} = -\text{Arg} \, z\), and
using this observation we could deduce part (c) from part (b).)

**Exercise 2.5**

For each of the following complex numbers \(z\), write down \(\text{Arg} \, z\) and
express \(z\) in polar form.

(a) \(-4\) \hspace{1cm} (b) \(3\sqrt{3} + 3i\) \hspace{1cm} (c) \(\sqrt{3} - i\) \hspace{1cm} (d) \(-1 - i\)

We finish this subsection with a remark on equality of complex numbers in
polar form. Suppose that \(z_1 = r_1(\cos \theta_1 + i \sin \theta_1)\) and
\(z_2 = r_2(\cos \theta_2 + i \sin \theta_2)\) are non-zero complex numbers, and suppose also
that \(z_1 = z_2\). It follows that the moduli \(r_1\) and \(r_2\) must be equal, because
\(r_1 = |z_1| = |z_2| = r_2\). In contrast, the arguments \(\theta_1\) and \(\theta_2\) need not be
equal – they may differ by an integer multiple of \(2\pi\); that is, \(\theta_1 = \theta_2 + 2k\pi\),
for some integer \(k\). However, the *principal* arguments \(\text{Arg} \, z_1\) and \(\text{Arg} \, z_2\)
must be equal, because they are uniquely specified by \(z_1\) and \(z_2\). Since \(|z|\)
and \(\text{Arg} \, z\) themselves uniquely specify the non-zero complex number \(z\), we
obtain the following conclusion.

Two non-zero complex numbers \(z_1\) and \(z_2\) are equal if and only if
\(|z_1| = |z_2|\) and \(\text{Arg} \, z_1 = \text{Arg} \, z_2\).
2.3 A geometric interpretation of multiplication and division

A geometric interpretation of the multiplication of complex numbers can be given using the polar form of complex numbers. Indeed, if \( z_1 \) and \( z_2 \) are non-zero complex numbers with polar forms

\[
z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2),
\]

then, by using the formulas for the sine and cosine of the sum of two angles, we see that

\[
\begin{align*}
z_1 z_2 & = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\
& = r_1 r_2 ((\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)) \\
& = r_1 r_2 (\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)).
\end{align*}
\]

So we have the following formula.

\[
z_1 z_2 = r_1 r_2 (\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)). \quad (2.1)
\]

This formula shows that \(|z_1 z_2| = r_1 r_2 = |z_1||z_2|\), which we knew already, and also that the number \( \theta_1 + \theta_2 \) is an argument of \( z_1 z_2 \). Thus we can describe in words the effect of multiplying \( z_1 \) by \( z_2 \) (both non-zero) as follows.

- The product of the modulus of \( z_1 \) and the modulus of \( z_2 \) is the modulus of \( z_1 z_2 \).
- The sum of an argument of \( z_1 \) and an argument of \( z_2 \) is an argument of \( z_1 z_2 \).

So the geometric effect on \( z_1 \) of multiplying it by \( z_2 \) is to scale it by the factor \(|z_2|\) and rotate it about 0 through the angle \( \text{Arg} \ z_2 \). (This rotation is anticlockwise if \( \text{Arg} \ z_2 > 0 \) and clockwise if \( \text{Arg} \ z_2 < 0 \).) This is illustrated in Figure 2.17 for the case where \( z_1 \), \( z_2 \) and \( z_1 z_2 \) are in the upper-right quadrant, and \( \theta_1 \), \( \theta_2 \) and \( \theta_1 + \theta_2 \) are their principal arguments.

Notice that it is not always true that the principal argument of \( z_1 z_2 \) is the sum of the principal arguments of \( z_1 \) and \( z_2 \). It may differ from this sum by \( \pm 2\pi \).

For example, if \( \text{Arg} \ z_1 = \pi/2 \) and \( \text{Arg} \ z_2 = 3\pi/4 \), then

\[
\text{Arg} \ z_1 + \text{Arg} \ z_2 = 5\pi/4.
\]

Thus \( 5\pi/4 \) is an argument of \( z_1 z_2 \), but \( 5\pi/4 > \pi \) so it is not the principal argument of \( z_1 z_2 \). In fact, since \( -\pi < \text{Arg}(z_1 z_2) \leq \pi \),

\[
\text{Arg}(z_1 z_2) = 5\pi/4 - 2\pi = -3\pi/4.
\]
Similarly, if $\text{Arg } z_1 = -\pi/4$ and $\text{Arg } z_2 = -7\pi/8$, then

$$\text{Arg } z_1 + \text{Arg } z_2 = -9\pi/8,$$

which is an argument of $z_1z_2$, but

$$\text{Arg}(z_1z_2) = -9\pi/8 + 2\pi = 7\pi/8.$$

In general, since $-2\pi < \text{Arg } z_1 + \text{Arg } z_2 \leq 2\pi$, we have the following property of $\text{Arg } z$.

If $z_1$ and $z_2$ are (non-zero) complex numbers, then

$$\text{Arg}(z_1z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2n\pi,$$

where $n$ is $-1$, $0$ or $1$, depending on whether $\text{Arg } z_1 + \text{Arg } z_2$ is greater than $\pi$, lies in the interval $(-\pi, \pi]$, or is less than or equal to $-\pi$.

**Exercise 2.6**

Use polar forms of the complex numbers

$$z_1 = -1 - \sqrt{3}i \quad \text{and} \quad z_2 = 3\sqrt{3} + 3i$$

To evaluate $z_1z_2$ and $z_1^2$.

(You will find Example 2.2(b) and Exercise 2.5(b) useful.)

**Exercise 2.7**

Describe the geometric effect on a complex number $z$ of multiplying $z$ by $2i$.

As you might expect, the polar form of complex numbers is also useful for division. Indeed, if $z_1$ and $z_2$ are non-zero complex numbers with polar forms

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2),$$

then, by using the formula $\cos^2 \theta_2 + \sin^2 \theta_2 = 1$ and the formulas for the sine and cosine of the difference of two angles, we see that

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} = \frac{r_1}{r_2} \left( \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \right) = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$
So we have the following formula.

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).
\]  
(2.2)

This formula shows that \( |z_1/z_2| = r_1/r_2 = |z_1|/|z_2| \) and also that the number \( \theta_1 - \theta_2 \) is an argument of \( z_1/z_2 \). Thus we can describe the effect of dividing non-zero complex numbers as follows.

The modulus of \( z_1 \) divided by the modulus of \( z_2 \) is the modulus of \( z_1/z_2 \).

An argument of \( z_1 \) minus an argument of \( z_2 \) is an argument of \( z_1/z_2 \).

Thus the geometric effect on \( z_1 \) of dividing it by \( z_2 \) is to scale it by the factor \( 1/|z_2| \) and rotate it about 0 through the angle \(-\text{Arg } z_2\). (This rotation is clockwise if \( \text{Arg } z_2 > 0 \) and anticlockwise if \( \text{Arg } z_2 < 0 \).)

**Exercise 2.8**

Use polar forms of the complex numbers

\[ z_1 = 1 + \sqrt{3}i \quad \text{and} \quad z_2 = \sqrt{3} - i \]

to evaluate \( z_1/z_2 \).

**Exercise 2.9**

Describe the geometric effect on a complex number \( z \) of dividing \( z \) by \( 2i \).

An important special case of formula (2.2) for the quotient \( z_1/z_2 \) is obtained when

\[ z_1 = 1 \quad \text{and} \quad z_2 = r(\cos \theta + i \sin \theta), \]

so

\[ r_1 = 1, \quad \theta_1 = 0, \quad r_2 = r, \quad \theta_2 = \theta. \]

In this case we find that

\[
\frac{1}{r(\cos \theta + i \sin \theta)} = \frac{1}{r} \frac{1 + i}{\cos(0 - \theta) + i \sin(0 - \theta)}
\]

\[ = \frac{1}{r} \frac{1}{\cos(-\theta) + i \sin(-\theta)}. \]
Thus the reciprocal of a non-zero complex number $z$ can be described as follows.

The reciprocal of the modulus of $z$ is the modulus of $z^{-1}$.
The negative of an argument of $z$ is an argument of $z^{-1}$.

Notice that if $z$ lies outside the circle of radius 1 centred at 0, then $|z| > 1$, so $z^{-1}$ lies inside this circle (because $|z^{-1}| < 1$), as shown in Figure 2.18, and vice versa. If $z$ lies on this circle, then $|z| = 1$, so $z$ is of the form $z = \cos \theta + i \sin \theta$ and

$$z^{-1} = (\cos \theta + i \sin \theta)^{-1} = \cos(-\theta) + i \sin(-\theta).$$

Hence $z^{-1}$ also lies on the circle (see Figure 2.19); moreover, in this case

$$z^{-1} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta = \overline{z}.$$ 

In general, for all non-zero complex numbers $z$,

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{1}{|z|^2} \overline{z},$$

so (since $1/|z|^2$ is real and positive)

$$\text{Arg } z^{-1} = \text{Arg } \overline{z}.$$ 

Also, if $-\pi < \arg z < \pi$, then

$$\text{Arg } \overline{z} = - \text{Arg } z,$$

since $\overline{z}$ is the reflection of $z$ in the real axis. Thus we have the following properties of $\text{Arg } z$.

If $z$ is non-zero and $-\pi < \arg z < \pi$, then

$$\text{Arg } \overline{z} = \text{Arg } z^{-1} = - \text{Arg } z.$$ 

**Exercise 2.10**

Use a polar form of $1 + i$ to evaluate $(1 + i)^{-1}$.

The product of several complex numbers $z_1, z_2, \ldots, z_n$ has an interpretation similar to the product of two complex numbers.
The product of the moduli of \( z_1, z_2, \ldots, z_n \) is the modulus of \( z_1 z_2 \cdots z_n \).

The sum of arguments of \( z_1, z_2, \ldots, z_n \) is an argument of \( z_1 z_2 \cdots z_n \).

In other words, the product of the \( n \) complex numbers

\[
z_k = r_k (\cos \theta_k + i \sin \theta_k), \quad k = 1, 2, \ldots, n,
\]

is given by

\[
z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n (\cos (\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin (\theta_1 + \theta_2 + \cdots + \theta_n)).
\]

This formula can be obtained by applying the reasoning for the case \( n = 2 \) repeatedly (and the formula can be proved in general using the Principle of Mathematical Induction).

**Exercise 2.11**

Use polar forms of the complex numbers

\[
z_1 = 1 + i, \quad z_2 = 1 + \sqrt{3}i, \quad z_3 = \sqrt{3} + i,
\]

to evaluate \( z_1 z_2 z_3 \).

In the next subsection, polar form is used to calculate powers.

### 2.4 De Moivre’s Theorem

An important special case of the formula from the end of the previous subsection for the product \( z_1 z_2 \cdots z_n \) is obtained when

\[
r_1 = r_2 = \cdots = r_n = 1 \quad \text{and} \quad \theta_1 = \theta_2 = \cdots = \theta_n = \theta,
\]

so

\[
z_1 = z_2 = \cdots = z_n = \cos \theta + i \sin \theta.
\]

In this case, the product formula becomes

\[
(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad n = 1, 2, \ldots.
\]

This identity is due to de Moivre (pronounced ‘duh mwah-vr’, with a short ‘uh’).

**Abraham de Moivre**

Abraham de Moivre (1667–1754) was a French mathematician who lived much of his life in England. He first discovered a rather complicated-looking version of the identity that bears his name in 1707, and later, in 1722, refined it to give the version stated here. de Moivre also made important contributions to the theory of probability, publishing an influential text on the subject in 1711.
Figure 2.20 shows a geometric interpretation of de Moivre’s identity. The powers of $\cos \theta + i \sin \theta$ are spaced around the circle with centre 0 and radius 1, the angle between adjacent powers being $\theta$. Each multiplication by $\cos \theta + i \sin \theta$ gives rise to a rotation through angle $\theta$ about 0.

In Figure 2.21, the position of $(\cos \theta + i \sin \theta)^{-1}$ suggests that de Moivre’s identity holds also for negative integer powers; we now show that this is true.

**Theorem 2.2 De Moivre’s Theorem**

If $n$ is an integer and $\theta$ is a real number, then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$  

**Proof** We have already proved De Moivre’s Theorem for a positive integer $n$, and it is also true for

- $n = 0$, since $(\cos \theta + i \sin \theta)^0 = 1 = \cos 0 + i \sin 0$,
- $n = -1$, since $(\cos \theta + i \sin \theta)^{-1} = \cos(-\theta) + i \sin(-\theta)$. 
To complete the proof, note that if $m$ is a positive integer, then
\[(\cos \theta + i \sin \theta)^{-m} = ((\cos \theta + i \sin \theta)^{-1})^m\]
\[= (\cos(-\theta) + i \sin(-\theta))^m\]
\[= \cos(-m\theta) + i \sin(-m\theta).\]
Hence De Moivre’s Theorem holds also if $n = -m$, where $m$ is a positive integer.

**Exercise 2.12**

Use De Moivre’s Theorem to evaluate the following powers.

(a) $(\sqrt{3} + i)^4$  
(b) $(1 - \sqrt{3}i)^3$  
(c) $(1 + i)^{10}$  
(d) $(-1 + i)^{-8}$  
(e) $(\sqrt{3} + i)^{-6}$

**Further exercises**

**Exercise 2.13**

Plot each of the following complex numbers, and express each one in polar form, using the principal argument in each case.

(a) 5  
(b) $i$  
(c) $-3i$  
(d) $2 + 2i$  
(e) $-2 + 2i$  
(f) $-\sqrt{3} - i$  
(g) $3 + 4i$  
(h) $3 - 4i$

**Exercise 2.14**

Plot each of the following complex numbers, and express each one in Cartesian form.

(a) $\cos \pi + i \sin \pi$  
(b) $4(\cos(-\pi/2) + i \sin(-\pi/2))$  
(c) $3(\cos 3\pi/4 + i \sin 3\pi/4)$  
(d) $3(\cos \pi/6 + i \sin \pi/6)$  
(e) $\cos(-2\pi/3) + i \sin(-2\pi/3)$

**Exercise 2.15**

Find the distance from $z_1$ to $z_2$ in each of the following cases.

(a) $z_1 = 1 + i$, $z_2 = 2 + 3i$  
(b) $z_1 = -2 + 3i$, $z_2 = 1 - 7i$  
(c) $z_1 = i$, $z_2 = -i$

**Exercise 2.16**

Use polar form and De Moivre’s Theorem to evaluate the following expressions, giving your answers in Cartesian form.

(a) $(1 + \sqrt{3}i)^5$  
(b) $(1 + i)^{-4}$  
(c) $\frac{(1 + i)^6}{(\sqrt{3} - i)^3}$
Exercise 2.17

Use De Moivre’s Theorem and the Binomial Theorem to prove that

\[ \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta, \]

where \( \theta \) is a real number.

Exercise 2.18

Prove that if \((x + iy)^4 = a + ib\), where \(x + iy\) and \(a + ib\) are in Cartesian form, then

\[ (x^2 + y^2)^4 = a^2 + b^2. \]

Exercise 2.19

Prove that if \(\bar{z} = z^{-1}\), then \(|z| = 1\).

3 Solving equations with complex numbers

After working through this section, you should be able to:

- calculate the \( n \)th roots of a complex number
- solve certain polynomial equations with complex coefficients.

As we discussed in the Introduction, the use of complex numbers allows both quadratic and cubic equations with real coefficients to be solved. You will see in this module that complex numbers enable us to solve many equations that do not have real solutions. In this section we describe various polynomial equations whose complex solutions can be found explicitly.

3.1 Calculating \( n \)th roots

If \( a \) is a non-negative real number and \( n \) is a positive integer, then \( \sqrt[n]{a} \) or \( a^{1/n} \) denotes the non-negative \( n \)th root of \( a \), that is, the unique non-negative number \( x \) such that \( x^n = a \). In this subsection we discuss the \( n \)th roots of a complex number, beginning with square roots.

The simplest quadratic equation that has a complex solution but no real solutions is

\[ z^2 + 1 = 0, \]  
that is, \( z^2 = -1 \).

One solution of this equation is \( z = i \), since \( i^2 = -1 \); another solution is \( z = -i \), since \(( -i)^2 = (-1)^2 i^2 = -1 \).
A more general quadratic equation is
\[ z^2 = w, \]
where \( w \) is a given complex number. Any solution \( z \) of this equation is called a **square root** of \( w \); for example, both \( i \) and \( -i \) are square roots of \( -1 \).

In fact, we will show shortly that each non-zero complex number \( w \) has exactly two square roots. Then later we will introduce the notation \( \sqrt{w} \) or \( w^{1/2} \) to denote a particular square root of \( w \).

The following example shows how to find square roots geometrically.

### Example 3.1
Find the two solutions of the equation
\[ z^2 = i. \]

**Solution**

By the geometric properties of multiplication of complex numbers, described in Subsection 2.3,

- the square of the modulus of \( z \) is the modulus of \( z^2 \),
- an argument of \( z \) multiplied by 2 is an argument of \( z^2 \).

Since \( i \) has modulus 1 and argument \( \pi/2 \), one solution of \( z^2 = i \) is obtained by taking \( z \) to have modulus \( \sqrt{1} = 1 \) and argument \( \frac{1}{2}(\pi/2) = \pi/4 \) (see Figure 3.1). This gives

\[
z = 1 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.
\]

(Check: \( \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)^2 = \frac{1}{2} + 2 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}i = \frac{1}{2} = i \).)

Since \( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \) is a solution of \( z^2 = i \), and \((-1)^2 = 1 \),

\[
z = -\left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)
\]

is another solution of \( z^2 = i \). Therefore the required solutions are

\[
z = \pm \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right),
\]

illustrated in Figure 3.2.
Remarks

1. Notice that the second solution could also have been found geometrically. For example, if we had begun by taking \( i \) to have modulus 1 and argument \( \pi/2 + 2\pi = 5\pi/2 \), then the corresponding solution \( z \) would have modulus \( \sqrt{1} = 1 \) and argument \( \frac{1}{2}(5\pi/2) = 5\pi/4 \) (see Figure 3.3). This gives the solution

\[
z = 1 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i.
\]

Applying the same procedure but with other arguments for \( i \) (such as \( \pi/2 - 2\pi = -3\pi/2 \)) just gives us repeats of the two solutions we have obtained already.

2. Note that if \( z \) is a square root of a complex number \( w \), then \( -z \) is also a square root of \( w \).

3. An alternative method of solving \( z^2 = i \) is to write \( z = x + iy \), equate the real parts and imaginary parts of

\[
(x + iy)^2 = x^2 - y^2 + 2xyi = i
\]

(remembering that \( z_1 = z_2 \) means that \( \text{Re } z_1 = \text{Re } z_2 \) and \( \text{Im } z_1 = \text{Im } z_2 \)), and then solve the resulting equations for \( x \) and \( y \) (as you will do in Exercise 3.7). This method is, however, not suitable for finding the \( n \)th roots of complex numbers if \( n > 2 \).

Exercise 3.1

Find the two solutions of the equation

\[
z^2 = -1 + \sqrt{3}i.
\]

We now turn to the more general equation

\[
z^n = w,
\]

where \( w \) is a given complex number and \( n \) is any positive integer with \( n \geq 2 \). Each solution of \( z^n = w \) is called an \( n \)th root of \( w \). If \( w = 0 \), then \( z = 0 \) is the only solution. We will show shortly that each non-zero complex number \( w \) has exactly \( n \) \( n \)th roots.

As a simple example, consider the equation

\[
z^3 = -8.
\]

We will find solutions of this equation by expressing \(-8\) in polar form. First, we can write

\[
-8 = 8(\cos \pi + i \sin \pi),
\]

so a solution of \( z^3 = -8 \) is obtained by taking \( z \) to have modulus \( \sqrt[3]{8} = 2 \) and argument \( \pi/3 \). This gives

\[
z = 2(\cos \pi/3 + i \sin \pi/3) = 1 + \sqrt{3}i.
\]
But there are other ways of writing \(-8\) in polar form, such as

\[-8 = 8(\cos 3\pi + i \sin 3\pi),\]

so another solution of \(z^3 = -8\) is obtained by taking \(z\) to have modulus 2 (as before) and argument \(3\pi/3 = \pi\). This is the real solution

\[z = 2(\cos \pi + i \sin \pi) = -2.\]

By writing \(-8\) in polar form in yet another way, as

\[-8 = 8(\cos 5\pi + i \sin 5\pi),\]

we obtain a third solution of \(z^3 = -8\), namely

\[z = 2(\cos 5\pi/3 + i \sin 5\pi/3) = 1 - \sqrt{3}i.\]

We have now found the three solutions

\[z = -2, \quad 1 + \sqrt{3}i, \quad 1 - \sqrt{3}i,\]

and there are no more; any other polar form representations of \(-8\) will give repeats of solutions we have obtained already.

Notice that these solutions all lie on the circle with centre 0 and radius 2 (see Figure 3.4), and that the angle between adjacent solutions is \(2\pi/3\). Thus these three cube roots form the vertices of an equilateral triangle. This is a special case of a general result for \(n\)th roots.

\[\text{Figure 3.4} \quad \text{The three cube roots of } -8\]

In the following theorem we use the Greek letters \(\rho\) and \(\phi\) in place of \(r\) and \(\theta\), because \(r\) and \(\theta\) are needed in the proof.

**Theorem 3.1**

Let \(w = \rho(\cos \phi + i \sin \phi)\) be a non-zero complex number in polar form. Then \(w\) has exactly \(n\) \(n\)th roots, given by

\[z_k = \rho^{1/n} \left( \cos \left( \frac{\phi}{n} + \frac{2\pi k}{n} \right) + i \sin \left( \frac{\phi}{n} + \frac{2\pi k}{n} \right) \right),\]

where \(k = 0, 1, \ldots, n - 1\). These roots form the vertices of an \(n\)-sided regular polygon inscribed in the circle of radius \(\rho^{1/n}\) centred at 0.
Proof We seek the solutions of \( z^n = w \) in polar form, 
\[ z = r(\cos \theta + i \sin \theta). \] Since \( w = \rho(\cos \phi + i \sin \phi) \), the equation \( z^n = w \) takes the form 
\[ r^n(\cos \theta + i \sin \theta)^n = \rho(\cos \phi + i \sin \phi); \]
that is,
\[ r^n(\cos n\theta + i \sin n\theta) = \rho(\cos \phi + i \sin \phi), \]
by De Moivre's Theorem.

We now determine \( r \) and \( \theta \) by ‘equating moduli and arguments’. That is, we equate the moduli of both sides, and use the fact that the arguments of the two sides differ by an integer multiple of \( 2\pi \), to obtain
\[ r^n = \rho \quad \text{and} \quad n\theta = \phi + 2k\pi, \quad \text{where} \ k \in \mathbb{Z}. \]

Since \( r \) is non-negative, the only possible value of \( r \) is \( \rho^{1/n} \) (recall that for \( a \geq 0 \), \( a^{1/n} \) means the non-negative \( n \)th root of \( a \)), and the only possible values of \( \theta \) are
\[ \theta = \frac{\phi}{n} + k\frac{2\pi}{n}, \quad \text{where} \ k \in \mathbb{Z}. \]

Hence the solutions of \( z^n = w \) are all of the form
\[ z_k = \rho^{1/n} \left( \cos \left( \frac{\phi}{n} + k\frac{2\pi}{n} \right) + i \sin \left( \frac{\phi}{n} + k\frac{2\pi}{n} \right) \right), \quad \text{where} \ k \in \mathbb{Z}. \]

At first sight, it might appear that we have found infinitely many solutions, one for each value of \( k \). However, not all these solutions are distinct. Indeed, if \( k_1 \) and \( k_2 \) differ by an integer multiple of \( n \), say
\[ k_2 = k_1 + mn, \quad \text{where} \ m \in \mathbb{Z}, \]
then
\[ \frac{\phi}{n} + k_2\frac{2\pi}{n} = \frac{\phi}{n} + (k_1 + mn)\frac{2\pi}{n} = \left( \frac{\phi}{n} + k_1\frac{2\pi}{n} \right) + 2\pi m, \]
so
\[ \frac{\phi}{n} + k_2\frac{2\pi}{n} \quad \text{and} \quad \frac{\phi}{n} + k_1\frac{2\pi}{n} \]
differ by an integer multiple of \( 2\pi \).

Hence the solutions arising from \( k_1 \) and \( k_2 \) are identical. So all possible solutions of \( z^n = w \) arise from the integers \( k = 0, 1, \ldots, n - 1 \). These \( n \) solutions are clearly distinct, since they lie on the circle of radius \( \rho^{1/n} \) centred at 0, with the angle \( 2\pi/n \) between adjacent solutions. Thus they do form the vertices of a regular \( n \)-sided polygon. (Figure 3.5 illustrates this in the case \( n = 6 \).)

\[ \Box \]

If \( w = \rho(\cos \phi + i \sin \phi) \), where \( \phi \) is the principal argument of \( w \), then
\[ z_0 = \rho^{1/n} \left( \cos \frac{\phi}{n} + i \sin \frac{\phi}{n} \right). \]
is called the **principal $n$th root** of $w$, denoted by $\sqrt[n]{w}$ or $w^{1/n}$. In particular, the principal square root of $w$ is denoted by $\sqrt{w}$ or $w^{1/2}$. Note that if $w$ is a positive real number (with principal argument $\phi = 0$), then the principal $n$th root of $w$ also has argument 0, so it is positive. Hence this use of the notation $\sqrt[n]{w}$ is consistent with the familiar real case. This consistency is taken further because for $0 \in \mathbb{C}$, $\sqrt[0]{0}$ or $0^{1/n}$ is defined to be 0.

A particularly important case of Theorem 3.1 occurs when $w = 1$, so $\rho = 1$ and $\phi = 0$.

**Corollary**

The number 1 has exactly $n$ $n$th roots, given by

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, \ldots, n - 1.$$  

These are called the **$n$th roots of unity**.

Note that $z_0 = 1$ is the principal $n$th root of unity for each $n$.

The $n$th roots of unity lie on the circle of radius 1 centred at 0, with the angle $2\pi/n$ between adjacent roots. The cases $n = 2, 3, 4$ are illustrated in Figure 3.6.

**Figure 3.6** The $n$th roots of unity for (a) $n = 2$, (b) $n = 3$, (c) $n = 4$

**Example 3.2**

Determine the fourth roots of $-8 + 8\sqrt{3}i$ in polar and Cartesian forms, plot them in the complex plane, and specify the principal fourth root.

**Solution**

Since

$$|-8 + 8\sqrt{3}i| = |8||-1 + \sqrt{3}i| = 8\sqrt{(-1)^2 + (\sqrt{3})^2} = 16$$

and $-8 + 8\sqrt{3}i$ has principal argument

$$\pi - \tan^{-1}\frac{8\sqrt{3}}{8} = \pi - \frac{\pi}{3} = \frac{2\pi}{3},$$
we deduce that
\[-8 + 8\sqrt{3}i = 16\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\].

Hence, by Theorem 3.1 with \(\rho = 16\) and \(\phi = 2\pi/3\), the four fourth roots of \(-8 + 8\sqrt{3}i\) are
\[
z_k = 16^{1/4}\left(\cos \left(\frac{2\pi/3}{4} + \frac{k \cdot 2\pi}{4}\right) + i \sin \left(\frac{2\pi/3}{4} + \frac{k \cdot 2\pi}{4}\right)\right)
\]
\[= 2\left(\cos \left(\frac{\pi}{6} + \frac{k \cdot \pi}{2}\right) + i \sin \left(\frac{\pi}{6} + \frac{k \cdot \pi}{2}\right)\right), \quad k = 0, 1, 2, 3.\]

Thus the arguments of the four fourth roots of \(-8 + 8\sqrt{3}i\) are
\[
\frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{7\pi}{6}, \quad \frac{5\pi}{3},
\]
so the polar and Cartesian forms of the fourth roots are as given below.

<table>
<thead>
<tr>
<th>(z_k)</th>
<th>Polar form</th>
<th>Cartesian form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_0)</td>
<td>2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})</td>
<td>(\sqrt{3} + i)</td>
</tr>
<tr>
<td>(z_1)</td>
<td>2(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})</td>
<td>(-1 + \sqrt{3}i)</td>
</tr>
<tr>
<td>(z_2)</td>
<td>2(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6})</td>
<td>(-\sqrt{3} - i)</td>
</tr>
<tr>
<td>(z_3)</td>
<td>2(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})</td>
<td>(1 - \sqrt{3}i)</td>
</tr>
</tbody>
</table>

The fourth roots are plotted in Figure 3.7.

Since the principal argument of \(-8 + 8\sqrt{3}i\) is \(2\pi/3\), its principal fourth root is \(z_0 = \sqrt{3} + i\).

The solution above illustrates the following strategy.

**Strategy for finding \(n\)th roots**

To find the \(n\) \(n\)th roots \(z_0, z_1, \ldots, z_{n-1}\) of a non-zero complex number \(w\), apply the following steps.

1. Express \(w\) in polar form, with modulus \(\rho\) and argument \(\phi\).
2. Substitute the values of \(\rho\) and \(\phi\) in the formula
   \[
z_k = \rho^{1/n}\left(\cos \left(\frac{\phi}{n} + \frac{k \cdot 2\pi}{n}\right) + i \sin \left(\frac{\phi}{n} + \frac{k \cdot 2\pi}{n}\right)\right),
   \]
   where \(k = 0, 1, \ldots, n - 1\).
3. Convert the roots to Cartesian form, if required.
Remarks

1. In the first step you should normally choose \( \phi \) to be \( \text{Arg} \ w \) (as in Example 3.2); this has the advantage that the root \( z_0 \) obtained in the second step is the principal \( n \)th root of \( w \).

One disadvantage of this choice is the appearance of minus signs when \( \text{Arg} \ w \) is negative. This can be avoided by choosing \( \phi \) to be \( \text{Arg} \ w + 2\pi \) (which is positive), but with this choice, \( z_0 \) in step 2 will not be the principal \( n \)th root of \( w \), which has to be identified separately. (See the solution to Exercise 3.2(b).)

2. Where possible you should try to use the fact that the \( n \)th roots of \( w \) form a regular \( n \)-sided polygon to check your calculation of \( n \)th roots. For example, in Example 3.2 note that

\[
z_1 = iz_0, \quad z_2 = i^2z_0 = -z_0 \quad \text{and} \quad z_3 = i^3z_0 = -iz_0,
\]

corresponding to the fact that multiplying \( z \) by \( i \) rotates \( z \) about 0 through \( \pi/2 \) anticlockwise.

Exercise 3.2

(a) Determine the cube roots of \( 8i \) in Cartesian form, plot them in the complex plane, and specify the principal cube root.

(b) Determine the sixth roots of \( -i \) in polar form, plot them in the complex plane, and specify the principal sixth root.

Exercise 3.3

(a) Use the Geometric Series Identity to prove that if \( z \) is an \( n \)th root of unity (\( n \geq 2 \)) and \( z \neq 1 \), then

\[
1 + z + z^2 + \cdots + z^{n-1} = 0.
\]

(b) Deduce from part (a) that the \( n \) \( n \)th roots of unity have sum 0.

The result of Exercise 3.3 has the following physical interpretation. Consider \( n \) identical point masses distributed evenly around a circle in a plane, at positions marked by the \( n \)th roots of unity. Then the centre of mass of this collection of point masses is at the origin.

3.2 Solutions of polynomial equations

The quadratic equation

\[
a z^2 + b z + c = 0,
\]

where \( a, b, c \) are complex numbers and \( a \neq 0 \), can be solved by the methods for solving real quadratic equations. For example, we may be able to factorise the quadratic expression, as in the following cases:

\[
z^2 + 9 = (z - 3i)(z + 3i) = 0, \quad \text{so} \quad z = \pm 3i,
\]
and
\[ z^2 + (1 - i)z - i = (z + 1)(z - i) = 0, \quad \text{so } z = -1, i. \]

If there is no easy factorisation, then the formula
\[ z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.1} \]
can be used. The justification of this formula (by completing the square and rearranging) is identical to that in the real case.

**Exercise 3.4**

Solve the following equations.

(a) \[ z^2 - 7iz + 8 = 0 \]
(b) \[ z^2 + 2z + 1 - i = 0 \]

In the previous subsection we saw how to find the \( n \) solutions of the equation
\[ z^n - w = 0. \]

However, it is only in exceptional cases that we can find an explicit algebraic solution of the polynomial equation (of degree \( n \geq 3 \))
\[ a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \]
where \( a_0, a_1, \ldots, a_n \) are complex numbers and \( a_n \neq 0 \). For example, it may be possible to reduce a given polynomial equation to a quadratic equation by making a substitution, as in the next example.

**Example 3.3**

Solve the equation
\[ z^4 + 4z^2 + 8 = 0. \]

**Solution**

Substituting \( w = z^2 \) gives
\[ w^2 + 4w + 8 = 0, \]
which has solutions
\[ w = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i. \]

Thus \( z = \pm \sqrt{-2 + 2i} \) or \( z = \pm \sqrt{-2 - 2i} \).

Since
\[ -2 + 2i = \sqrt{8} (\cos 3\pi/4 + i \sin 3\pi/4), \]
we have
\[ \sqrt{-2 + 2i} = 8^{1/4} (\cos 3\pi/8 + i \sin 3\pi/8). \]
Solving equations with complex numbers

(This is the principal square root of $-2 + 2i$, because $3\pi/4$ is the principal argument of $-2 + 2i$.) Thus two solutions of $z^4 + 4z^2 + 8 = 0$ are $\pm 8^{1/4}(\cos 3\pi/8 + i \sin 3\pi/8)$.

Similarly, since

$$-2 - 2i = \sqrt{8}(\cos(-3\pi/4) + i \sin(-3\pi/4)),$$

we have

$$\sqrt{-2 - 2i} = 8^{1/4}(\cos(-3\pi/8) + i \sin(-3\pi/8));$$

thus two further solutions are $\pm 8^{1/4}(\cos(-3\pi/8) + i \sin(-3\pi/8))$.

So the four solutions are

$$\pm 8^{1/4}(\cos 3\pi/8 + i \sin 3\pi/8)$$

and

$$\pm 8^{1/4}(\cos(-3\pi/8) + i \sin(-3\pi/8)).$$

**Remark**

The solutions in Example 3.3 are presented using the $\pm$ notation to indicate the relationship between them. (As a consequence, they are not all in polar form.) In fact, since $\cos(-3\pi/8) = \cos 3\pi/8$ and $\sin(-3\pi/8) = -\sin 3\pi/8$, the four solutions form two complex conjugate pairs. It can be shown that non-real roots of a polynomial equation with real coefficients must occur in complex conjugate pairs; you will see this in Exercise 3.9.

**Exercise 3.5**

(a) Solve the equation

$$z^6 - 7iz^3 + 8 = 0.$$  

(*Hint:* Use Exercise 3.4(a) and then Exercise 3.2(a). Also, you may find the following fact useful: if $z$ is a cube root of $8i$, then $-\frac{1}{2}z$ is a cube root of $-i$.)

(b) Solve the equation

$$z^4 + 4iz^2 + 8 = 0.$$
Further exercises

Exercise 3.6

For each of the following complex numbers determine, in Cartesian form where convenient, the $n$th roots indicated, and plot them. In each case specify the principal $n$th root.

(a) The square roots of
   (i) $-i$  (ii) $4i$.

(b) The cube roots of
   (i) $-1$  (ii) $-2 + 2i$.

(c) The fourth roots of
   (i) $\frac{1}{\sqrt{2}}(-1 - i)$  (ii) $-1 + i$.

(d) The fifth roots of
   (i) $-1$  (ii) $-16 + 16\sqrt{3}i$.

Exercise 3.7

Use the method of equating real parts and imaginary parts to solve each of the following equations.

(a) $(x + iy)^2 = 3 + 4i$  (b) $(x + iy)^2 = -5 + 12i$

Exercise 3.8

Solve each of the following equations, and plot their solutions.

(a) $z^4 - z^2 + 1 + i = 0$  (b) $z^3 - 4z^2 + 6z - 4 = 0$

Exercise 3.9

Let $p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers. Prove that if $z$ satisfies $p(z) = 0$, then $p(\overline{z}) = 0$.

(This shows that non-real roots of a polynomial equation with real coefficients must occur in complex conjugate pairs.)
4 Sets of complex numbers

After working through this section, you should be able to:

- understand the meaning of an inequality between real expressions involving complex numbers
- understand the specification of subsets of the complex plane in terms of such inequalities
- recognise certain basic open and closed sets.

4.1 Inequalities

Throughout the module we will use many inequalities involving complex numbers, and you will need to become adept at interpreting them. Here are some simple inequalities involving a complex number $z$ and some examples of values of $z$ for which they are true (√) or false (×).

<table>
<thead>
<tr>
<th></th>
<th>$1 + i$</th>
<th>$2 - i$</th>
<th>$-\frac{1}{2} + \frac{1}{2}i$</th>
<th>$-1 - 3i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Re } z &gt; 1$</td>
<td>×</td>
<td>✓</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>$</td>
<td>z</td>
<td>\leq 1$</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>$</td>
<td>\text{Im } z</td>
<td>&gt; 2$</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>$\text{Arg } z &lt; \pi/2$</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
</tbody>
</table>

Notice that these four inequalities are all between expressions that are real-valued. We never write inequalities between complex-valued expressions such as $2 + i$ or $z^2 + 1$.

The inequalities

$$z_1 < z_2 \ \text{and} \ \ z_1 \leq z_2$$

have no meaning unless both $z_1$ and $z_2$ are real.

The reason why we can use inequalities with real numbers but not with complex numbers is because $\mathbb{R}$ is an ordered field, but $\mathbb{C}$ is not.

Exercise 4.1

Complete the following true/false table.

<table>
<thead>
<tr>
<th></th>
<th>$1 + 2i$</th>
<th>$-1 - 2i$</th>
<th>$i$</th>
<th>$-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Re } z &lt; 0$</td>
<td>×</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>z</td>
<td>&gt; 2$</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>$</td>
<td>\text{Im } z</td>
<td>\leq -1$</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>$\text{Arg } z \geq 0$</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\( \mathbb{C} \) is not an ordered field

Roughly speaking, an ordered field is a field whose elements can be ordered using inequalities which satisfy certain rules of the type that you will meet in Section 5. Here is a quick and informal explanation of why \( \mathbb{C} \) is not an ordered field.

If \( \mathbb{C} \) were an ordered field, then we would have either \( i > 0 \) or \( i < 0 \). In the first case, after multiplying both sides by the ‘positive’ number \( i \), we obtain \( i^2 > 0 \), or \( -1 > 0 \), which is a contradiction! There is a similar argument for the second case.

4.2 Sketching subsets of the complex plane

In this subsection we will use set notation to describe various subsets of the complex plane. Before we do so, it will be helpful to recall some conventions from real analysis for representing intervals. For example, the open interval with endpoints 1,3 is

\[ (1,3) = \{ x : 1 < x < 3 \} , \]

and the closed interval with endpoints −2,2 is

\[ [-2,2] = \{ x : -2 \leq x \leq 2 \} . \]

(Here \( x \) is a real variable.) Intervals such as

\[ (-\pi, \pi) = \{ x : -\pi < x \leq \pi \} \quad \text{or} \quad [-\pi, \pi) = \{ x : -\pi \leq x < \pi \} \]

are called half-open (or half-closed), and it is often convenient to use unbounded open and closed intervals, such as

\[ (0, \infty) = \{ x : x > 0 \} \quad \text{(open)} \]

and

\[ [1, \infty) = \{ x : x \geq 1 \} \quad \text{(closed)} . \]

Let us now proceed with sketching in the complex plane. We begin with the example of two straight lines, illustrated in Figure 4.1.

\[ \begin{align*}
(1) & \quad \{ z : \Re z = 1 \} , \\
(2) & \quad \{ z : \Re z + 2 \Im z = 3 \} 
\end{align*} \]

Figure 4.1 Two straight lines: (a) \( \{ z : \Re z = 1 \} \), (b) \( \{ z : \Re z + 2 \Im z = 3 \} \)
Figure 4.1(a) depicts the vertical straight line with equation \( x = 1 \). If we were working with Cartesian coordinates \((x, y)\), then that line would be made up of all points \((x, y)\) with \(x = 1\). However, we are working in the complex plane, so the line consists of the set of complex numbers \( z = x + iy \) such that \(x = 1\). This set can be described in set notation as

\[
\{ z = x + iy : x = 1 \}.
\]

We can use the formula \( x = \text{Re} z \) to write this set without \(x\) and \(y\) as

\[
\{ z : \text{Re} z = 1 \}.
\]

Similarly, the line in Figure 4.1(b) can be described as the set

\[
\{ z = x + iy : x + 2y = 3 \}.
\]

Using the formulas \( x = \text{Re} z \) and \( y = \text{Im} z \), we can write this set as

\[
\{ z : \text{Re} z + 2 \text{Im} z = 3 \}.
\]

Consider now the sets represented by shading in Figure 4.2. Each set comprises all points lying to one side of a straight line (possibly including the line itself). Such sets are called half-planes.

**Figure 4.2** (a) Open half-plane \( \{ z : \text{Re} z > 1 \} \) (b) Closed half-plane \( \{ z : \text{Im} z \leq 2 \} \)

The half-plane in Figure 4.2(a) consists of all points that lie to the right of the line \(x = 1\), *excluding* the line itself. In set notation, this set is

\[
\{ z : \text{Re} z > 1 \}.
\]

The boundary line \(x = 1\) is drawn as a broken line, to indicate that it is excluded from the set. The line crosses the \(x\)-axis at the point 1. This point is represented by a hollow dot (a small, empty circle), to indicate that it too is excluded from the set.
The half-plane in Figure 4.2(b) is made up of all points that lie below the line \( y = 2 \), including the line itself. This set is

\[ \{ z : \text{Im} \, z \leq 2 \} \]

This time, the boundary line \( y = 2 \) is drawn as an unbroken line, to show that it is included in the set. Also, the point \( 2i \) on the boundary is shown as a solid dot (a small, filled-in circle), to indicate that it is included in the set.

Since the boundary is not included in the half-plane \( \{ z : \text{Re} \, z > 1 \} \), we describe this half-plane as an open half-plane, in the same way that we describe an interval of the real line as an open interval if it does not include its boundary within the real line. (The boundary of an interval within the real line consists of its endpoints, if it has any.) In contrast, we say that the half-plane \( \{ z : \text{Im} \, z \leq 2 \} \) is a closed half-plane because it does include its boundary – again, this terminology corresponds to the terminology we use for closed intervals.

Two more examples of open and closed half-planes are shown in Figure 4.3.

![Figure 4.3](image)

(a) Open half-plane \( \{ z : \text{Re} \, z + 2 \text{Im} \, z > 3 \} \)

(b) Closed half-plane \( \{ z : \text{Re} \, z - \text{Im} \, z \geq -1 \} \)

The equation of the broken line in Figure 4.3(a) is \( x + 2y = 3 \), or

\[ \text{Re} \, z + 2 \text{Im} \, z = 3. \]

The shaded region represents points \( z \) that lie above this line, excluding the line itself; such points satisfy

\[ \text{Re} \, z + 2 \text{Im} \, z > 3. \]

Points below the broken line satisfy

\[ \text{Re} \, z + 2 \text{Im} \, z < 3. \]

For example, the point \( z = 0 \) lies below the line because

\[ \text{Re} \, 0 + 2 \text{Im} \, 0 = 0 < 3. \]

Figure 4.3(b) displays the half-plane consisting of all points that lie below the line \( x - y = -1 \), including the line itself. After writing this line as

\[ \text{Re} \, z - \text{Im} \, z = -1, \]
we see that points in the half-plane satisfy
\[ \text{Re } z - \text{Im } z \geq -1, \]
and points outside the half-plane satisfy
\[ \text{Re } z - \text{Im } z < -1. \]
For example, the point \( z = 0 \) lies in the half-plane because
\[ \text{Re } 0 - \text{Im } 0 = 0 \geq -1. \]
The half-plane in Figure 4.3(a) is an open half-plane, because the boundary is excluded, whereas the half-plane in Figure 4.3(b) is a closed half-plane, because the boundary is included. The general definitions of open and closed half-planes are as follows.

**Definitions**

An open half-plane is a set of the form
\[ \{ z : a \text{Re } z + b \text{Im } z > c \}, \]
and a closed half-plane is a set of the form
\[ \{ z : a \text{Re } z + b \text{Im } z \geq c \}, \]
where \( a, b, c \in \mathbb{R} \) and \( a, b \) are not both zero.

Notice that an open half-plane can also be written in the form
\[ \{ z : a \text{Re } z + b \text{Im } z < c \} \]
(which is equally valid), by replacing \( a, b \) and \( c \) with their negatives. For example, by multiplying both sides of the inequality \( \text{Re } z > 1 \) by \(-1\), we can write it in the alternative form \(-\text{Re } z < -1\). It follows that the open half-plane shown in Figure 4.2(a) can also be written as
\[ \{ z : -\text{Re } z < -1 \}. \]

Similar comments apply to closed half-planes.

When asked to sketch a half-plane, you should first plot the boundary line \( ax + by = c \), using either a broken line for an open half-plane (given by a strict inequality, \(< \text{or} >\)) or an unbroken line for a closed half-plane (given by a weak inequality, \(\leq \text{or} \geq\)). Then shade in one of the half-planes separated by the line. To determine which half-plane to shade, choose just one point not on the line (for example, 0) and work out whether or not it lies in the set.

Try this procedure in the following exercise.
Exercise 4.2

(a) Sketch the following sets.
   (i) \( \{ z : 2 \text{Re} z - 3 \text{Im} z = -1 \} \)
   (ii) \( \{ z : \text{Re} z - \text{Im} z > 0 \} \)
   (iii) \( \{ z : \text{Re} z + \text{Im} z \leq -1 \} \)

(b) Use set notation to describe the set shaded in the following figure.

The four open half-planes shown in Figure 4.4 are particularly important in complex analysis. The half-planes above and below the real axis are called the **upper half-plane** and **lower half-plane**, respectively, and the half-planes to the left and right of the imaginary axis are called the **left half-plane** and **right half-plane**, respectively.

![Figure 4.4](image)

**Figure 4.4** Four half-planes

Complex numbers can be used to give particularly elegant formulas for circles in the complex plane. Consider the circles shown in Figure 4.5.
The circle $x^2 + y^2 = 1$ in Figure 4.5(a) is centred at 0 and has radius 1. Writing $z = x + iy$, we know that the modulus $|z|$ of $z$ satisfies $|z|^2 = x^2 + y^2$. Therefore the equation of the circle is $|z|^2 = 1$, or, more simply, $|z| = 1$. So the circle is given in set notation as

$$\{ z : |z| = 1 \}.$$

This particular circle is called the **unit circle**, and we will use it often.

We can obtain the equation $|z| = 1$ in another manner by thinking geometrically. The circle consists of those points $z$ in the complex plane that lie at a distance 1 from 0. As the distance from 0 to $z$ is $|z - 0| = |z|$, we again see that the equation of the circle is $|z| = 1$.

Let us apply this geometric method to obtain the equation of the circle with centre $i$ and radius $\frac{1}{2}$ shown in Figure 4.5(b). This circle is made up of those points $z$ that lie at a distance $\frac{1}{2}$ from $i$. The distance from $i$ to $z$ is $|z - i|$, so the equation of the circle is $|z - i| = \frac{1}{2}$. Therefore this circle is given in set notation as

$$\{ z : |z - i| = \frac{1}{2} \}.$$

In this case, the equation of the circle in complex form, namely

$$|z - i| = \frac{1}{2},$$

is more concise than the equation

$$x^2 + (y - 1)^2 = \frac{1}{4}$$

using $x$ and $y$.

More generally, we have the following observation, in which we use the Greek letter $\alpha$ to denote the centre of a circle (a complex number).

The circle with centre $\alpha \in \mathbb{C}$ and radius $r > 0$ can be written as

$$\{ z : |z - \alpha| = r \}.$$
Next we look at how to use complex numbers to represent discs. A **disc** is the set of points inside a circle, possibly including the circle itself. Figure 4.6 shows two discs, both centred at 0.

![Figure 4.6](image)

**Figure 4.6** (a) Open disc \{z : |z| < 1\}  (b) Closed disc \{z : |z| ≤ 2\}

The boundary circle of the disc in Figure 4.6(a) is drawn as a broken curve to indicate that it is not part of the set. It follows that this disc comprises those points that lie less than a distance 1 away from 0, so it has equation \(|z| < 1\). Discs that exclude their boundaries are called *open discs*.

The boundary of the disc in Figure 4.6(b) is drawn as an unbroken curve to show that it is included in the set. This disc consists of points \(z\) that lie a distance less than or equal to 2 away from 0, so it has equation \(|z| ≤ 2\). Discs that include their boundaries are called *closed discs*.

Two more examples of discs are shown in Figure 4.7.

![Figure 4.7](image)

**Figure 4.7** (a) Open disc \(\{z : |z - 1 - i| < 1\}\)  (b) Closed disc \(\{z : |z - i| ≤ \frac{1}{2}\}\)

The disc in Figure 4.7(a) is centred at \(1 + i\) and has radius 1. It is an open disc because the boundary circle, shown as a broken curve, is excluded from the set. In particular, the points \(z = 1\) and \(z = i\) where the circle touches the axes are not in the set. The disc is made up of those points \(z\)
that lie less than a distance 1 away from the centre $1 + i$, so it has equation

$$|z - (1 + i)| < 1, \quad \text{or} \quad |z - 1 - i| < 1.$$ 

The disc in Figure 4.7(b) with centre $i$ and radius $\frac{1}{2}$ is a closed disc, because the boundary circle, shown as an unbroken curve, is included in the set. For example, the points $\frac{1}{2}i$ and $\frac{3}{2}i$ where the circle intersects the imaginary axis are both included in the set. This disc has equation

$$|z - i| \leq \frac{1}{2}.$$ 

More generally, we have the following definitions of open and closed discs using set notation.

**Definitions**

An **open disc** is a set of the form

$$\{z : |z - \alpha| < r\},$$

and a **closed disc** is a set of the form

$$\{z : |z - \alpha| \leq r\},$$

where $\alpha \in \mathbb{C}$ is the centre of the disc and $r > 0$ is the radius.

**Exercise 4.3**

(a) Sketch the following sets.

(i) $\{z : |z - 1 + 2i| = 1\}$
(ii) $\{z : |z - 1 + 2i| < 1\}$
(iii) $\{z : |z + 2 - 3i| \leq 3\}$

(b) Use set notation to describe the set shaded in the following figure, which is a disc with centre $-1 - i$.

Next we will look at various other sets that are related to circles and discs. Figure 4.8 shows two shaded sets, each the outside of a disc.
The set in Figure 4.8(a) comprises those points that lie outside the circle $|z| = 1$, excluding the circle itself (because the boundary is drawn as a broken curve). Points in this set lie at a distance greater than 1 from 0, so the set is made up of points $z$ that satisfy the inequality $|z| > 1$.

The set in Figure 4.8(b) is made up of those points that lie outside the circle centred at $1 + i$ of radius 1, which, as you saw earlier, has equation $|z - 1 - i| = 1$. In this case, the set includes the circle itself, because the boundary is drawn as an unbroken curve. Points in this set lie at a distance greater than or equal to 1 from $1 + i$, so the set is made up of points $z$ that satisfy the inequality $|z - 1 - i| \geq 1$.

In both these figures, the centres of the circles are marked by solid dots. We use solid dots rather than hollow dots, even though the centres do not belong to the sets, because hollow dots are reserved for points on the boundary of a set that are excluded from that set.

Figure 4.9 introduces a new type of set called an annulus (the plural is annuli). An annulus is the set between two concentric circles, possibly including one or both of the boundary circles.
The annulus in Figure 4.9(a) consists of those points that lie strictly between the circles given by the equations $|z| = 1$ and $|z| = 2$. Both boundary circles are excluded, so the annulus is the set of points $z$ that satisfy the inequalities

$$1 < |z| < 2.$$ 

An annulus that excludes both its boundary circles is called an **open annulus**.

The annulus in Figure 4.9(b) has boundary circles given by the equations $|z - i| = \frac{1}{3}$ and $|z - i| = \frac{2}{3}$. This time, both circles are included in the set, so this annulus is the set of points $z$ that satisfy the inequalities

$$\frac{1}{3} \leq |z - i| \leq \frac{2}{3}.$$ 

An annulus that includes both its boundary circles is called a **closed annulus**.

**Definitions**

An **open annulus** is a set of the form

$$\{z : r_1 < |z - \alpha| < r_2\},$$

and a **closed annulus** is a set of the form

$$\{z : r_1 \leq |z - \alpha| \leq r_2\},$$

where $\alpha \in \mathbb{C}$ is the centre of the annulus and $r_2 > r_1 > 0$ are the radii of the boundary circles.

Some annuli include one boundary circle but not the other; such annuli are not referred to as either open or closed.

Figure 4.10 shows two **punctured discs**; these are discs from which the centre points have been removed. In each diagram, the centre point is indicated by a hollow dot, because it lies on the boundary of the set, but is not included in the set.

![Punctured discs](image)

**Figure 4.10** Punctured discs: (a) $\{z : 0 < |z| < 1\}$, (b) $\{z : 0 < |z + i| \leq \frac{1}{2}\}$
Unit A1  Complex numbers

Figure 4.10(a) is a punctured open disc. It consists of points in the disc \( \{ z : |z| < 1 \} \), except the centre 0. Therefore this punctured disc is the set of points \( z \) that satisfy the inequalities
\[
0 < |z| < 1.
\]

Figure 4.10(b) is a punctured closed disc. It is the set of points \( z \) that satisfy the inequalities
\[
0 < |z + i| \leq \frac{1}{2}.
\]

The next exercise gives you practice at sketching sets related to circles and discs. 

**Exercise 4.4**

(a) Sketch the following sets.
(i) \( \{ z : |z + i| > \frac{1}{2} \} \)
(ii) \( \{ z : \frac{1}{2} \leq |z + 1| < 2 \} \)
(iii) \( \{ z : 2 \leq |z + 2 - 3i| \leq 3 \} \)

(b) Use set notation to describe the set shaded in the following figure, which is a punctured disc with centre \( 1 - 2i \).

Each of the two diagrams in Figure 4.11 displays a ray or half-line, which is half a straight line with its endpoint missing (indicated by a hollow dot). 

**Figure 4.11**  Rays: (a) \( \{ z : \text{Arg} \ z = \pi/4 \} \), (b) \( \{ z : \text{Arg}(z - 1 - i) = \pi/4 \} \)
The ray in Figure 4.11(a) comprises those points of the complex plane with principal argument equal to $\pi/4$, so it has equation

$$\text{Arg } z = \pi/4.$$ 

For Figure 4.11(b), we see that a point $z$ lies on this ray if and only if the point $z - (1 + i)$ lies on the ray in Figure 4.11(a). As $z - (1 + i) = z - 1 - i$, it follows that the equation of Figure 4.11(b) is

$$\text{Arg}(z - 1 - i) = \pi/4.$$ 

More generally, we have the following definition of a ray.

**Definition**

A ray or half-line is a set of the form

$$\{z : \text{Arg}(z - \alpha) = \theta\},$$

where $\alpha \in \mathbb{C}$ and $-\pi < \theta \leq \pi$.

Figure 4.12 shows two examples of sets that we call sectors. A sector is a set bounded by two rays that share a common endpoint. The sector may or may not include one or both of the boundary rays.

![Figure 4.12 Sectors](image.png)

(a) \hspace{1cm} (b)

**Figure 4.12** Sectors: (a) $\{z : \pi/4 < \text{Arg } z < 3\pi/4\}$, (b) $\{z : |\text{Arg } z| \geq 5\pi/6\}$

The sector in Figure 4.12(a) consists of all points whose principal argument lies strictly between $\pi/4$ and $3\pi/4$, so it is the set of points $z$ for which

$$\pi/4 < \text{Arg } z < 3\pi/4.$$ 

This set is called an open sector because the boundary is excluded.

The sector in Figure 4.12(b) is made up of those points whose principal argument is either less than or equal to $-5\pi/6$, or greater than or equal to $5\pi/6$. That is, the sector contains those points $z$ that satisfy either

$$\text{Arg } z \leq -5\pi/6 \quad \text{or} \quad \text{Arg } z \geq 5\pi/6.$$ 

We can write these two inequalities as the single inequality

$$|\text{Arg } z| \geq 5\pi/6.$$
The sector in Figure 4.12(b) is not called a ‘closed sector’ because even though the boundary rays are included in the sector, the point 0, which also lies on the boundary, is excluded.

The four quadrants of the complex plane (defined in Subsection 2.1) are open sectors. For example, the upper-right quadrant is the set

\[ \{ z : 0 < \text{Arg} z < \pi/2 \}. \]

Two more sectors are shown in Figure 4.13. In each of these sectors the boundary rays meet at vertices away from the origin.

Figure 4.13  Open sectors:  (a) \( \{ z : |\text{Arg}(z - 1)| < 3\pi/4 \} \),
(b) \( \{ z : \text{Arg}(z - 1 - i) < -\pi/2 \text{ or } \text{Arg}(z - 1 - i) > 2\pi/3 \} \)

To find the inequality for the sector in Figure 4.13(a), first consider the set that is obtained by translating this sector by one unit to the left. This new set is itself a sector, comprising points \( z \) that satisfy the inequality

\[ |\text{Arg} z| < 3\pi/4. \]

Under the translation, the point \( z \) is moved to the point \( z - 1 \), so it follows that the original sector comprises points \( z \) that satisfy the inequality

\[ |\text{Arg}(z - 1)| < 3\pi/4. \]

Reasoning in a similar way, we find that the sector in Figure 4.13(b) consists of points \( z \) that satisfy either

\[ \text{Arg}(z - 1 - i) < -\pi/2 \text{ or } \text{Arg}(z - 1 - i) > 2\pi/3. \]

Both sectors in Figure 4.13 are open sectors. The general definition of an open sector is as follows.

**Definition**

An **open sector** is a set of one of the forms

\[ \{ z : a < \text{Arg}(z - \alpha) < b \} \]

or

\[ \{ z : \text{Arg}(z - \alpha) < a \text{ or } \text{Arg}(z - \alpha) > b \}, \]

where \( \alpha \in \mathbb{C} \) and \( -\pi < a < b \leq \pi. \)
The complex number $\alpha$ in this definition is the location of the vertex of the sector, and $a$ and $b$ are real numbers that determine the angles of the boundary rays.

**Exercise 4.5**

(a) Sketch the following sets.

(i) $\{ z : \operatorname{Arg} z = -2\pi/3 \}$  
(ii) $\{ z : \operatorname{Arg}(z - i) = 3\pi/4 \}$  
(iii) $\{ z : |\operatorname{Arg} z| < 2\pi/3 \}$

(b) Use set notation to describe the set shaded in the following figure.

![Diagram](image)

We can use standard operations on sets to create new subsets of the complex plane from old ones. Three of these operations are defined below.

**Definitions**

Let $A$ and $B$ be subsets of the complex plane.

The **union** of $A$ and $B$ is

$$A \cup B = \{ z : z \in A \text{ or } z \in B \}. $$

The **intersection** of $A$ and $B$ is

$$A \cap B = \{ z : z \in A \text{ and } z \in B \}. $$

The **difference** of $A$ and $B$ is

$$A - B = \{ z : z \in A \text{ and } z \notin B \}. $$

**Remarks**

1. The set $A \cap B$ is also commonly written as $\{ z : z \in A, z \in B \}$; the comma takes the place of the word ‘and’.
2. The difference $A - B$ should be read as ‘$A$ minus $B$’.
These operations are illustrated by the Venn diagrams in Figure 4.14; these are abstract depictions of the operations. (They are not sketches in the complex plane.) The sets $A$ and $B$ are enclosed by the circles in each diagram, and the shaded parts represent $A \cup B$, $A \cap B$ and $A - B$, respectively.

![Venn diagrams](image)

**Figure 4.14** Venn diagrams

Notice that, as shown in Figure 4.15, the three sets $A - B$, $A \cap B$ and $B - A$ are mutually disjoint, meaning that no two of them have any points in common, and their union is $A \cup B$.

We now examine the effects of these operations on the subsets $A$ and $B$ of the complex plane shown in Figure 4.16.

![Disjoint sets](image)

**Figure 4.15** Disjoint sets $A - B$, $A \cap B$ and $B - A$

$A = \{z : \text{Re } z > 1\}$

$B = \{z : |z - 1| \leq 1\}$

![An open half-plane A and a closed disc B](image)

**Figure 4.16** An open half-plane $A$ and a closed disc $B$

The sets $A \cup B$, $A \cap B$ and $A - B$ are displayed in Figure 4.17.

![The sets A U B, A intersect B and A - B](image)

**Figure 4.17** The sets $A \cup B$, $A \cap B$ and $A - B
Since $A$ is determined by the single condition $\text{Re} \, z > 1$, and $B$ is determined by the single condition $|z - 1| \leq 1$, the set $A \cup B$ consists of points $z$ for which $\text{Re} \, z > 1$ or $|z - 1| \leq 1$. The set $A \cap B$ is made up of points that satisfy both conditions, $\text{Re} \, z > 1$ and $|z - 1| \leq 1$. Finally, the set $A - B$ comprises points $z$ for which the condition $\text{Re} \, z > 1$ is true and the condition $|z - 1| \leq 1$ is false. That is, $A - B$ is determined by the inequalities $\text{Re} \, z > 1$ and $|z - 1| > 1$.

We finish here by discussing *complements* of subsets of the complex plane.

**Definition**

The **complement** of a subset $A$ of the complex plane is the set $\mathbb{C} - A$ of all the points of $\mathbb{C}$ that are *not* in $A$.

We have already seen some examples of complements. For instance, Figure 4.8(a) illustrates the complement of a closed disc, and Figure 4.8(b) illustrates the complement of an open disc. Figure 4.18 provides two more examples.

![Figure 4.18](image)

**Figure 4.18**  (a) A punctured plane $\mathbb{C} - \{1 + i\} = \{z : |z - 1 - i| > 0\}$
(b) The cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} = \{z : |\text{Arg} \, z| < \pi\}$

Figure 4.18(a) is the complex plane with a single point removed; such sets are called **punctured planes**. The punctured plane in Figure 4.18(a) is the complement of the single-point set $\{1 + i\}$. Figure 4.18(b) is the complex plane with the negative real axis and 0 removed. This set can be described by the equation $|\text{Arg} \, z| < \pi$; we call it a *cut plane*.

**Definition**

A **cut plane** is the complex plane $\mathbb{C}$ with a half-line from the origin and the origin itself removed.

In particular, the set $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ is a cut plane, and this set can also be specified as $\{z : |\text{Arg} \, z| < \pi\}$.

Note that some texts use the phrase *slit plane* instead of *cut plane*.
The next exercise gives you practice at applying set operations.

**Exercise 4.6**

Let

\[ A = \{ z : |\text{Re } z| < 1 \} \quad \text{and} \quad B = \{ z : |z| \leq 2 \}. \]

(a) Sketch the following sets (on separate diagrams).

(i) \( A \)  
(ii) \( B \)  
(iii) \( A \cup B \)  
(iv) \( A \cap B \)  
(v) \( A - B \)  
(vi) \( \mathbb{C} - A \)

(b) Complete the statement \( \mathbb{C} - (A \cup B) = \{ z : \} \).

We summarise the various conventions for sketching subsets of \( \mathbb{C} \).

**Sketching conventions**

- The interior of a set is shown by shading.
- Boundary curves that belong to the set are drawn unbroken.
- Boundary curves that do not belong to the set are drawn broken.
- Distinguished boundary points that belong to the set are drawn as solid dots (small, filled-in circles).
- Distinguished boundary points that do not belong to the set are drawn as hollow dots (small, empty circles).

When sketching, there is some latitude as to which points you should mark on your sketch and whether you should calculate the complex numbers corresponding to those points. As a general rule, you should always mark distinguished boundary points of your sketch, such as points where two boundary curves meet at a corner, or distinguished points that are excluded from the set. You should also include points to help specify exactly what the sketch represents; for example, you may include some intersection points of boundary curves with the axes. However, do not include too many points, or your diagram will appear cluttered. Usually, you should write down the complex number corresponding to a point that you have included, provided that the complex number is relatively straightforward to determine.

These conventions will remain in force throughout the module, although we will not always include shading. When sketching sets by hand you could consider replacing shading by hatching, as shown in Figure 4.19.


**5 Proving inequalities**

After working through this section, you should be able to:

- use the rules for rearranging inequalities and the rules for obtaining new inequalities from old ones
- prove inequalities involving the moduli of complex numbers by using various forms of the Triangle Inequality.

---

**Exercise 4.7**

Sketch the following sets, using the sketching conventions.

(a) \{z : \text{Im} z > 0\}

(b) \{z : |z + 1| \leq 1\}

(c) \{z : 0 < |z + 1 + 2i| < 1\}

(d) \{z : |\text{Arg}(z + 1 - i)| < \pi/3\}

(e) \{z : |z - 1| \leq |z - 2|\}

(f) \mathbb{C} - \{z : \text{Re} z \geq 1\}

(g) \{z : \text{Im} z > 0\} - \{z : |z + 1| \leq 1\}

(h) \{z : \text{Arg} z = \pi/6\} \cup \{z : \text{Arg}(z - \sqrt{3} - i) = 0\}

(i) \{z : \text{Arg} z = \pi/6\} \cap \{z : \text{Arg}(z - \sqrt{3} - i) = 0\}

(*Hint*: For part (e), interpret the inequality in terms of distances.)

---

**Further exercises**

**Exercise 4.8**

Sketch the following sets, using the sketching conventions.

(a) \{z : |\text{Re} z| < 1, |\text{Im} z| < 1\}

(b) \{z : |z - i| \leq 2, |z| \geq 1\}

(c) \{z : \text{Re} z + 2 \text{Im} z + 3 > 0\}

(d) \{z : \text{Re} z \geq 0\} \cup \{z : \text{Im} z > 0\}

(e) \{z : |z| > 1, |\text{Arg} z| \leq \pi/4\}

(f) \{z : |z + 1 + 2i| \leq 1\}

(g) \{z : \text{Re} z > 1, |z - i| < 2\}

(h) \{z : |z + i| < |z + 2i|\}

(i) \{z : |z| < 3\} - \{z : |z| \leq 2\}

(j) \mathbb{C} - \{z : z^2 + z - 2 = 0\}
5.1 Rules for rearranging inequalities

In Section 4 we used equalities and inequalities to define subsets of the complex plane. In this section you will see how to prove new inequalities by deducing them from simpler known inequalities (such as $|z| \geq 0$, which holds for all $z$) using various rules. We begin by reminding you of the rules for rearranging a given inequality into an\textit{ equivalent} form; such equivalent inequalities are linked by the symbol ‘$\iff$’, which may be read as ‘is equivalent to’ or ‘if and only if’.

### Rules for Rearranging Inequalities

For all $a, b, c \in \mathbb{R}$, the following rules apply.

**Rule 1** \quad $a < b \iff b - a > 0$.

**Rule 2** \quad $a < b \iff a + c < b + c$.

**Rule 3** If $c > 0$, then $a < b \iff ac < bc$.

If $c < 0$, then $a < b \iff ac > bc$.

**Rule 4** If $a, b > 0$, then $a < b \iff \frac{1}{a} > \frac{1}{b}$.

**Rule 5** If $a, b \geq 0$ and $p > 0$, then $a < b \iff a^p < b^p$.

**Rule 6** $|a| < b \iff -b < a < b$.

There are corresponding versions of Rules 1–6 in which the strict inequality ‘$<$’ is replaced by the weak inequality ‘$\leq$’.

The next two rules can be used to deduce new inequalities from given ones. Here, however, the new inequalities are not equivalent to the old ones, since the old inequalities cannot be deduced from the new ones. Such deductions are written using the symbol ‘$\implies$’, which may be read as ‘implies’.

### Transitive Rule

For all $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c.$$  

For example, if $x < 2$, then $x < 3$ (because $2 < 3$).

### Combination Rules for Inequalities

For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, the following rules apply.

(a) **Sum Rule** \quad $a + c < b + d$.

(b) **Product Rule** \quad $ac < bd$ \quad (provided that $a, c \geq 0$).
For example, if $0 \leq n < 5$, then (since $2 < 3$),

$$n + 2 < 5 + 3 = 8 \quad \text{and} \quad 2n < 3 \times 5 = 15.$$ 

There are also weak and weak/strict versions of the Transitive Rule and the Combination Rules, which you should be able to work out as they arise.

The following example illustrates how the various rules are used in practice.

**Example 5.1**

Prove that

$$2r^2 > (r + 1)^2, \quad \text{for } r \geq 3.$$ 

**Solution**

We rearrange the given inequality in order to find an equivalent, but simpler one:

\[
2r^2 > (r + 1)^2 \iff 2 > \left( \frac{r + 1}{r} \right)^2 \quad \text{(Rule 3)} \\
\iff \sqrt{2} > 1 + \frac{1}{r} \quad \text{(Rule 5)} \\
\iff \sqrt{2} - 1 > \frac{1}{r} \quad \text{(Rule 2)} \\
\iff r > \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 \quad \text{(Rule 4)}.
\]

Since we are given that $r \geq 3$, and $3 > \sqrt{2} + 1 = 2.414 \ldots$, it follows from the Transitive Rule that $r > \sqrt{2} + 1$. Therefore the final inequality is true, so the first inequality must be true for $r \geq 3$ also.

**Remarks**

1. Example 5.1 could be solved, alternatively, by using Rule 1 to obtain the equivalent inequality $r^2 - 2r - 1 > 0$ and then completing the square. There is often more than one way to deal with a given inequality.

2. In future we will not usually indicate which rule for rearranging a given inequality is being used.

**Exercise 5.1**

Prove that

$$\frac{3r}{r^2 + 2} < 1, \quad \text{for } r > 2.$$
5.2 The Triangle Inequality

Many inequalities have a geometric interpretation. For example, the two inequalities

\[ |x| \leq \sqrt{x^2 + y^2} \quad \text{and} \quad |y| \leq \sqrt{x^2 + y^2} \]

can be used to represent the statement that, in a right-angled triangle, the hypotenuse is the longest side. They can be written in complex form as follows (see Figure 5.1).

\[ |\text{Re } z| \leq |z| \quad \text{and} \quad |\text{Im } z| \leq |z|. \]

Or, equivalently, they can be written as

\[ -|z| \leq \text{Re } z \leq |z| \quad \text{and} \quad -|z| \leq \text{Im } z \leq |z|. \]

Another elementary fact from plane geometry is that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. If \( z_1, z_2 \) are complex numbers, then 0, \( z_1 \) and \( z_1 + z_2 \) form the vertices of a triangle (see Figure 5.2) with side lengths \( |z_1|, |z_2| \) and \( |z_1 + z_2| \), so

\[ |z_1 + z_2| \leq |z_1| + |z_2|. \]

This is one form of an inequality called the Triangle Inequality, which will be used frequently throughout the module.

**Theorem 5.1 Triangle Inequality**

If \( z_1, z_2 \in \mathbb{C} \), then

(a) \( |z_1 + z_2| \leq |z_1| + |z_2| \) (usual form)

(b) \( |z_1 - z_2| \geq ||z_1| - |z_2|| \) (backwards form).

An immediate consequence of the second part of this theorem is that

\[ |z_1 - z_2| \geq |z_1| - |z_2| \quad \text{and} \quad |z_1 - z_2| \geq |z_2| - |z_1|. \]

**Proof** Although part (a) follows from plane geometry, we give a proof using complex numbers that illustrates the use of several results about complex conjugates from this unit.

We have

\[ |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \]

\[ = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \] (Theorem 2.1(c))

\[ = z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \]

\[ = |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_2|^2 \] (Theorems 2.1(c), 1.1(a)(iii) and 1.1(b)(iii))
5 Proving inequalities

\[ |z_1|^2 + 2 \text{Re}(z_1 \overline{z_2}) + |z_2|^2 \quad \text{(Theorem 1.1(a)(i))} \]
\[ \leq |z_1|^2 + 2 |z_1| |z_2| + |z_2|^2 \quad \text{(since } |\text{Re } z| \leq |z|) \]
\[ = |z_1|^2 + 2 |z_1| |z_2| + |z_2|^2 \quad \text{(Theorem 2.1(b), (e))} \]
\[ = (|z_1| + |z_2|)^2 \]

and so part (a) follows.

Part (b) can be proved by a similar method; alternatively, note that

\[ |z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|, \]

by part (a), so

\[ |z_1 - z_2| \geq |z_1| - |z_2|. \tag{5.1} \]

Similarly,

\[ |z_2 - z_1| \geq |z_2| - |z_1|, \]

and since \(|z_2 - z_1| = |z_1 - z_2|\), we see that

\[ |z_1 - z_2| \geq |z_2| - |z_1|. \tag{5.2} \]

Part (b) then follows from inequalities (5.1) and (5.2).

The backwards form of the Triangle Inequality also has a useful geometric interpretation, concerning the two circles centred at 0 through \(z_1\) and \(z_2\). It says that the distance from \(z_2\) to \(z_1\) is at least as large as the difference between the radii of these circles, as shown in Figure 5.3 for the case \(|z_1| > |z_2|\).

Several other versions of the Triangle Inequality are given in the following corollary. Each is a variant of one of the forms of the Triangle Inequality.

**Corollary**

If \(z, z_1, z_2, \ldots, z_n \in \mathbb{C}\), then

(a) \(|z| \leq |\text{Re } z| + |\text{Im } z|\)

(b) \(|z_1 - z_2| \leq |z_1| + |z_2|\)

(c) \(|z_1 + z_2| \geq ||z_1| - |z_2||\)

(d) \(|z_1 \pm z_2 \pm \cdots \pm z_n| \leq |z_1| + |z_2| + \cdots + |z_n|\)

(e) \(|z_1 \pm z_2 \pm \cdots \pm z_n| \geq |z_1| - |z_2| - \cdots - |z_n|\).

Parts (a), (b) and (d) are variants of the usual form of the Triangle Inequality, whereas parts (c) and (e) are variants of the backwards form.

**Proof** Part (a) is obtained by taking \(z_1 = \text{Re } z\) and \(z_2 = i \text{Im } z\) in the usual form of the Triangle Inequality.

Parts (b) and (c) are obtained by substituting \(-z_2\) for \(z_2\) in Theorem 5.1.

Parts (d) and (e) are obtained from parts (b) and (c) of this corollary and Theorem 5.1 by applying the Principle of Mathematical Induction – we omit the details.
The Triangle Inequality can be used to obtain estimates (also known as bounds) for the modulus of a complex expression involving $z$ when we know that $z$ lies in a certain set (such as a circle). The next example includes some typical applications. As indicated in the solutions to the example, it is not usual to refer explicitly to any of the variants (in the corollary) of the Triangle Inequality. However, use of the backwards form should be distinguished.

**Example 5.2**

(a) Prove the following inequalities.

(i) \[ |z^2 - 4z - 3| \leq 15, \quad \text{for } |z| = 2 \]

(ii) \[ |z^2 - 7| \geq 3, \quad \text{for } |z| = 2 \]

(iii) \[ |z^2 + 2| \geq 2, \quad \text{for } |z| = 2 \]

(b) Find a number $M$ such that

\[ \left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| \leq M, \quad \text{for } |z| = 2. \]

**Solution**

(a) (i) By the Triangle Inequality,

\[ |z^2 - 4z - 3| \leq |z^2| + |-4z| + |-3| = |z|^2 + 4|z| + 3; \]

so, for $|z| = 2$,

\[ |z^2 - 4z - 3| \leq 4 + 8 + 3 = 15. \]

(ii) By the backwards form of the Triangle Inequality,

\[ |z^2 - 7| \geq ||z|^2 - 7|; \]

so, for $|z| = 2$,

\[ |z^2 - 7| \geq |4 - 7| = 3. \]

(iii) By the backwards form of the Triangle Inequality,

\[ |z^2 + 2| \geq ||z|^2 - 2|; \]

so, for $|z| = 2$,

\[ |z^2 + 2| \geq |4 - 2| = 2. \]

(b) From part (a) we have, for $|z| = 2$,

\[ |z^2 - 4z - 3| \leq 15, \quad |z^2 - 7| \geq 3, \quad |z^2 + 2| \geq 2. \]

Now

\[ \left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| = \frac{|z^2 - 4z - 3|}{|z^2 - 7| \times |z^2 + 2|} = \frac{1}{|z^2 - 7|} \times \frac{1}{|z^2 + 2|}. \]

So, for $|z| = 2$, using the inequalities from part (a), we have

\[ \left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| \leq 15 \times \frac{1}{3} \times \frac{1}{2} = \frac{5}{2}, \]
because
\[ |z^2 - 7| \geq 3 \implies 1/|z^2 - 7| \leq 1/3\]
and
\[ |z^2 + 2| \geq 2 \implies 1/|z^2 + 2| \leq 1/2.\]
Thus we can take \( M = 5/2. \)

**Remarks**

1. Example 5.2(b) illustrates the fact that to obtain an upper estimate for a quotient, we need an upper estimate (in this case, 15) for the numerator and a *lower estimate* (in this case, 3 \times 2) for the denominator.

2. Equality is attained in the inequality
\[ |z^2 + 2| \geq 2, \quad \text{for } |z| = 2,\]
when \( z = 2i \) (or \( z = -2i \)), because
\[ |(2i)^2 + 2| = |-4 + 2| = 2.\]
In contrast, it is not possible to attain equality in the inequality
\[ |z^2 - 4z - 3| \leq 15, \quad \text{for } |z| = 2,\]
because it can be shown that the inequality remains true if 15 is replaced by certain smaller numbers, the smallest possible one of which is \( 7\sqrt{7/3} = 10.69 \ldots \).

**Exercise 5.2**

Prove the following inequalities.
(a) \[ \frac{1}{7} \leq \left| \frac{1}{3 + 4z^2} \right| \leq 1, \quad \text{for } |z| = 1\]
(b) \[ 2 \leq \left| \frac{z^3 + 2z + 1}{z^2 + 1} \right| \leq \frac{17}{4}, \quad \text{for } |z| = 3\]

**Further exercises**

**Exercise 5.3**

For \( |z| = 2 \), find an upper estimate for each of the following moduli.
(a) \( |z + 3| \) \hspace{1cm} (b) \( |z - 4i| \) \hspace{1cm} (c) \( |3z + 2| \) \hspace{1cm} (d) \( |3z^2 - 5| \)
(e) \( |z^2 + z + 1| \)
**Exercise 5.4**

For $|z| = 5$, find a positive lower estimate for each of the following moduli.

(a) $|z - 2|$  
(b) $|z + 3i|$  
(c) $|z - 7|$  
(d) $|2z - 7|$

**Exercise 5.5**

Find positive numbers $m$ and $M$ such that

$$m \leq \left| \frac{z^3 + 1}{z^3 - 1} \right| \leq M,$$

for $|z| = 4$.

---

**Quaternions**

As you have seen, the complex numbers are formed by adjoining to the real numbers a new symbol $i$ with the property $i^2 = -1$. In 1843 the Irish mathematician William Rowan Hamilton (1805–1865), whom you met in the Introduction, realised that by adjoining to the real numbers three new symbols $i$, $j$ and $k$ with the properties

$$i^2 = j^2 = k^2 = ijk = -1$$

we obtain a collection of numbers $a + bi + cj + dk$ (where $a, b, c, d \in \mathbb{R}$) with almost all of the algebraic properties of a field. Hamilton called these numbers **quaternions**. The quaternions are peculiar in that multiplication of quaternions is not commutative; for example, it can be proved that

$$ij = k, \quad \text{whereas} \quad ji = -k.$$ 

Since they satisfy all the other properties of a field (all but property M5 from the ‘Arithmetic in C’ table after Theorem 1.1) the quaternions can be described as a **non-commutative field**.

You have learned in this unit how the algebra of complex arithmetic is complemented by two-dimensional geometry in the complex plane. Quaternions are four-dimensional numbers, and they can be used to represent geometric objects in three and four dimensions. They play an important role in computer graphics, where quaternion algebra is used to manipulate three-dimensional images. They also feature in quantum mechanics, as an algebraic tool for representing the spin of elementary particles.
Solutions to exercises

Solution to Exercise 1.1
(a) (i) \((2 + i) + 3i(-1 + 3i) = 2 + i - 3i - 9 = -7 - 2i\)
(ii) \((2 + i)(-1 + 3i) = -2 + 6i - i - 3 = -5 + 5i\)
(iii) \((-1 + 3i)(-1 - 3i) = 1 + 3i - 3i + 9 = 10\)
(b) By part (a)(i), \(\text{Re } z = -7\) and \(\text{Im } z = -2\).

Solution to Exercise 1.2
(a) \((x + iy_1) + (x + iy_2) = (x + x) + i(y_1 + y_2)\)
(b) \((x + iy_1) - (x + iy_2) = (x - x) + i(y_1 - y_2)\)
(c) \((x + iy_1)(x + iy_2) = x^2 + ix_1y_2 + iy_1x_2 + iyx + y^2\)
(d) \((x + iy)(x - iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2\)

Solution to Exercise 1.3
(a) (i) \(\frac{1}{i} = \frac{-i}{1} = -i\)
(ii) \(\frac{1 + i}{2 + 3i} = \frac{(1 + i)(2 - 3i)}{4 + 9} = \frac{1 - i}{13} = \frac{1}{13} - \frac{1}{13}i\)
(iii) \(\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}\)
\[= \frac{x_1x_2 + iy_1y_2 + iyx - y_1x_2}{x_2^2 + y_2^2}\]
\[= \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)\]

Solution to Exercise 1.4
Theorem 1.1(b)(i)
Let \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Then
\[
\frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)
\]
Also, \(z_1 = x_1 - iy_1\) and \(z_2 = x_2 - iy_2\), so
\[
\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2).
\]

Solution to Exercise 1.5
(a) \((z_1 + z_2)^3\)
\[= (z_1 + z_2)(z_1^2 + 2z_1z_2 + z_2^2)\]
\[= z_1^3 + 2z_1^2z_2 + z_1z_2^2 + z_2^3 + z_2z_1z_2 + z_1z_2^2 + z_2^3\]
\[= z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3\]
(b) \((z_1 - z_2)(z_1^2 + z_2z_1 + z_2^2)\)
\[= z_1^3 + z_1z_2^2 + z_1^2z_2 - z_1^2z_2 - z_1z_2^2 - z_1z_2^2 - z_2^3\]
\[= z_1^3 - z_2^3\]
(c) \((z_1 + z_2)(z_1^2 - z_1z_2 + z_2^2)\)
\[= z_1^3 - z_2^3\]
\[= z_1^3 + z_1^2z_2 + z_1z_2^2 - z_1^2z_2 - z_1z_2^2 - z_1z_2^2 - z_2^3\]
\[= z_1^3 + z_2^3\]
Alternatively, apply part (b) with \(z_2\) replaced by \(-z_2\).
Solution to Exercise 1.6
(a) By the Binomial Theorem,
\[(1 + i)^4 = 1 + 4i + 6i^2 + 4i^3 + i^4 = 1 + 4i - 6 - 4i + 1 = -4.\]
(b) By the Binomial Theorem,
\[(3 + 2i)^3 = 3^3 + 3 \times 3^2 \times 2i + 3 \times 3 \times (2i)^2 + (2i)^3 = 27 + 54i - 36 - 8i = -9 + 46i.\]

Solution to Exercise 1.7
(a) By the Geometric Series Identity,
\[1 + (1 + i) + (1 + i)^2 + (1 + i)^3 = \frac{1 - (1 + i)^4}{1 - (1 + i)} = \frac{1 - (-4)}{-i} = \frac{5 \times i}{-i \times i} = 5i.\]
(b) By the Geometric Series Identity and the hint,
\[z^5 - i = z^5 - i^5 = (z - i)(z^4 + z^3i + z^2i^2 + zi^3 + i^4) = (z - i)(z^4 + z^3i - z^2 - zi + 1).\]
So \(z - i\) is one factor.

Solution to Exercise 1.8
<table>
<thead>
<tr>
<th>(z)</th>
<th>Re (z)</th>
<th>Im (z)</th>
<th>(-z)</th>
<th>(\bar{z})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 3i</td>
<td>2</td>
<td>3</td>
<td>-2 - 3i</td>
<td>2 - 3i</td>
</tr>
<tr>
<td>-3 - i</td>
<td>-3</td>
<td>-1</td>
<td>3 + i</td>
<td>-3 + i</td>
</tr>
<tr>
<td>4i</td>
<td>0</td>
<td>4</td>
<td>-4i</td>
<td>-4i</td>
</tr>
<tr>
<td>5i</td>
<td>5</td>
<td>0</td>
<td>-5</td>
<td>5</td>
</tr>
<tr>
<td>0i</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Solution to Exercise 1.9
(a) \(i^3 = (i^2)i = -i\)
(b) \(i^4 = (i^2)(i^2) = 1\)
(c) \((1 + i)^2 = 1 + 2i + i^2 = 2i\)
(d) \((1 - i)^2 = 1 - 2i + (-i)^2 = -2i\)
(e) \(\frac{1}{1 - i} = \frac{1 + i}{(1 - i)(1 + i)} = \frac{1 + i}{2} = \frac{1}{2} + \frac{1}{2}i\)
(f) \(\frac{1 + i}{1 - i} = \frac{(1 + i)(1 + i)}{(1 - i)(1 + i)} = \frac{2i}{2} = i\)
(g) \((1 + i)^3 = (1 + i)^2(1 + i) = 2i(1 + i) = -2 + 2i\)

Alternatively, use the Binomial Theorem.
(h) Use the identity \(a^2 - b^2 = (a + b)(a - b)\):
\[(2 + i)^2 - (2 - i)^2 = ((2 + i) + (2 - i))((2 + i) - (2 - i)) = 4 \times 2i = 8i.\]

Alternatively, expand \((2 + i)^2\) and \((2 - i)^2\) separately.
(i) \(\frac{3 + 5i}{2 - 3i} = \frac{(3 + 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{6 + 19i - 15}{4 + 9} = -\frac{9}{13} + \frac{19}{13}i\)
(j) \(\frac{3 + 2i}{1 + 4i} = \frac{(3 + 2i)(1 - 4i)}{(1 + 4i)(1 - 4i)} = \frac{3 - 10i + 8}{1 + 16} = \frac{11}{17} - \frac{10}{17}i\)
(k) By the Binomial Theorem,
\[(3 + 4i)^4 - (3 - 4i)^4 = (3^4 + 4 \times 3^3 \times (4i) + 6 \times 3^2 \times (4i)^2 + 4 \times 3 \times (4i)^3 + (4i)^4) - (3^4 + 4 \times 3^3 \times (-4i) + 6 \times 3^2 \times (-4i)^2 + 4 \times 3 \times (-4i)^3 + (-4i)^4)\]
\[= 2(432i - 768i) = -672i.\]

Alternatively, write
\[(3 + 4i)^4 - (3 - 4i)^4 = ((3 + 4i)^2 - (3 - 4i)^2)((3 + 4i)^2 + (3 - 4i)^2),\]
and then observe that \((3 + 4i)^2 = -7 + 24i\) and \((3 - 4i)^2 = -7 - 24i\), and simplify.
(l) By the Geometric Series Identity,
\[1 + i + i^2 + \ldots + i^{10} = \frac{1 - i^{11}}{1 - i} = \frac{1 + i}{2}(1 + i)^2 = i.\]
(m) By the Geometric Series Identity,
\[
1 - i + i^2 + \cdots + i^{10} = \frac{1 - (-i)^{11}}{1 - (-i)} = \frac{1 - i}{1 + i} = \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{-2i}{2} = -i.
\]
Alternatively, observe that
\[
i^n = (i)^n = (1)i^n = (1)\cdot 1^n,
\]
so by taking the complex conjugate of both sides of the equation
\[
1 + i + i^2 + \cdots + i^{10} = i
\]
found in part (l), we obtain
\[
1 - i + i^2 + \cdots + i^{10} = -i.
\]

Solution to Exercise 1.10

<table>
<thead>
<tr>
<th></th>
<th>Re z</th>
<th>Im z</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) −i</td>
<td>0</td>
<td>−1</td>
<td>i</td>
</tr>
<tr>
<td>(c) 1/2 + i/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2 - i/2</td>
</tr>
<tr>
<td>(g) −2 + 2i</td>
<td>−2</td>
<td>2</td>
<td>−2 − 2i</td>
</tr>
</tbody>
</table>

Solution to Exercise 1.11

Let \( z = x + iy \). Then \( \overline{z} = x - iy \) and

\[
\text{Im} \; \overline{z} = \text{Im}(x - iy) = -y \\
= -\text{Im}(x + iy) \\
= -\text{Im} \; z.
\]

Solution to Exercise 2.1

(a) With \( z_1 = 3 + i, z_2 = -1 + 2i \), we have

\[
\begin{align*}
-z_1 &= -3 - i, \\
-z_2 &= 1 - 2i, \\
z_1 + z_2 &= 2 + 3i, \\
z_1 - z_2 &= 4 - i.
\end{align*}
\]

(b) With \( z_1 = 3 + i, z_2 = -1 + 2i \), we have

\[
\begin{align*}
\overline{z_1} &= 3 - i, \\
\overline{z_2} &= -1 - 2i, \\
z_1 + z_2 &= 2 + 3i, \\
\overline{z_1 + z_2} &= 2 - 3i.
\end{align*}
\]

Solution to Exercise 2.2

(a) (i) \( |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2} \)
(ii) \( |2 - 4i| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5} \)
(iii) \( |i| = \sqrt{1^2} = 1 \)
(iv) \( |-5 + 12i| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13 \)
(b) Let \( z = x + iy \). Then \( \overline{z} = x - iy \) and

\[
-z = -x - iy. \text{ Hence}
\]

\[
|\overline{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|,
\]

\[
|-z| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|.
\]

Note that these results are ‘obvious’ geometrically.

Solution to Exercise 2.3

(a) Since \( z_1 - z_2 = 4 - i \),

\[
|z_1 - z_2| = \sqrt{4^2 + (-1)^2} = \sqrt{17}.
\]

(b) Since \( z_1 + z_2 = 2 + 3i \),

\[
|z_1 + z_2| = \sqrt{2^2 + 3^2} = \sqrt{13}.
\]

(c) The distance from \( z_2 \) to \( -z_1 \) is

\[
|(-z_1) - z_2| = |z_1 + z_2| = \sqrt{13}.
\]
Solution to Exercise 2.4

(a) Here $r = |i| = 1$, and the obvious choice for an argument of $i$ is $\theta = \pi/2$. Thus
\[
i = 1(\cos \pi/2 + i \sin \pi/2) = \cos \pi/2 + i \sin \pi/2.
\]

(b) (i) $2(\cos \pi/3 + i \sin \pi/3)$
\[
= 2(1/2 + i\sqrt{3}/2) = 1 + \sqrt{3}i.
\]
(ii) $3(\cos(-\pi/4) + i \sin(-\pi/4))$
\[
= 3(1/\sqrt{2} + i(-1/\sqrt{2})) = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i.
\]

Solution to Exercise 2.5

In each case we use the strategy for determining principal arguments to find $\text{Arg } z$.

(a) $-4$ lies on the negative real axis, so $\text{Arg}(-4) = \pi$ (see Figure 2.12). Since $|-4| = 4$, a polar form of $-4$ is
\[
4(\cos \pi + i \sin \pi).
\]

(b) $3\sqrt{3} + 3i$ lies in the upper-right quadrant, and
\[
\phi = \tan^{-1} \frac{3}{3\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \pi/6.
\]
Thus the principal argument $\theta$ is $\theta = \phi$ (see Figure 2.13(b))
\[
= \pi/6.
\]
Since $|3\sqrt{3} + 3i| = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27 + 9} = 6$, a polar form of $3\sqrt{3} + 3i$ is
\[
6(\cos \pi/6 + i \sin \pi/6).
\]

(An alternative way to calculate the modulus is to observe that
\[
|3\sqrt{3} + 3i| = |3||\sqrt{3} + i|
\]
\[
= 3\sqrt{(\sqrt{3})^2 + 1^2}
\]
\[
= 3 \times 2 = 6.
\]

(c) $\sqrt{3} - i$ lies in the lower-right quadrant, and
\[
\phi = \tan^{-1} \frac{|-1|}{\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}.
\]
Thus the principal argument $\theta$ is $\theta = -\phi$ (see Figure 2.13(b))
\[
= -\pi/6.
\]
Since $|\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, a polar form of $\sqrt{3} - i$ is
\[
2(\cos(-\pi/6) + i \sin(-\pi/6)).
\]

(d) $-1 - i$ lies in the lower-left quadrant, and
\[
\phi = \tan^{-1} \frac{|-1|}{|-1|} = \tan^{-1} 1 = \frac{\pi}{4}.
\]
Thus the principal argument $\theta$ is $\theta = -(\pi - \phi)$ (see Figure 2.13(b))
\[
= -(\pi - \pi/4) = -3\pi/4.
\]
Since $|-1 - i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, a polar form of $-1 - i$ is
\[
\sqrt{2}(\cos(-3\pi/4) + i \sin(-3\pi/4)).
\]
Solution to Exercise 2.6

\[ |z_1| = |-1 - \sqrt{3}i| = 2 \text{ and, from Example 2.2(b), an argument of } z_1 \text{ is } -2\pi/3; \text{ so } \]
\[ z_1 = 2(\cos(-2\pi/3) + i \sin(-2\pi/3)). \]

From Exercise 2.5(b),
\[ z_2 = 6(\cos(\pi/6) + i \sin(\pi/6)). \]

Thus, from formula (2.1),
\[ z_1z_2 = 2 \times 6 \cos(-2\pi/3 + \pi/6) + i \sin(-2\pi/3 + \pi/6) \]
\[ = 12 \left( \cos(-\pi/2) + i \sin(-\pi/2) \right) \]
\[ = -12i \]

and
\[ z_2^2 = z_1z_1 \]
\[ = 2 \times 2 \cos(-2\pi/3 - 2\pi/3) + i \sin(-2\pi/3 - 2\pi/3) \]
\[ = 4 \left( \cos(-4\pi/3) + i \sin(-4\pi/3) \right) \]
\[ = 4 \left( -1/2 + i \sqrt{3}/2 \right) \]
\[ = -2 + 2\sqrt{3}i. \]

Solution to Exercise 2.7

Since \(|2i| = 2\) and \(\text{Arg}(2i) = \pi/2\), multiplying \(z\) by \(2i\) scales \(z\) by the factor \(1/2\) and rotates it clockwise through \(\pi/2\) about 0.

Solution to Exercise 2.8

A polar form of \(1 + \sqrt{3}i\) is
\[ z_1 = 2(\cos\pi/3 + i \sin\pi/3) \]

and, from Exercise 2.5(c), a polar form of \(\sqrt{3} - i\) is
\[ z_2 = 2(\cos(-\pi/6) + i \sin(-\pi/6)). \]

Solution to Exercise 2.9

Since \(|2i| = 2\) and \(\text{Arg}(2i) = \pi/2\), dividing \(z\) by \(2i\) scales \(z\) by the factor \(1/2\) and rotates it anticlockwise through \(\pi/2\) about 0.

Solution to Exercise 2.10

Since \(1 + i = \sqrt{2}(\cos\pi/4 + i \sin\pi/4)\), we have
\[ (1 + i)^{-1} = \frac{1}{\sqrt{2}} \left( \cos(-\pi/4) + i \sin(-\pi/4) \right) \]
\[ = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \]
\[ = \frac{1}{2} - \frac{1}{2}i. \]

Solution to Exercise 2.11

Since
\[ z_1 = 1 + i = \sqrt{2}(\cos\pi/4 + i \sin\pi/4), \]
\[ z_2 = 1 + \sqrt{3}i = 2(\cos\pi/3 + i \sin\pi/3), \]
\[ z_3 = \sqrt{3} + i = 2(\cos\pi/6 + i \sin\pi/6), \]
we have
\[ z_1z_2z_3 = \sqrt{2} \times 2 \times 2 \cos(\pi/4 + \pi/3 + \pi/6) + i \sin(\pi/4 + \pi/3 + \pi/6) \]
\[ = 4\sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4) \]
\[ = 4\sqrt{2}\left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \]
\[ = -4 + 4i. \]
Solution to Exercise 2.12

(a) Since
\[ \sqrt{3} + i = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) , \]
De Moivre’s Theorem gives
\[ (\sqrt{3} + i)^4 = 2^4(\cos 4\pi/6 + i \sin 4\pi/6) = 16(\cos 2\pi/3 + i \sin 2\pi/3) \]
\[ = 16 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \]
\[ = -8 + 8\sqrt{3}i. \]

(b) Since
\[ 1 - \sqrt{3}i = 2(\cos(-\pi/3) + i \sin(-\pi/3)) , \]
De Moivre’s Theorem gives
\[ (1 - \sqrt{3}i)^3 = 2^3(\cos(-3\pi/3) + i \sin(-3\pi/3)) = 8(\cos(-\pi) + i \sin(-\pi)) = -8. \]

(c) Since
\[ 1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4) , \]
De Moivre’s Theorem gives
\[ (1 + i)^{10} = (\sqrt{2})^{10}(\cos 10\pi/4 + i \sin 10\pi/4) = 2^5(\cos \pi/2 + i \sin \pi/2) = 32i. \]

(d) Since
\[ -1 + i = \sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4) , \]
De Moivre’s Theorem gives
\[ (-1 + i)^{-8} = (\sqrt{2})^{-8}(\cos(-24\pi/4) + i \sin(-24\pi/4)) = 2^{-4}(\cos(-6\pi) + i \sin(-6\pi)) = \frac{1}{16}. \]

(e) Since
\[ \sqrt{3} + i = 2(\cos \pi/6 + i \sin \pi/6) , \]
De Moivre’s Theorem gives
\[ (\sqrt{3} + i)^{-6} = 2^{-6}(\cos(-6\pi/6) + i \sin(-6\pi/6)) = 2^{-6}(\cos(-\pi) + i \sin(-\pi)) = -\frac{1}{64}. \]
Solutions to exercises

(d) \[2 + 2i = \sqrt{8} = 2\sqrt{2}, \quad \text{so} \quad 2 + 2i = 2\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).\]

(e) \[-2 + 2i = \sqrt{8} = 2\sqrt{2}, \quad \text{so} \quad -2 + 2i = 2\sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right).\]

(f) \[-\sqrt{3} - i = 2, \quad \text{where} \quad \theta = \tan^{-1} \frac{1}{\sqrt{3}} \approx 0.927 \text{ rad.}\]

(g) \[\text{Arg}(3 + 4i) = \tan^{-1} \frac{4}{3}, \quad \text{so} \quad 3 + 4i = 5(\cos \theta + i \sin \theta), \quad \text{where} \quad \theta = \tan^{-1} \frac{4}{3} \approx 0.927 \text{ rad.}\]

(h) \[\text{Alternatively, part (g) tells us that} \quad 3 + 4i = 5(\cos \theta + i \sin \theta), \quad \text{where} \quad \theta = \tan^{-1} \frac{4}{3}, \quad \text{so} \quad 3 - 4i = 3 + 4i = 5(\cos(-\theta) + i \sin(-\theta)).\]

\[\text{Arg}(-\sqrt{3} - i) = -\left( \pi - \tan^{-1} \frac{1}{\sqrt{3}} \right) = -\frac{5\pi}{6}, \quad \text{so} \quad -\sqrt{3} - i = 2 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right).\]
Solution to Exercise 2.14
(a) $\cos \pi + i \sin \pi = -1$

(b) $4 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right) = -4i$

(c) $3 \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$

(d) $3 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \frac{3\sqrt{3}}{2} + \frac{3}{2} i$

(e) $\cos \left( -\frac{2\pi}{3} \right) + i \sin \left( -\frac{2\pi}{3} \right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

Solution to Exercise 2.15
(a) $|z_2 - z_1| = |(2 + 3i) - (1 + i)|$
   $= |1 + 2i|$
   $= \sqrt{1 + 4} = \sqrt{5}$

(b) $|z_2 - z_1| = |(1 - 7i) - (-2 + 3i)|$
   $= |3 - 10i|$
   $= \sqrt{9 + 100} = \sqrt{109}$

(c) $|z_2 - z_1| = | -i - i |$
   $= | -2i |$
   $= 2$

Solution to Exercise 2.16
(a) Since $1 + \sqrt{3}i = 2(\cos \pi/3 + i \sin \pi/3)$, De Moivre’s Theorem gives
   $(1 + \sqrt{3}i)^5 = (2(\cos \pi/3 + i \sin \pi/3))^5$
   $= 2^5(\cos 5\pi/3 + i \sin 5\pi/3)$
   $= 32 \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i \right)$
   $= 16 - 16\sqrt{3}i$.

(b) Since $1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$, De Moivre’s Theorem gives
   $(1 + i)^{-4} = \left( \sqrt{2}(\cos \pi/4 + i \sin \pi/4) \right)^{-4}$
   $= 2^{-4/2}(\cos(-4\pi/4) + i \sin(-4\pi/4))$
   $= \frac{1}{4}(\cos(-\pi) + i \sin(-\pi))$
   $= -\frac{1}{4}$. 
Since
\[ 1 + i = \sqrt{2} (\cos \pi/4 + i \sin \pi/4), \]
De Moivre’s Theorem gives
\[ (1 + i)^6 = \left( \sqrt{2} (\cos \pi/4 + i \sin \pi/4) \right)^6 \]
\[ = 2^3 (\cos 3\pi/2 + i \sin 3\pi/2). \]

Also,
\[ \sqrt{3} - i = 2(\cos(-\pi/6) + i \sin(-\pi/6)), \]
so De Moivre’s Theorem gives
\[ (\sqrt{3} - i)^{-3} = (2(\cos(-\pi/6) + i \sin(-\pi/6)))^{-3} \]
\[ = 2^{-3} (\cos \pi/2 + i \sin \pi/2). \]

Hence
\[ \frac{(1 + i)^6}{(\sqrt{3} - i)^3} = \frac{2^3 (\cos 3\pi/2 + i \sin 3\pi/2)}{2^{-3} (\cos \pi/2 + i \sin \pi/2)} \]
\[ = \cos 2\pi + i \sin 2\pi = 1. \]

**Solution to Exercise 2.17**

By De Moivre’s Theorem,
\[ (\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta. \]

By the Binomial Theorem,
\[ (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3 \cos^2 \theta \sin \theta + 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta). \]

The two expressions we have obtained for \( (\cos \theta + i \sin \theta)^3 \) are equal, so their real parts are equal and their imaginary parts are equal. Equating the two imaginary parts gives
\[ \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta. \]

Since \( \cos^2 \theta = 1 - \sin^2 \theta \), we have
\[ \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta, \]
as required.

**Solution to Exercise 2.18**

Since \( (x + iy)^4 = a + ib \), it follows that
\[ |x + iy|^4 = |(x + iy)^4| = |a + ib|. \]

Squaring the left-hand side gives
\[ (|x + iy|^4)^2 = (|x + iy|^4)^4 = (x^2 + y^2)^4, \]
and squaring the right-hand side gives \( a^2 + b^2 \).

Hence
\[ (x^2 + y^2)^4 = a^2 + b^2. \]

**Solution to Exercise 2.19**

If \( z = z^{-1} \), then
\[ z\overline{z} = z z^{-1} = 1. \]

Hence, by Theorem 2.1(c),
\[ |z|^2 = z\overline{z} = 1, \]
so \( |z| = 1 \), as required.

**Solution to Exercise 3.1**

Since \(-1 + \sqrt{3}i = 2(\cos 2\pi/3 + i \sin 2\pi/3)\), a solution of \( z^2 = -1 + \sqrt{3}i \) is obtained by taking \( z \) to have modulus \( \sqrt{2} \) and argument \( \frac{1}{2}(2\pi/3) = \pi/3 \).

This gives
\[ z = \sqrt{2} (\cos \pi/3 + i \sin \pi/3) \]
\[ = \sqrt{2} \left( \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \]
\[ = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i. \]

Therefore the required solutions are
\[ z = \pm \left( \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i \right). \]

**Solution to Exercise 3.2**

(a) Using the principal argument of \( 8i \), we have
\[ 8i = 8(\cos \pi/2 + i \sin \pi/2), \]
and, using the strategy for finding \( n \)th roots, we deduce that the cube roots of \( 8i \) are
\[ z_k = 8^{1/3} \left( \cos \left( \frac{\pi}{6} + \frac{2\pi k}{3} \right) + i \sin \left( \frac{\pi}{6} + \frac{2\pi k}{3} \right) \right), \]
where \( k = 0, 1, 2 \); that is,
\[ z_0 = 2(\cos \pi/6 + i \sin \pi/6) = \sqrt{3} + i, \]
\[ z_1 = 2(\cos 5\pi/6 + i \sin 5\pi/6) = -\sqrt{3} + i, \]
\[ z_2 = 2(\cos 3\pi/2 + i \sin 3\pi/2) = -2i. \]
Since the principal argument of $8i$ is $\pi/2$, the principal cube root of $8i$ is $z_0$.

(a) The principal argument of $-i$ is $-\pi/2$; to avoid negative signs, we use the argument $3\pi/2$. Thus

$$-i = \cos 3\pi/2 + i \sin 3\pi/2,$$

and, using the strategy for finding $n$th roots, we deduce that the sixth roots of $-i$ are

$$z_k = \cos \left(\frac{3\pi/2}{6} + k\frac{2\pi}{6}\right) + i \sin \left(\frac{3\pi/2}{6} + k\frac{2\pi}{6}\right),$$

where $k = 0, 1, \ldots, 5$; that is,

$$z_0 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4},$$
$$z_1 = \cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12},$$
$$z_2 = \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12},$$
$$z_3 = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4},$$
$$z_4 = \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12},$$
$$z_5 = \cos \frac{23\pi}{12} + i \sin \frac{23\pi}{12}.$$

Since the principal argument of $-i$ is $-\pi/2$, the principal sixth root of $-i$ has argument $-\pi/12$ and hence it is $z_5$.

Alternatively, using the principal argument, we have

$$-i = \cos(-\pi/2) + i \sin(-\pi/2),$$

and, using the strategy, we deduce that the sixth roots of $-i$ are

$$z_k = \cos \left(-\frac{\pi}{12} + k\frac{\pi}{3}\right) + i \sin \left(-\frac{\pi}{12} + k\frac{\pi}{3}\right),$$

where $k = 0, 1, \ldots, 5$; that is,

$$z_0 = \cos(-\pi/12) + i \sin(-\pi/12),$$
$$z_1 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4},$$
$$z_2 = \cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12},$$
$$z_3 = \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{2},$$
$$z_4 = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4},$$
$$z_5 = \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12}.$$

(b) The principal argument of $-i$ is $-\pi/2$; to avoid negative signs, we use the argument $3\pi/2$.

By the Geometric Series Identity,

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1}).$$

Thus if

$$z^n = 1 \quad \text{and} \quad z \neq 1,$$

then

$$1 - z^n = 0 \quad \text{and} \quad 1 - z \neq 0,$$

so

$$1 + z + z^2 + \cdots + z^{n-1} = 0,$$

as required.

By De Moivre’s Theorem and the corollary to Theorem 3.1, the $n$th roots of unity are of the form

$$1, z, z^2, \ldots, z^{n-1},$$

where

$$z = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Hence, by part (a), the sum of the $n$th roots of unity is 0.
Solution to Exercise 3.4

(a) The factorisation
\[ z^2 - 7iz + 8 = (z + i)(z - 8i) = 0 \]
shows that the solutions are \( z = -i, 8i \).

(b) Formula (3.1) with \( a = 1, b = 2, c = 1 - i \) gives
\[ z = \frac{-2 \pm \sqrt{4 - 4(1 - i)}}{2} = -1 \pm \sqrt{1/2}(1 + i) \]
(by Example 3.1).

So the solutions are
\[ z = \left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i, \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i. \]

Solution to Exercise 3.5

(a) Putting \( w = z^3 \) gives
\[ w^2 - 7iw + 8 = 0, \]
so \( w = -i \) or \( w = 8i \) (see Exercise 3.4(a)). Thus the solutions are the cube roots of \(-i\) and \(8i\).

We found the cube roots of \(8i\) in Exercise 3.2(a); these are
\[ \sqrt{3} + i, \quad -\sqrt{3} + i \quad \text{and} \quad -2i. \]

By a similar method, or using the hint, we find that the cube roots of \(-i\) are
\[ -\sqrt{3} + i, \quad \frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad \text{and} \quad i. \]

Hence the six solutions are
\[ \sqrt{3} + i, \quad -\sqrt{3} + i, \quad -2i, \]
\[ -\sqrt{3} + i, \quad \frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad \text{and} \quad i. \]

(b) Putting \( w = z^2 \) gives
\[ w^2 + 4iw + 8 = 0, \]
so
\[ w = -4i \pm \sqrt{-16 - 32}, \]
that is,
\[ w = (\sqrt{12} - 2)i \quad \text{or} \quad w = - (\sqrt{12} + 2)i. \]

Since
\[ (\sqrt{12} - 2)i = (\sqrt{12} - 2)(\cos \pi/2 + i \sin \pi/2) \]
and
\[ - (\sqrt{12} + 2)i = (\sqrt{12} + 2)(\cos 3\pi/2 + i \sin 3\pi/2), \]
the four solutions are
\[ z = \pm (\sqrt{12} - 2)^{1/2} (\cos \pi/4 + i \sin \pi/4) \]
\[ = \pm (\sqrt{12} - 2)^{1/2} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \]
\[ = \pm (\sqrt{3} - 1)^{1/2} (1 + i) \]
and
\[ z = \pm (\sqrt{12} + 2)^{1/2} (\cos 3\pi/4 + i \sin 3\pi/4) \]
\[ = \pm (\sqrt{12} + 2)^{1/2} \left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \]
\[ = \pm (\sqrt{3} + 1)^{1/2} (-1 + i). \]

Solution to Exercise 3.6

In each case we will express the given complex number in polar form using the principal argument.

(a) (i) As \(-i = \cos(-\pi/2) + i \sin(-\pi/2)\), the square roots of \(-i\) are
\[ z_k = \cos(-\pi/4 + k\pi) + i \sin(-\pi/4 + k\pi), \]
k = 0, 1. Thus the principal square root is
\[ z_0 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}} (1 - i), \]
and the other root is
\[ z_1 = -z_0 = -\frac{1}{\sqrt{2}} (1 - i). \]

\[ z_1 = -\frac{1}{\sqrt{2}} (1 - i) \quad \text{and} \quad z_0 = \frac{1}{\sqrt{2}} (1 - i). \]
(ii) As $4i = 4\cos(\pi/2 + i\sin\pi/2)$, the square roots of $4i$ are

$z_k = 2\cos(\pi/4 + k\pi) + i\sin(\pi/4 + k\pi)$,

$k = 0, 1$. Thus the principal square root is

$z_0 = \sqrt{2} + \sqrt{2}i = \sqrt{2}(1 + i)$,

and the other root is

$z_1 = -z_0 = -\sqrt{2}(1 + i)$.

(b) (i) As $-1 = \cos\pi + i\sin\pi$, the cube roots of $-1$ are

$z_k = \cos\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right)$,

$k = 0, 1, 2$. The principal cube root is

$z_0 = \cos\pi/3 + i\sin\pi/3 = \frac{1 + \sqrt{3}i}{2}$,

and the other roots are

$z_1 = \cos\pi + i\sin\pi = -1$,

$z_2 = \cos5\pi/3 + i\sin5\pi/3 = \frac{1 - \sqrt{3}i}{2}$.

(c) (i) As $\frac{1}{\sqrt{2}}(-1 - i) = \cos(-3\pi/4) + i\sin(-3\pi/4)$,

the fourth roots of $\frac{1}{\sqrt{2}}(-1 - i)$ are

$z_k = \cos\left(-\frac{3\pi}{16} + k\frac{\pi}{2}\right) + i\sin\left(-\frac{3\pi}{16} + k\frac{\pi}{2}\right)$,

$k = 0, 1, 2, 3$. The principal fourth root is

$z_0 = \cos(-3\pi/16) + i\sin(-3\pi/16)$,

and the other roots are

$z_1 = \cos5\pi/16 + i\sin5\pi/16$,

$z_2 = \cos13\pi/16 + i\sin13\pi/16$,

$z_3 = \cos21\pi/16 + i\sin21\pi/16$.
(ii) As \(-1 + i = \sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4)\), the fourth roots of \(-1 + i\) are 
\[ z_k = 2^{1/8} \left( \cos \left( \frac{3\pi}{16} + \frac{k\pi}{2} \right) + i \sin \left( \frac{3\pi}{16} + \frac{k\pi}{2} \right) \), \]
\[ k = 0, 1, 2, 3. \] The principal fourth root is 
\[ z_0 = 2^{1/8}(\cos 3\pi/16 + i \sin 3\pi/16), \] 
and the other roots are 
\[ z_1 = 2^{1/8}(\cos 11\pi/16 + i \sin 11\pi/16), \]
\[ z_2 = 2^{1/8}(\cos 19\pi/16 + i \sin 19\pi/16), \]
\[ z_3 = 2^{1/8}(\cos 27\pi/16 + i \sin 27\pi/16). \]

(d) (i) As \(-1 = \cos \pi + i \sin \pi\), the fifth roots of \(-1\) are 
\[ z_k = \cos \left( \frac{\pi}{5} + \frac{2k\pi}{5} \right) + i \sin \left( \frac{\pi}{5} + \frac{2k\pi}{5} \right), \]
\[ k = 0, 1, \ldots, 4. \] The principal fifth root is 
\[ z_0 = \cos (\pi/5 + i \sin \pi/5), \] 
and the other roots are 
\[ z_1 = \cos (3\pi/5 + i \sin 3\pi/5), \]
\[ z_2 = \cos (\pi + i \sin \pi) = -1, \]
\[ z_3 = \cos (7\pi/5 + i \sin 7\pi/5), \]
\[ z_4 = \cos (9\pi/5 + i \sin 9\pi/5). \]

Solution to Exercise 3.7

(a) Since \((x + iy)^2 = x^2 - y^2 + 2xyi\), equating the real parts and imaginary parts of 
\((x + iy)^2 = 3 + 4i\) 
gives 
\[ x^2 - y^2 = 3, \quad 2xy = 4. \] (S1) 
From the second equation, \(y = 2/x\), and substituting this in the first equation, we obtain 
\[ x^2 - \frac{4}{x^2} = 3; \]
that is, 
\[ x^4 - 3x^2 - 4 = 0 \]
or 
\[ (x^2 - 4)(x^2 + 1) = 0. \] 
Since \(x^2 + 1 \neq 0\), it follows that \(x^2 = 4\), so \(x = \pm 2\). By equations (S1), when \(x = 2\) we have \(y = 1\), and when \(x = -2\) we have \(y = -1\). The two solutions are therefore 
\[ x + iy = \pm(2 + i). \]
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(b) Since \((x + iy)^2 = x^2 - y^2 + 2xyi\), equating the real parts and imaginary parts of
\[(x + iy)^2 = -5 + 12i\]
gives
\[x^2 - y^2 = -5, \quad 2xy = 12. \tag{S2}\]
From the second equation, \(y = 6/x\), and substituting this in the first equation, we obtain
\[x^2 - 36/x^2 = -5;\]
that is,
\[x^4 + 5x^2 - 36 = 0\]
or
\[(x^2 + 9)(x^2 - 4) = 0.\]
Since \(x^2 + 9 \neq 0\), it follows that \(x^2 = 4\), so \(x = \pm 2\).
By equations (S2), when \(x = 2\) we have \(y = 3\), and when \(x = -2\) we have \(y = -3\). The two solutions are therefore
\[x + iy = \pm (2 + 3i).\]

Solution to Exercise 3.8

(a) Substituting \(w = z^2\) in \(z^4 - z^2 + 1 + i = 0\) gives
\[w^2 - w + 1 + i = 0,\]
which has solutions
\[w = \frac{1 \pm \sqrt{1 - 4(1 + i)}}{2} = \frac{1 \pm i}{2} \sqrt{-3 + 4i} = \frac{1 \pm i}{2} \sqrt{3 + 4i}.\]
From Exercise 3.7(a), \(\sqrt{3 + 4i} = 2 + i\), and hence
\[w = \frac{1}{2} \pm \frac{i}{2} (2 + i),\]
so \(w = i\) or \(w = 1 - i\).
(You may have spotted the factorisation
\[w^2 - w + 1 + i = (w - i)(w - 1 + i).\])
Thus \(z = \pm \sqrt{i}\) or \(z = \pm \sqrt{1 - i}\). Since
\[i = \cos \pi/2 + i \sin \pi/2\]
and
\[1 - i = \sqrt{2}(\cos(-\pi/4) + i \sin(-\pi/4)),\]
we obtain the four solutions
\[z = \pm (\cos \pi/4 + i \sin \pi/4)\]
and
\[z = \pm 2^{1/4}(\cos(-\pi/8) + i \sin(-\pi/8)).\]

(b) Since the polynomial \(z^3 - 4z^2 + 6z - 4\) has real coefficients and is of odd degree, we expect there to be at least one real solution (because the graph of the real function \(f(x) = x^3 - 4x^2 + 6x - 4\) crosses the \(x\)-axis). By trial and error (trying factors of 4), we find that \(z = 2\) is a solution and hence
\[z^3 - 4z^2 + 6z - 4 = (z - 2)(z^2 - 2z + 2).\]
Since the solutions of \(z^2 - 2z + 2 = 0\) are
\[z = \frac{2 \pm \sqrt{4 - 4 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i,\]
the solutions of the original equation are
\[z = 2, 1 + i, 1 - i.\]
Solution to Exercise 3.9

If \( z \) satisfies \( p(z) = 0 \), then
\[
a_n z^n + \cdots + a_1 z + a_0 = 0.
\]
Taking complex conjugates of both sides and using Theorem 1.1(b) gives
\[
0 = a_n z^n + \cdots + a_1 z + a_0 = a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0,
\]
since \( a_0, a_1, \ldots, a_n \) are real. Hence \( p(\bar{z}) = 0 \), as required.

Remark: The solution to Exercise 3.8(b) provides an example of this result, whereas Exercise 3.8(a) shows that if the coefficients of a polynomial \( p \) are not all real, then solutions of \( p(z) = 0 \) need not occur in complex conjugate pairs.

Solution to Exercise 4.1

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</tr>
<tr>
<td>( \text{Arg} , z \geq 0 )</td>
<td>( \checkmark )</td>
<td>( \times )</td>
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</tr>
</tbody>
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Solution to Exercise 4.2

(a) (i) \( \{ z : 2 \text{Re} \, z - 3 \text{Im} \, z = -1 \} \)

(b) The half-plane includes its boundary, which has equation
\[
\frac{1}{2} x - y = -1.
\]
At \( z = 0 \), \( \frac{1}{2} x - y > -1 \), and since 0 is not in this half-plane, the half-plane is the set
\[
\{ z : \frac{1}{2} \text{Re} \, z - \text{Im} \, z \leq -1 \}.
\]
Solution to Exercise 4.3

(a) (i) \( \{ z : |z - 1 + 2i| = 1 \} \)

(ii) \( \{ z : |z - 1 + 2i| < 1 \} \)

(iii) \( \{ z : |z + 2 - 3i| \leq 3 \} \)

(b) This set is an open disc with centre \(-1 - i\) and radius \(|-1 - i| = \sqrt{2}\), so it is the set \( \{ z : |z + 1 + i| < \sqrt{2} \} \).

Solution to Exercise 4.4

(a) (i) \( \{ z : |z + i| > \frac{1}{2} \} \)

(ii) \( \{ z : \frac{1}{2} \leq |z + 1| < 2 \} \)

(iii) \( \{ z : 2 \leq |z + 2 - 3i| \leq 3 \} \)
(b) This set is a punctured open disc with centre $1 - 2i$ and radius 1, so it is the set
   \[ \{ z : 0 < |z - 1 + 2i| < 1 \} \].

**Solution to Exercise 4.5**

(a) (i) \( \{ z : \operatorname{Arg} z = -2\pi/3 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.5} \\
(a) (i) \{ z : \operatorname{Arg} z = -2\pi/3 \} \\
\end{array} \]

(ii) \( \{ z : \operatorname{Arg}(z - i) = 3\pi/4 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.5} \\
(a) (ii) \{ z : \operatorname{Arg}(z - i) = 3\pi/4 \} \\
\end{array} \]

(iii) \( \{ z : |\operatorname{Arg} z| < 2\pi/3 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.5} \\
(a) (iii) \{ z : |\operatorname{Arg} z| < 2\pi/3 \} \\
\end{array} \]

(b) This set is a sector (not an open one) with boundary rays \( \{ z : \operatorname{Arg}(z + 2i) = 0 \} \) and \( \{ z : \operatorname{Arg}(z + 2i) = \pi/4 \} \), so it is
   \[ \{ z : 0 \leq \operatorname{Arg}(z + 2i) \leq \pi/4 \} \].

**Solution to Exercise 4.6**

(a) (i) \( A = \{ z : |\operatorname{Re} z| < 1 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.6} \\
(a) (i) \{ z : |\operatorname{Re} z| < 1 \} \\
\end{array} \]

(ii) \( B = \{ z : |z| \leq 2 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.6} \\
(a) (ii) \{ z : |z| \leq 2 \} \\
\end{array} \]

(iii) \( A \cup B = \{ z : |\operatorname{Re} z| < 1 \text{ or } |z| \leq 2 \} \)

\[ \begin{array}{c}
\text{Solution to Exercise 4.6} \\
(a) (iii) \{ z : |\operatorname{Re} z| < 1 \text{ or } |z| \leq 2 \} \\
\end{array} \]
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(iv) \( A \cap B = \{ z : |\text{Re} z| < 1, |z| \leq 2 \} \)

(v) \( A - B = \{ z : |\text{Re} z| < 1, |z| > 2 \} \)

(vi) \( \mathbb{C} - A = \{ z : |\text{Re} z| \geq 1 \} \)

(b) \( \mathbb{C} - (A \cup B) \) is the set of points \( z \) that lie in neither \( A \) nor \( B \); that is,

\[ \mathbb{C} - (A \cup B) = \{ z : |\text{Re} z| \geq 1, |z| > 2 \} \]

Solution to Exercise 4.7

(a) \( \{ z : \text{Im} z > 0 \} \)

(b) \( \{ z : |z + 1| \leq 1 \} \)

(c) \( \{ z : 0 < |z + 1 + 2i| < 1 \} \)
(d) \( \{ z : \text{Arg}(z + 1 - i) < \pi/3 \} \)

\( -1 + i \)

\( \frac{\pi/3}{3/6} \)

(e) \( \{ z : |z - 1| \leq |z - 2| \} \)

This set consists of all points \( z \) whose distance from 1 is less than or equal to its distance from 2.

(f) \( \mathbb{C} - \{ z : \text{Re} z \geq 1 \} \)

(g) \( \{ z : \text{Im} z > 0 \} - \{ z : |z + 1| \leq 1 \} \)

(h) \( \{ z : \text{Arg} z = \pi/6 \} \cup \{ z : \text{Arg}(z - \sqrt{3} - i) = 0 \} \)

(i) The sets \( \{ z : \text{Arg} z = \pi/6 \} \) and \( \{ z : \text{Arg}(z - \sqrt{3} - i) = 0 \} \) have no points in common: their intersection is the empty set, \( \emptyset \). We make no attempt to sketch \( \emptyset \)!

Solution to Exercise 4.8

(a) \( \{ z : |\text{Re} z| < 1, |\text{Im} z| < 1 \} \)
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(b) \( \{ z : |z - i| \leq 2, |z| \geq 1 \} \)

(c) \( \{ z : \text{Re } z + 2 \text{Im } z + 3 > 0 \} \)

(d) \( \{ z : \text{Re } z \geq 0 \} \cup \{ z : \text{Im } z > 0 \} \)

(e) \( \{ z : |z| > 1, |\text{Arg } z| \leq \pi/4 \} \)

(f) \( \{ z : |z + 1 + 2i| \leq 1 \} \)

(g) \( \{ z : \text{Re } z > 1, |z - i| < 2 \} \)
Solutions to exercises

(h) \( \{ z : |z + i| < |z + 2i| \} \)

(i) \( \{ z : |z| < 3 \} - \{ z : |z| \leq 2 \} \)

(j) Since \( z^2 + z - 2 = (z + 2)(z - 1) \), we see that \( \{ z : z^2 + z - 2 = 0 \} = \{ 1, -2 \} \), so the set is \( \mathbb{C} - \{ 1, -2 \} \).

Solution to Exercise 5.1
Rearranging the given inequality, we obtain
\[
\frac{3r}{r^2 + 2} < 1 \iff 3r < r^2 + 2 \\
(\text{since } r^2 + 2 > 0, \text{ for all } r) \\
\iff 0 < r^2 - 3r + 2 \\
\iff 0 < (r - 1)(r - 2). 
\]
Since the final inequality is true for \( r > 2 \), the first inequality must be true for \( r > 2 \).

Solution to Exercise 5.2
(a) By the Triangle Inequality,
\[
|3 + 4z^2| \leq |3| + |4z^2| = 3 + 4|z|^2. 
\]
Hence, for \( |z| = 1 \),
\[
|3 + 4z^2| \leq 7, 
\]
so
\[
\left| \frac{1}{3 + 4z^2} \right| \geq \frac{1}{7}. 
\]
Now, by the backwards form of the Triangle Inequality,
\[
|3 + 4z^2| \geq |4|z|^2 - |3|. 
\]
Hence, for \( |z| = 1 \),
\[
|3 + 4z^2| \geq |4 - 3| = 1, 
\]
so
\[
\left| \frac{1}{3 + 4z^2} \right| \leq \frac{1}{1} = 1. 
\]
Thus
\[
\frac{1}{7} \leq \left| \frac{1}{3 + 4z^2} \right| \leq 1, \text{ for } |z| = 1, 
\]
as required.
(b) We first establish the right-hand inequality. By the Triangle Inequality,
\[
|z^3 + 2z + 1| \leq |z|^3 + 2|z| + 1, 
\]
and, by the backwards form of the Triangle Inequality,
\[
|z^2 + 1| \geq |z|^2 - 1. 
\]
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Hence, for $|z| = 3$,
\[ |z^3 + 2z + 1| \leq 34 \quad \text{and} \quad |z^2 + 1| \geq 8, \]
so
\[ \left| \frac{z^3 + 2z + 1}{z^2 + 1} \right| \leq \frac{34}{8} = \frac{17}{4}, \]
as required.

Next we establish the left-hand inequality. By the backwards form of the Triangle Inequality,
\[ |z^3 + 2z + 1| \geq |z|^3 - 2|z| - 1, \]
and, by the usual form of the Triangle Inequality,
\[ |z^2 + 1| \leq |z|^2 + 1. \]

Hence, for $|z| = 3$,
\[ |z^3 + 2z + 1| \geq 20 \quad \text{and} \quad |z^2 + 1| \leq 10, \]
so
\[ \left| \frac{z^3 + 2z + 1}{z^2 + 1} \right| \geq \frac{20}{10} = 2, \]
as required.

**Solution to Exercise 5.3**

Throughout this solution and the next two solutions, the appropriate version of the Triangle Inequality is given in parentheses.

(a) $|z + 3| \leq |z| + |3|$ (usual form)
\[ = 2 + 3 = 5, \quad \text{for } |z| = 2. \]

(b) $|z - 4i| \leq |z| + |4i|$ (usual form)
\[ = 2 + 4 = 6, \quad \text{for } |z| = 2. \]

(c) $|3z + 2| \leq |3z| + |2|$ (usual form)
\[ = 3|z| + 2 \]
\[ = 3 \times 2 + 2 = 8, \quad \text{for } |z| = 2. \]

(d) $|3z^2 - 5| \leq |3z^2| + |5|$ (usual form)
\[ = 3|z|^2 + 5 \]
\[ = 3 \times 2^2 + 5 = 17, \quad \text{for } |z| = 2. \]

(e) $|z^2 + z + 1| \leq |z^2| + |z| + |1|$ (usual form)
\[ = |z|^2 + |z| + 1 \]
\[ = 2^2 + 2 + 1 = 7, \quad \text{for } |z| = 2. \]

**Solution to Exercise 5.4**

(a) $|z - 2| \geq ||z| - |2||$ (backwards form)
\[ = |5 - 2| = 3, \quad \text{for } |z| = 5. \]

(b) $|z + 3i| \geq ||z| - |3i||$ (backwards form)
\[ = |5 - 3| = 2, \quad \text{for } |z| = 5. \]

(c) $|z - 7| \geq ||z| - |7||$ (backwards form)
\[ = |5 - 7| = 2, \quad \text{for } |z| = 5. \]

(d) $|2z - 7| \geq ||2z| - |7||$ (backwards form)
\[ = |10 - 7| = 3, \quad \text{for } |z| = 5. \]

**Solution to Exercise 5.5**

First we obtain upper and lower estimates for $|z^3 + 1|$, for $|z| = 4$, using appropriate versions of the Triangle Inequality. We have
\[ |z^3 + 1| \leq |z^3| + |1| \quad \text{(usual form)} \]
\[ = |z|^3 + 1 \]
\[ = 64 + 1 = 65, \quad \text{for } |z| = 4. \]

Also,
\[ |z^3 + 1| \geq |z^3| - |1| \quad \text{(backwards form)} \]
\[ = |z|^3 - 1 \]
\[ = 64 - 1 = 63, \quad \text{for } |z| = 4. \]

Next we obtain upper and lower estimates for $|z^3 - 1|$, for $|z| = 4$. We have
\[ |z^3 - 1| \leq |z^3| + |1| \quad \text{(usual form)} \]
\[ = |z|^3 + 1 \]
\[ = 64 + 1 = 65, \quad \text{for } |z| = 4. \]

Also,
\[ |z^3 - 1| \geq |z^3| - |1| \quad \text{(backwards form)} \]
\[ = |z|^3 - 1 \]
\[ = 64 - 1 = 63, \quad \text{for } |z| = 4. \]

So
\[ \left| \frac{z^3 + 1}{z^3 - 1} \right| = \left| \frac{z^3 + 1}{z^3 - 1} \right| \leq 65 \times \frac{1}{63} = \frac{65}{63}, \quad \text{for } |z| = 4, \]
and
\[ \left| \frac{z^3 + 1}{z^3 - 1} \right| = \left| \frac{z^3 + 1}{z^3 - 1} \right| \geq 63 \times \frac{1}{65} = \frac{63}{65}, \quad \text{for } |z| = 4. \]

Hence the given inequalities hold with $m = 63/65$ and $M = 65/63$. 