Diagnostic Quiz for the Open University Mathematics Msc

Contents

1 Introduction

This quiz is designed to help you see if you have sufficient mathematical background to enrol on the MSc in Mathematics programme, in particular the entry level course M820 *The Calculus of Variations and Advanced Calculus*.

Ideally you should work through all sections of the quiz, identifying any weak areas or gaps in your knowledge. The questions are designed to test the main subject areas needed for M820, and even if you find them straightforward you should read through the answers as they may contain extra useful information. However, you should not worry if you are unfamiliar with one or two subjects (for example, you might not have encountered PDEs before); the important thing is that you can follow the answers and, having done so, would be able to answer similar questions in the future. By the same token, this quiz cannot cover all the subject areas needed for the MSc programme, but if you are able to complete most of the questions here then you ought to be in a good position to succeed on the MSc.

If you find that you do have large gaps in your knowledge then you are advised either to enrol on one of our undergraduate courses, or to do some background reading prior to starting the MSc. Suggestions are given at the end of the quiz, before the answers.

If after working through the quiz you would like to discuss your study plans then you should contact Maths-Msc@open.ac.uk .

2 Differentiation

Exercise 2.1

(a) If $x + e^x = t$, find dx/dt and d^2x/dt^2.

(b) If $z = xe^{-y},\ x = \cosh t$ and $y = \cos t$, find dz/dt.

(c) If $z = (x + y)^5$ and $y = \sin 10x$, find dz/dx.

Exercise 2.2

(a) If $w = (r\cos\theta)^{r\sin\theta}$, for $r > 0$ and $0 \le \theta < \pi/2$, find $\partial w/\partial\theta$.

(b) Let $u = f(x, y)$. Given $x = e^s\cos t$ and $y = e^s\sin t$, find expressions for $\partial u/\partial s$ and $\partial u/\partial t$ in terms of $\partial u/\partial x$ and $\partial u/\partial y$. Hence prove that

$$\frac{\partial u}{\partial y} = e^{-s}\left(\sin t\frac{\partial u}{\partial s} + \cos t\frac{\partial u}{\partial t}\right).$$

Exercise 2.3

If $w = f(ax + by)$, where a and b are constants, show that $b\dfrac{\partial w}{\partial x} - a\dfrac{\partial w}{\partial y} = 0$.
(Hint: Let $z = ax + by$.)

Exercise 2.4

If $z = \dfrac{1}{x} f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = 0$.

Exercise 2.5

Find the Taylor series to second order about $x = -1,\ y = 1$ for the function

$$f(x, y) = x^2 + y^3 + \exp(xy).$$

Exercise 2.6

Locate and classify the stationary points of the function

$$f(x, y) = x^3 - 12x - y^3 + 3y.$$

3 Integration

Exercise 3.1

Evaluate the following integrals:

(a) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$;

(b) $\displaystyle\int \frac{x}{x^2 + 3}\,dx$;

(c) $\displaystyle\int \frac{x^3 + 3x - 2}{x^2 - 3x + 2}\,dx$, for $x > 2$;

(d) $\displaystyle\int e^{2x} \cos x\,dx$;

(e) $\displaystyle\int \sqrt{(1 - 4x^2)}\,dx$, for $|x| \leq 1/2$ (hint: use the substitution $2x = \sin\theta$);

(f) $\displaystyle\int_2^3 \frac{x + 1}{\sqrt{(x^2 - 4)}}\,dx$.

Exercise 3.2
Given that
$$I_{p,n} = \int_0^1 (1-x)^p x^n dx, \quad p \geq 0,\ n \geq 0,$$
show that $I_{p,n} = I_{n,p}$.
Prove that, for $p \geq 1$,
$$(n+1)I_{p,n} = pI_{p-1,n+1}$$
and also that
$$(p+n+1)I_{p,n} = pI_{p-1,n}.$$
Hence prove that, if p and n are positive integers,
$$I_{p,n} = \frac{p!n!}{(p+n+1)!}.$$

Exercise 3.3
The Fourier Transform $\widehat{f}(\omega)$ of a function $f(x)$ may be defined as
$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx.$$
Using the result $\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = 1$, find the Fourier Transform of $f(x) = e^{-x^2/(2\sigma^2)}$.
(Hint: you might find it helpful to complete the square in the exponent.)

Exercise 3.4
By changing the order of integration, show that
$$\int_0^x \left(\int_0^t f(p)\ dp \right) dt = \int_0^x (x-p)f(p)\ dp.$$

Exercise 3.5
Consider
$$S[y] = \int_a^b \left(\int_a^b K(s,t)y(s)y(t)\ dt \right) ds,$$
where a and b are constants, y is a function of a single variable and K is a function of two variables. ($S[y]$ is termed a *functional*, with the square brackets around the y emphasising the fact that S depends on the choice of the function y used to evaluate the integral.) Assuming that the order of integration may be interchanged, show that
$$\left. \frac{d}{d\epsilon} S[y+\epsilon h] \right|_{\epsilon=0} = \int_a^b \left(h(s) \int_a^b \left[K(s,t) + K(t,s) \right] y(t)\ dt \right) ds.$$

4 Differential Equations

4.1 First Order ODEs

Exercise 4.1
Find the general solutions of the following differential equations:

(a) $xy\dfrac{dy}{dx} = y^2 + 1;$

(b) $y\dfrac{dy}{dx} = e^{x+y^2};$

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} - \tan\left(\dfrac{y}{x}\right).$ (Hint: use the substitution $y = xu(x)$.)

Exercise 4.2
Find the particular solution of the following initial-value problem:

$t\dfrac{dy}{dt} + 2y = t^2, \quad y(1) = 1.$

4.2 Higher Order ODEs

Exercise 4.3
Find the general solutions of the following differential equations:

(a) $\dfrac{d^2y}{dx^2} - 4y = x^2;$

(b) $4x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - y = 0;$ (Hint: use the substitution $x = e^t$;)

(c) $\dfrac{d^4y}{dx^4} - 4y = 8.$

Exercise 4.4
Find the solutions of the following problems:

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0, \quad y(0) = 0, \ y'(0) = 1,$

(b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 5, \quad y(0) = 1, \ y(\pi) = 1.$

4.3 PDEs

Exercise 4.5

Find the ordinary differential equation satisfied by $f(r)$ if $\frac{1}{r} f(r)\cos\omega t$ is a solution of the PDE

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where ω and c are constants.

Hence show that a solution of this PDE is

$$u = \frac{1}{r}\left(A\cos nr + B\sin nr\right)\cos\omega t,$$

where $n = \omega/c$, and A and B are arbitrary constants.

For a particular solution the following three conditions all hold: u is finite at $r = 0$ for all t, $\partial u/\partial r = 0$ at $r = a$ for all t, and u is not identically zero. Show that $A = 0$ and obtain the equation which must be satisfied by ω.

Exercise 4.6

The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

Show that

$$u(x,t) = A\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right)$$

is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$$

where A and k are constants.

5 Linear Algebra

Exercise 5.1

(a) Show that if $\mathbf{A}$ and $\mathbf{B}$ are any two square matrices of the same size, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Let $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$.

(b) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

(c) Verify that $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

(d) Verify that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Exercise 5.2
Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Exercise 5.3
Evaluate the Vandermonde determinant

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \ldots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

6 Mathematical Proof

Exercise 6.1
Using mathematical induction prove that, if f is the function

$$f(x) = xe^x,$$

then, for all $n = 1, 2, \ldots$, the nth derivative of f is given by the formula

$$f^{(n)}(x) = (n + x)e^x.$$

Exercise 6.2
Prove by induction that $3^n < n!$ for all integers $n \geq 7$.

Exercise 6.3
Use proof by contradiction to prove that there are no integers m and n such that $5m + 15n = 357$.

Exercise 6.4
For each of the conditions given in (i)-(v), decide whether it is necessary, sufficient, necessary and sufficient, or neither necessary nor sufficient, in order that $x \equiv y(\mathrm{mod} 10)$:

(i) $x - y = 50$, (ii) $x - y = 5$, (iii) $x - y$ is divisible by 10,

(iv) $x - y$ is divisible by 5, (v) $x - y$ is divisible both by 5 and by 2,

(vi) x and y are both even.

7 Complex Numbers

Exercise 7.1
Find the roots of $z^3 - 7z^2 + 31z - 25 = 0$.

Exercise 7.2
Use the polar forms of $1 + i$ and $\sqrt{3} - i$ to evaluate

$$\frac{(1+i)^6}{(\sqrt{3}-i)^3},$$

giving your answer in Cartesian form.

8 Analysis

Exercise 8.1
Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$,

(b) $\displaystyle\lim_{x\to \pi/2} \frac{\ln(2-\sin x)}{\ln(1+\cos x)}$,

(c) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$,

(d) $\displaystyle\lim_{x\to 0} x \ln 2x$.

Exercise 8.2
Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $\displaystyle\lim_{n\to\infty} a_n = 0$ and $\displaystyle\lim_{n\to\infty} b_n = 0$. Show, from the definition of the limit of a sequence, that $\displaystyle\lim_{n\to\infty} a_n b_n = 0$.

Exercise 8.3
For $|z| = 2$, prove that

(a) $|z^2 - 4z - 3| \leq 15$,

(b) $|z^2 - 7| \geq 3$,

(c) $|z^2 + 2| \geq 2$.

Hence find a number M such that

$$\left|\frac{z^2 - 4z - 3}{(z^2-7)(z^2+2)}\right| \leq M, \quad \text{for } |z| = 2.$$

Exercise 8.4

The Squeeze Rule for continuity says that a function f is continuous at a if f, g and h are functions defined on an open interval I, $a \in I$ and the following three criteria hold: (i) $g(x) \leq f(x) \leq h(x), x \in I$, (ii) $g(a) = f(a) = h(a)$ and (iii) g and h are continuous at a.

Use the Squeeze Rule to prove that the following function is continuous at 0:

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

9 What to do now?

If you have been able to make reasonable attempts at most of the questions in this quiz then you should be in a strong position to make good progress on the MSc.

If however you feel that there are large gaps in your knowledge then you would be well advised to take one or more of the OU's undergraduate courses first. Alongside each solution in the quiz we have listed the OU undergraduate course which covers the relevant mathematics in that question. The courses that are particularly suitable preparation are:

- MST121 Using Mathematics,
- MS221 Exploring Mathematics,
- M208 Pure Mathematics,
- MST209 Mathematical methods and models,
- MST326 Mathematical methods and fluid mechanics,
- MS324 Waves, diffusion and variational principles,
- MS325 Computer algebra, chaos and simulations,
- M337 Complex analysis,
- M338 Topology.

These courses are available for viewing at your local Regional Office; contact the librarian there for details.

Alternatively you might prefer to do some background reading and the following list contains some suggestions.

- Kreyszig *Advanced Engineering Mathematics* (Wiley),
- Boas *Mathematical Methods in the Physical Sciences* (Wiley),
- Riley, Hobson and Bence *Mathematical Methods for Physics and Engineering* (Cambridge),

- Bronson *Differential equations* (Schaum),
- Priestly *Introduction to Complex Analysis*(Oxford),
- Sutherland *Introduction to Metric and Topological Spaces* (OUP),
- Binmore *Mathematical Analysis, a straightforward approach* (Cambridge).

As stated in the Introduction to this quiz, if you have any further questions as to your suitability for the MSc programme then please contact Maths-Msc@open.ac.uk .

10 Answers

Differentiation

If you have problems attempting the questions in this section then you should consider studying MST209 and MST326.

Solution for Exercise 2.1

(a) Differentiating implicitly with respect to t we have

$$\frac{dx}{dt} + e^x \frac{dx}{dt} = 1, \tag{1}$$

from which we obtain

$$\frac{dx}{dt} = \frac{1}{1+e^x}.$$

Differentiating (1) again with respect to t, we obtain

$$\frac{d^2x}{dt^2} + e^x \frac{d^2x}{dt^2} + e^x \left(\frac{dx}{dt}\right)^2 = 0,$$

from which we have

$$\frac{d^2x}{dt^2} = \frac{-e^x \left(dx/dt\right)^2}{1+e^x} = \frac{-e^x}{(1+e^x)^3}.$$

(b) The easiest way is probably to use the chain rule. We have

$$\frac{\partial z}{\partial x} = e^{-y}, \frac{\partial z}{\partial y} = -xe^{-y}, \frac{dx}{dt} = \sinh t \quad \text{and} \quad \frac{dy}{dt} = -\sin t,$$

so the chain rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^{-y}\sinh t + xe^{-y}\sin t \\ &= e^{-\cos t}\sinh t + \cosh t \sin t e^{-\cos t}. \end{aligned}$$

(Note that this question can also be answered using direct substitution; substitute for x and y to obtain z as a function of t, and then differentiate that directly.)

(c) Again although this could be answered using direct substitution we will use the chain rule, this time with

$$\frac{\partial z}{\partial x} = 5(x+y)^4, \frac{\partial z}{\partial y} = 5(x+y)^4 \quad \text{and} \quad \frac{dy}{dx} = 10\cos 10x.$$

So

$$\begin{aligned} \frac{dz}{dx} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 5(x+y)^4[1+10\cos 10x] \\ &= 5(x+\sin 10x)^4[1+10\cos 10x]. \end{aligned}$$

Solution for Exercise 2.2

(a) Taking logs, we obtain $\ln w = r\sin\theta\ln(r\cos\theta)$. Differentiating implicitly gives

$$\frac{1}{w}\frac{\partial w}{\partial \theta} = r\sin\theta\left(\frac{-r\sin\theta}{r\cos\theta}\right) + r\cos\theta\ln(r\cos\theta),$$

so that

$$\frac{\partial w}{\partial \theta} = (r\cos\theta)^{r\sin\theta}\left[r\cos\theta\ln(r\cos\theta) - \frac{r\sin^2\theta}{\cos\theta}\right].$$

(b) Using the chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\frac{\partial u}{\partial y}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = -e^s\sin t\frac{\partial u}{\partial x} + e^s\cos t\frac{\partial u}{\partial y}.$$

Hence

$$\begin{aligned} e^{-s}\frac{\partial u}{\partial s} &= \cos t\frac{\partial u}{\partial x} + \sin t\frac{\partial u}{\partial y}, \\ e^{-s}\frac{\partial u}{\partial t} &= \cos t\frac{\partial u}{\partial y} - \sin t\frac{\partial u}{\partial x}; \end{aligned}$$

multiplying the first of these by $\sin t$ and the second by $\cos t$ and then adding them together gives

$$\frac{\partial u}{\partial y} = e^{-s}\left(\sin t\frac{\partial u}{\partial s} + \cos t\frac{\partial u}{\partial t}\right).$$

Solution for Exercise 2.3
Following the hint we let $z = ax + by$ so that $w = f(z)$. Then

$$\frac{\partial w}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = a\frac{dw}{dz},$$

and similarly $\partial w/\partial y = b\ dw/dz$. Hence

$$b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = ab\frac{dw}{dz} - ab\frac{dw}{dz} = 0.$$

Solution for Exercise 2.4
Using the product and chain rules, and letting $u = y/x$ so that $z = 1/x\ f(u)$, we have

$$\frac{\partial z}{\partial x} = \frac{1}{x}\frac{df}{du}\frac{\partial u}{\partial x} + f(u)\left(\frac{-1}{x^2}\right) = -\frac{y}{x^3}f'(u) - \frac{1}{x^2}f(u)$$

and

$$\frac{\partial z}{\partial y} = \frac{1}{x}\frac{df}{du}\frac{\partial u}{\partial y} = \frac{1}{x^2}f'(u).$$

Hence

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + z = -\frac{y}{x^2}f'(u) - \frac{1}{x}f(u) + \frac{y}{x^2}f'(u) + \frac{1}{x}f(u) = 0.$$

Solution for Exercise 2.5
We differentiate $f(x, y) = x^2+y^3+\exp(xy)$ and evaluate the derivatives at $x = -1,\ y = 1$. This gives

$$\begin{aligned}
f_x &= 2x + y\exp(xy), \quad f_x(-1,1) = -2 + \exp(-1);\\
f_y &= 3y^2 + x\exp(xy), \quad f_y(-1,1) = 3 - \exp(-1);\\
f_{xx} &= 2 + y^2\exp(xy), \quad f_{xx}(-1,1) = 2 + \exp(-1);\\
f_{yy} &= 6y + x^2\exp(xy), \quad f_{yy}(2,1) = 6 + \exp(-1);\\
f_{xy} &= \exp(xy) + xy\exp(xy), \quad f_{xy}(-1,1) = 0.
\end{aligned}$$

The general formula for the Taylor series for two variables (x, y) about the point (a, b) is

$$f(x,y) = \sum_{n=0}^{\infty}\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^n f(a,b);$$

hence the required Taylor series (to order two) is given by

$$\begin{aligned}
f(x,y) &= f(-1,1) + f_x(-1,1)(x+1) + f_y(-1,1)(y-1)\\
&\quad + f_{xx}(-1,1)(x+1)^2/2! + f_{xy}(-1,1)(x+1)(y-1)\\
&\quad + f_{yy}(-1,1)(y-1)^2/2!\\
&= 2 + \exp(-1) + (-2+\exp(-1))(x+1) + (3-\exp(-1))(y-1)\\
&\quad + (2+\exp(-1))(x+1)^2/2\\
&\quad + (6+\exp(-1))(y-1)^2/2.
\end{aligned}$$

Solution for Exercise 2.6
We have $f_x = 3x^2 - 12$ and $f_y = -3y^2 + 3$, so the stationary points (found by putting $f_x = f_y = 0$) are at $(x, y) = (2, 1),\ (2, -1),\ (-2, 1)$ and $(-2, -1)$.

Also, $f_{xx} = 6x,\ f_{xy} = 0$ and $f_{yy} = -6y$.

Thus at $(2, 1)$ and at $(-2, -1)$ we have $f_{xx}f_{yy} - f_{xy}^2 = -36xy = -72 < 0$, and these are saddle points.

At $(2, -1)$ and at $(-2, 1)$ we have $f_{xx}f_{yy} - f_{xy}^2 = -36xy = 72 > 0$. Since $f_{xx} > 0$ at $(2, -1)$ this is a local minimum; since $f_{xx} < 0$ at $(-2, 1)$ this is a local maximum.

Integration

If you have problems attempting the questions in this section then you should consider studying MST121 and MS221.

Solution for Exercise 3.1
In the following answers, c is an arbitrary real constant.

(a) $\displaystyle\int_0^\infty xe^{-x^2}dx = \left[\frac{e^{-x^2}}{-2}\right]_0^\infty = \frac{1}{2};$

(b) $\displaystyle\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \frac{1}{2}\ln(x^2+3) + c;$

(c) $\displaystyle\int \frac{x^3+3x-2}{x^2-3x+2}dx = \int\left(x+3+\frac{10x-8}{(x-1)(x-2)}\right)dx = \int\left(x+3+\frac{12}{x-2}-\frac{2}{x-1}\right)dx$

$\displaystyle= \frac{x^2}{2} + 3x + 12\ln(x-2) - 2\ln(x-1) + c;$

(d) This question can be solved using integration by parts. However, an alternative neater (and arguably shorter) method is as follows:

$$\begin{aligned}\int e^{2x}\cos x\ dx &= \mathrm{Re}\left\{\int e^{2x}e^{ix}dx\right\} = \mathrm{Re}\left\{\int e^{(2+i)x}dx\right\} = \mathrm{Re}\left\{\frac{1}{2+i}e^{(2+i)x}\right\} + c \\ &= \mathrm{Re}\left\{\frac{2-i}{5}e^{2x}e^{ix}\right\} + c = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + c;\end{aligned}$$

(e) Let $2x = \sin\theta$; then

$\displaystyle\int \surd(1-4x^2)dx = \int \surd(1-\sin^2\theta)\frac{1}{2}\cos\theta\ d\theta = \frac{1}{2}\int\cos^2\theta\ d\theta = \frac{1}{4}\int(1+\cos 2\theta)\ d\theta$

$\displaystyle= \frac{1}{4}(\theta + \frac{1}{2}\sin 2\theta) + c = \frac{1}{4}(\theta + \sin\theta\cos\theta) + c = \frac{1}{4}(\arcsin(2x) + 2x\surd(1-4x^2)) + c;$

(f) Observe there is a singularity at $x = 2$ so the integral is improper. However we will integrate as normal and see what happens.

Let $x = 2\cosh u$; then

$$\int_2^3 \frac{x+1}{\sqrt{(x^2-4)}}\ dx = \int_{x=2}^{x=3} \frac{2\cosh u + 1}{\sqrt{(4\cosh^2 u - 4)}} 2\sinh u\ du = \int_{x=2}^{x=3} (2\cosh u + 1)\ du$$

$$= [2\sinh u + u]_{x=2}^{x=3} = \left[2\sqrt{\left((x/2)^2 - 1\right)} + \operatorname{arcosh}\left(\frac{x}{2}\right)\right]_2^3 = \sqrt{5} + \operatorname{arcosh}(3/2).$$

The evaluation worked because the primitive was defined and continuous at $x = 2$.

Solution for Exercise 3.2

Using the substitution $u = 1 - x$,

$$\begin{aligned} I_{n,p} &= \int_0^1 (1-x)^n x^p dx = \int_1^0 u^n (1-u)^p (-du) = \int_0^1 u^n (1-u)^p du \\ &= \int_0^1 x^n (1-x)^p dx = I_{p,n} \end{aligned}$$

as required.

For the next part of the question,

$$I_{p,n} = \int_0^1 (1-x)^p x^n dx = \left[(1-x)^p \frac{x^{n+1}}{n+1}\right]_0^1 + \int_0^1 \frac{x^{n+1}}{n+1} p(1-x)^{p-1} dx,$$

so that

$$(n+1) I_{p,n} = p I_{p-1,n+1} \tag{2}$$

as required. Using this result, we then have

$$\begin{aligned} (p+n+1) I_{p,n} &= p I_{p,n} + (n+1) I_{p,n} = p I_{p,n} + p I_{p-1,n+1} \\ &= p \int_0^1 \left((1-x)^p x^n + (1-x)^{p-1} x^{n+1}\right) dx \\ &= p \int_0^1 (1-x)^{p-1} x^n [1 - x + x]\ dx \end{aligned}$$

so that

$$(p+n+1) I_{p,n} = p I_{p-1,n} \tag{3}$$

as required.

From (3) we have

$$I_{p,n} = \frac{p}{(p+n+1)} I_{p-1,n}.$$

From (2),

$$I_{p-1,n+1} = \frac{(n+1)}{p} I_{p,n} = \frac{(n+1)}{(p+n+1)} I_{p-1,n},$$

which is equivalent to

$$I_{p-1,n} = \frac{n}{p+n} I_{p-1,n-1}.$$

So overall this gives

$$\begin{aligned}
I_{p,n} &= \frac{p}{(p+n+1)}\frac{n}{(p+n)}I_{p-1,n-1} \\
&= \frac{p}{(p+n+1)}\frac{n}{(p+n)}\frac{p-1}{(p+n-1)}\frac{n-1}{(p+n-2)}I_{p-2,n-2} \\
&= \frac{p!n!}{(p+n+1)!}I_{0,0},
\end{aligned}$$

and $I_{0,0} = \displaystyle\int_0^1 dx = 1$ gives the final result.

Solution for Exercise 3.3
If you have problems attempting this question then you should consider studying MS324.

$\widehat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}-i\omega x}dx = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x+\sigma^2 i\omega)^2+\sigma^4\omega^2\right]}dx,$
on completing the square.

Now let $y = x + \sigma^2 i\omega$ so that

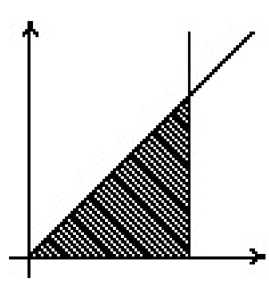
$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}e^{-\frac{\sigma^2\omega^2}{2}}dy = \frac{2}{\sqrt{2\pi}}e^{-\frac{\sigma^2\omega^2}{2}}\int_0^{\infty} e^{-(\frac{y}{\sqrt{2}\sigma})^2}dy \\
&= \sqrt{\frac{2}{\pi}}e^{-\frac{\sigma^2\omega^2}{2}}\int_0^{\infty} e^{-t^2}\sqrt{2}\sigma\ dt,
\end{aligned}$$

on letting $y = \sqrt{2}\sigma t$. Use of the integral result given in the question gives the Fourier Transform as

$$\widehat{f}(\omega) = \sigma e^{-\frac{\sigma^2\omega^2}{2}}.$$

Solution for Exercise 3.4
If you have problems attempting this question then you should consider studying MS324.

With $\displaystyle\int_0^x \left(\int_0^t f(p)\ dp\right) dt$, we first integrate with respect to p for p from 0 to t, and then with respect to t for t from 0 to x. If we change the order of integration then we will integrate with respect to t for t from p to x, and then with respect to p for p from 0 to x.



Hence we have

$$\begin{aligned}\int_0^x \left(\int_0^t f(p)\ dp \right) dt &= \int_0^x \left(\int_p^x f(p)\ dt \right) dp = \int_0^x \left[t f(p) \right]_{t=p}^{t=x} dp \\ &= \int_0^x (x-p) f(p)\ dp.\end{aligned}$$

Solution for Exercise 3.5

First note that

$$S[y+\epsilon h] = \int_a^b \left(\int_a^b K(s,t)(y(s)+\epsilon h(s))(y(t)+\epsilon h(t))\ dt \right) ds,$$

so that

$$\frac{d}{d\epsilon} S[y+\epsilon h] = \int_a^b \left(\int_a^b K(s,t) \left[y(s)h(t) + h(s)y(t) + O(\epsilon) \right]\ dt \right) ds.$$

Taking the limit as $\epsilon \to 0$ gives

$$\begin{aligned}\left. \frac{d}{d\epsilon} S[y+\epsilon h] \right|_{\epsilon=0} &= \int_a^b \left(\int_a^b K(s,t) \left[y(s)h(t) + h(s)y(t) \right]\ dt \right) ds \\ &= \int_a^b \left(\int_a^b K(s,t)y(s)h(t)\ dt \right) ds + \int_a^b \left(\int_a^b K(s,t)h(s)y(t)\ dt \right) ds.\end{aligned}$$

Now, we can interchange the order of integration for the first term, so that

$$\int_a^b \left(\int_a^b K(s,t)y(s)h(t)\ dt \right)\ ds = \int_a^b \left(\int_a^b K(s,t)y(s)h(t)\ ds \right) dt,$$

and the second term can be written as follows:

$$\int_a^b \left(\int_a^b K(s,t)h(s)y(t)\ dt \right) ds = \int_a^b \left(\int_a^b K(t',s')h(t')y(s')\ ds' \right) dt',$$

where the dummy variable s has been replaced by t' and similarly t has been replaced by s'. Hence

$$\left. \frac{d}{d\epsilon} S[y+\epsilon h] \right|_{\epsilon=0} = \int_a^b \left(\int_a^b K(s,t)y(s)h(t)\ ds \right) dt + \int_a^b \left(\int_a^b K(t',s')h(t')y(s')\ ds' \right) dt'.$$

Finally, we can change dummy variable again in the second term, replacing s' by s and t' by t to obtain

$$\begin{aligned}\left. \frac{d}{d\epsilon} S[y+\epsilon h] \right|_{\epsilon=0} &= \int_a^b \left(\int_a^b K(s,t)y(s)h(t)\ ds \right) dt + \int_a^b \left(\int_a^b K(t,s)h(t)y(s)\ ds \right) dt \\ &= \int_a^b \left(\int_a^b \left[K(s,t)y(s)h(t) + K(t,s)h(t)y(s) \right]\ ds \right) dt,\end{aligned}$$

and as $h(t)$ is independent of s we can bring it out of the s integral to get

$$\left. \frac{d}{d\epsilon} S[y+\epsilon h] \right|_{\epsilon=0} = \int_a^b \left(h(t) \int_a^b \left[K(s,t) + K(t,s) \right] y(s)\ ds \right) dt,$$

which, on changing variables one more time, gives the required result.

Differential Equations

If you have problems attempting the questions in this section then you should consider studying MST209 and MST326.

Solution for Exercise 4.1

(a) Separating the variables and integrating gives $\int \frac{y}{y^2+1} dy = \int \frac{1}{x} dx$ so that

$$\frac{1}{2}\ln(y^2+1) = \ln x + \ln c,$$

where c is a constant and we use $\ln c$ for the constant of integration to make subsequent manipulations easier. Hence $\ln(y^2+1) = \ln(xc)^2$ so that

$$y = \pm\sqrt{(dx^2 - 1)},$$

where d is a constant (equal to c^2).

(b) We can write the differential equation as $y\frac{dy}{dx} = e^x e^{y^2}$ so that on separating the variables we get $\int ye^{-y^2} dy = \int e^x dx$. Integrating, we obtain

$$-\frac{1}{2}e^{-y^2} = e^x + c,$$

so that $y = \pm\sqrt{(-\ln(-2e^x + d))}$ (for c and d both constants).

(c) Following the hint, let $y = xu$ (where u is a function of x). Then the product rule for differentiation gives $y' = xu' + u$, the prime denoting differentiation with respect to x. Substituting this, along with $y = xu$, into the original equation gives $xu' + u = u - \tan u$ so that the original differential equation has transformed into

$$x\frac{du}{dx} = -\tan u.$$

This can be solved by separating the variables:

$$\int \frac{\cos u}{\sin u}\, du = -\int \frac{1}{x}\, dx$$

so that $\ln(\sin u) = -\ln x + \ln c = \ln(c/x)$, where c is an arbitrary constant. Hence $u = \arcsin(c/x)$, and so the solution to the original differential equation is

$$y = x\arcsin(c/x).$$

Solution for Exercise 4.2
After division by t, the given equation can be written as

$$\frac{dy}{dt} + \frac{2}{t}y = t. \tag{4}$$

(To avoid division by zero, we can set $t > 0$, say, which is consistent with the given initial condition.)
The integrating factor is

$$p = \exp\left(\int \frac{2}{t}\, dt\right) = \exp(2\ln t) = t^2,$$

and multiplying (4) through by p gives

$$t^2\frac{dy}{dt} + 2ty = t^3.$$

Thus the differential equation can be written as

$$\frac{d}{dt}\left(t^2 y\right) = t^3,$$

and on integrating we obtain the general solution $t^2 y = t^4/4 + c$, or equivalently $y = t^2/4 + ct^{-2}$, where c is an arbitrary constant.
From the initial condition $y(1) = 1$ we have $1 = 1/4 + c$, so $c = 3/4$ and hence the solution of the initial-value problem is

$$y = \frac{1}{4}(t^2 + 3t^{-2}).$$

Solution for Exercise 4.3

(a) The auxiliary equation is $m^2 - 4 = 0$ which has solutions $m = \pm 2$ and so the complementary function is $y = Ae^{2x} + Be^{-2x}$ or $y = C\sinh(2x) + D\cosh(2x)$ where A, B, C and D are arbitrary constants (either form is perfectly acceptable).

For the particular solution try $y = ax^2 + bx + c$ so that $y'' = 2a$ and on substituting these into the original differential equation we obtain

$$2a - 4(ax^2 + bx + c) = x^2.$$

On equating coefficients we get $-4a = 1$ so that $a = -1/4$, $2a - 4c = 0$ so that $c = -1/8$, and $b = 0$. Hence the general solution is

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{4}x^2 - \frac{1}{8}$$

or

$$y = C\sinh(2x) + D\cosh(2x) - \frac{1}{4}x^2 - \frac{1}{8}.$$

(b) For $x = e^t$, note that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

and that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt} \; .$$

Substituting these into the original differential equation and simplifying results in

$$4\frac{d^2y}{dt^2} - y = 0,$$

the general solution of which is $y = Ae^{t/2} + Be^{-t/2}$, where A and B are arbitrary constants.

Hence the general solution of the original differential equation is

$$y = A\sqrt{x} + \frac{B}{\sqrt{x}} \; .$$

(c) Here the auxiliary equation is $m^4 - 4 = 0$, which has solutions $m = \pm\sqrt{2}, \pm\sqrt{2}i$. So the complementary function is

$$y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x),$$

where A, B, C and D are arbitrary constants.
Alternatively this solution may be written as

$$y = A\sinh(\sqrt{2}x) + B\cosh(\sqrt{2}x) + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x).$$

For the particular integral, let $y = a$. Then $y^{(iv)} = 0$ and so $-4a = 8$. Hence the general solution is

$$y = A\sinh(\sqrt{2}x) + B\cosh(\sqrt{2}x) + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x) - 2.$$

Solution for Exercise 4.4

(a) The auxiliary equation is $m^2-3m+2=0$, which has roots $m=1$ and $m=2$. Hence the complementary function is $y=Ce^x+De^{2x}$, where C and D are arbitrary constants. Since the differential equation is homogeneous, this is also the general solution.

Differentiating the general solution we obtain $y'=Ce^x+2De^{2x}$, so the initial conditions $y(0)=0$ and $y'(0)=1$ give

$$\begin{aligned} 0 &= Ce^0+De^0=C+D, \\ 1 &= Ce^0+2De^0=C+2D. \end{aligned}$$

Solving these gives $C=-1, D=1$, so the required solution to this initial-value problem is

$$y=-e^x+e^{2x}.$$

(b) The auxiliary equation is $m^2+4m+5=0$, which has solutions $m=-2\pm i$. So the complementary function is

$$y_c=e^{-2x}(C\cos x+D\sin x),$$

for C and D arbitrary constants.
To find a particular integral, try $y=a$. Substituting this into the differential equation gives $a=1$, so a particular integral is $y_p=1$ and the general solution is

$$y=e^{-2x}(C\cos x+D\sin x)+1.$$

The boundary conditions are $y(0)=1$ and $y(\pi)=1$; substituting each of these in turn into the general solution gives

$$\begin{aligned} 1 &= e^0(C\cos 0+D\sin 0)+1=C+1, \\ 1 &= e^{-2\pi}(C\cos\pi+D\sin\pi)+1=-Ce^{-2\pi}+1. \end{aligned}$$

Both of these reduce to $C=0$, but D can take any value, so any solution of the form

$$y=De^{-2x}\sin x+1$$

satisfies the differential equation and the boundary conditions. This is therefore an example of a boundary-value problem which does not have a unique solution; there is an infinite family of possible solutions.

Solution for Exercise 4.5

With $u=\dfrac{1}{r}f(r)\cos\omega t$, we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \cos\omega t\left[\frac{1}{r}f'(r)-\frac{1}{r^2}f(r)\right], \\ \frac{\partial^2 u}{\partial r^2} &= \cos\omega t\left[\frac{1}{r}f''(r)-\frac{2}{r^2}f'(r)+\frac{2}{r^3}f(r)\right], \\ \frac{\partial^2 u}{\partial t^2} &= -\frac{\omega^2}{r}f(r)\cos\omega t. \end{aligned}$$

Substituting these into the given PDE, dividing through by $\cos\omega t$ and rearranging gives

$$f''(r) + \frac{\omega^2}{c^2} f(r) = 0,$$

which is the required ODE. This has solution $f(r) = A\cos nr + B\sin nr$ (where $n = \omega/c$) so that the required solution to the PDE is

$$u = \frac{1}{r}(A\cos nr + B\sin nr)\cos\omega t.$$

For u to be finite at $r = 0$ requires $A = 0$, so $u = B/r \sin nr \cos\omega t$. Also,

$$\frac{\partial u}{\partial r} = \left(\frac{B}{r} n\cos nr - \frac{B}{r^2}\sin nr\right)\cos\omega t,$$

and for this to be zero at $r = a$ for all t requires

$$\frac{B}{a} n\cos na - \frac{B}{a^2}\sin na = 0;$$

excluding the trivial solution $u \equiv 0$ results in $\tan na = na$, so the required equation for ω is

$$\tan\left(\frac{\omega a}{c}\right) = \frac{\omega a}{c}.$$

Solution for Exercise 4.6

If you have problems attempting this question then you should consider studying MS324. From

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

we have

$$\frac{d}{dx}(\operatorname{erf}(x)) = \frac{2}{\sqrt{\pi}} e^{-x^2};$$

hence for $u = A - A\operatorname{erf}\left(\dfrac{x}{2\sqrt{kt}}\right)$ we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{2A}{\sqrt{\pi}} e^{-x^2/(4kt)} \frac{1}{2\sqrt{kt}},\\
\frac{\partial^2 u}{\partial x^2} &= \frac{A}{\sqrt{\pi kt}} \frac{2x}{4kt} e^{-x^2/(4kt)},\\
\frac{\partial u}{\partial t} &= -\frac{2A}{\sqrt{\pi}} e^{-x^2/(4kt)} \frac{x}{2\sqrt{k}} \frac{-1}{2t\sqrt{t}}.
\end{aligned}$$

Substituting these into the given PDE gives

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{k}\frac{\partial u}{\partial t} = \frac{2Ax}{4kt\sqrt{\pi kt}} e^{-x^2/(4kt)} - \frac{2Ax}{4kt\sqrt{\pi kt}} e^{-x^2/(4kt)} = 0.$$

Linear algebra

If you have problems attempting the questions in this section then you should consider studying MS221 and M208.

Solution for Exercise 5.1
(a) We have

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

and

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

(we need to check both of these as matrix multiplication is not commutative).

Hence $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Since $\det \mathbf{A} = 2$, we have

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{bmatrix}.$$

Similarly

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

(c)

$$(\mathbf{A}+\mathbf{B})^{-1} = \begin{bmatrix} 5 & 6 \\ 3 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -\frac{5}{3} \end{bmatrix}$$

and

$$\mathbf{A}^{-1}+\mathbf{B}^{-1} = \begin{bmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & 1 \\ -\frac{5}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1}+\mathbf{B}^{-1}$.

(d)

$$(\mathbf{AB})^{-1} = \left(\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{bmatrix}$$

and

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{bmatrix}.$$

Thus $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Solution for Exercise 5.2
The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0.$$

Expanding the determinant by the first row gives

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+1) = 0,$$

so $1-\lambda = 0$ or $\lambda^2+1=0$. Thus the eigenvalues are $\lambda = 1,\ \lambda = i$ and $\lambda = -i$.

The eigenvector equations are

$$\begin{aligned} (1-\lambda)x_1 \qquad\qquad &= 0, \\ -\lambda x_2 + \quad x_3 &= 0, \\ -x_2 - \lambda x_3 &= 0. \end{aligned}$$

When $\lambda = 1$ the eigenvector equations become $0 = 0,\ -x_2 + x_3 = 0$ and $-x_2 - x_3 = 0$, which reduce to the equations $x_2 = 0$ and $x_3 = 0$. There is no constraint on x_1, so we may choose it as we please. Hence a corresponding eigenvector is $[1\ \ 0\ \ 0]^T$.

When $\lambda = i$ the eigenvector equations become $(1-i)x_1 = 0,\ -ix_2 + x_3 = 0$ and $-x_2 - ix_3 = 0$, which reduce to the equations $x_1 = 0$ and $ix_2 = x_3$. It follows that a corresponding eigenvector is $[0\ \ 1\ \ i]^T$.

When $\lambda = -i$ the eigenvector equations become $(1+i)x_1 = 0,\ ix_2 + x_3 = 0$ and $-x_2 + ix_3 = 0$, which reduce to the equations $x_1 = 0$ and $-ix_2 = x_3$. It follows that a corresponding eigenvector is $[0\ \ 1\ \ -i]^T$.

Solution for Exercise 5.3

We subtract x_1 times the penultimate row from the last row, then x_1 times the $(n-2)$-th row from the penultimate row, then x_1 times the $(n-3)$-th row from the $(n-2)$-th row, etc., ending with x_1 times the first row from the second row. This gives us

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1 \\ 0 & x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \ldots & x_n^2 - x_1x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & x_2^{n-2} - x_1x_2^{n-3} & x_3^{n-2} - x_1x_3^{n-3} & \ldots & x_n^{n-2} - x_1x_n^{n-3} \\ 0 & x_2^{n-1} - x_1x_2^{n-2} & x_3^{n-1} - x_1x_3^{n-2} & \ldots & x_n^{n-1} - x_1x_n^{n-2} \end{vmatrix}.$$

Evaluating this using the first column and factorising each element gives

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \ldots & x_n(x_n - x_1) \\ \ldots & \ldots & \ldots & \ldots \\ x_2^{n-3}(x_2 - x_1) & x_3^{n-3}(x_3 - x_1) & \ldots & x_n^{n-3}(x_n - x_1) \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \ldots & x_n^{n-2}(x_n - x_1) \end{vmatrix}.$$

Factoring out from the first column the common factor $x_2 - x_1$, from the second column

$x_3 - x_1$, etc., ending with $x_n - x_1$ from the $(n-1)$-th column, we obtain

$$V_n(x_1, x_2, \ldots, x_n) = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_2 & x_3 & \ldots & x_n \\ x_2^2 & x_3^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ x_2^{n-2} & x_3^{n-2} & \ldots & x_n^{n-2} \end{vmatrix}.$$

Following the same procedures eventually results in

$$V_n(x_1, x_2, \ldots, x_n) = \Pi_{n \geq k > i \geq 1}(x_k - x_i).$$

Mathematical Proof

If you have problems attempting the questions in this section then you should consider studying M208.

Solution for Exercise 6.1
Let $P(n)$ be the proposition $f^{(n)}(x) = (n + x)e^x$.

We first check that $P(1)$ is true. Using the product rule,

$$f^{(1)}(x) = f'(x) = e^x + xe^x = (1 + x)e^x.$$

Hence $P(1)$ is true.
Now we assume that $P(k)$ is true for an integer $k \geq 1$, so that $f^{(k)}(x) = (k+x)e^x$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}(f^{(k)}(x)) = \frac{d}{dx}((k + x)e^x) \\ &= e^x + (k + x)e^x = (k + 1 + x)e^x. \end{aligned}$$

That is we have shown that $P(k) \Rightarrow P(k + 1)$. Hence by induction $P(n)$ is true for $n = 1, 2, \ldots$.

Solution for Exercise 6.2
Let $P(n)$ be the statement $3^n < n!$. Then $P(7)$ is true because $3^7 = 2187 < 5040 = 7!$. (We note that $P(n)$ is false for $n = 1, 2, \ldots, 6$.)

Now assume that $P(k)$ is true for an integer $k \geq 7$, that is that $3^k < k!$.
We wish to deduce that $P(k + 1)$ is true, that is that $3^{k+1} < (k + 1)!$.
Then,

$$\begin{aligned} 3^{k+1} &= 3 \times 3^k < 3 \times k! \text{ (by } P(k)) \\ &< (k + 1)k! \text{ (because } k \geq 7, \text{ and hence } k + 1 \geq 8 > 3) \\ &= (k + 1)!. \end{aligned}$$

That is we have shown that $P(k) \Rightarrow P(k + 1)$. Hence by induction $P(n)$ is true for $n = 7, 8, \ldots$.

Solution for Exercise 6.3
Suppose that there exist integers m and n with $5m + 15n = 357$. The left-hand side of this equation is a multiple of 5, so the right-hand side is also. But this is a contradiction, so our supposition must be false. Hence there are no such integers m and n.

Solution for Exercise 6.4
(i) This is a sufficient condition. If $x - y = 50$, then x is congruent to y modulo 10. But it is not a necessary condition. For example, $51 \equiv 31 \pmod{10}$, but $51 - 31$ is not equal to 50.

(ii) This condition is neither necessary nor sufficient.

(iii) This condition is both necessary and sufficient that $x \equiv y \pmod{10}$.

(iv) This is a necessary condition, but not sufficient. If $x \equiv y$ (mod10), then $x - y$ is divisible by 10, and so must be divisible by 5. However, this condition is not enough to ensure that $x \equiv y \pmod{10}$; for example, we might have $x - y = 15$, and x would then not be congruent to y modulo 10.

(v) This condition is equivalent to that in (iii), and is again a necessary and sufficient condition that $x \equiv y \pmod{10}$.

(vi) This condition is neither necessary nor sufficient.

Complex numbers

If you have problems attempting the questions in this section then you should consider studying M208.

Solution for Exercise 7.1
By inspection we see that $z = 1$ is one root of the given cubic, and so we factorise the cubic as follows:

$$z^3 - 7z^2 + 31z - 25 = (z - 1)(z^2 - 6z + 25).$$

Solving the quadratic equation $z^2 - 6z + 25 = 0$ gives the other two roots as $z = 3 \pm 4i$.

Solution for Exercise 7.2
We have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

so by de Moivre's Theorem

$$(1 + i)^6 = 2^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right).$$

Similarly, as

$$\sqrt{3} - 1 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right),$$

we have

$$(\sqrt{3} - i)^{-3} = 2^{-3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

Hence

$$\begin{aligned}\frac{(1+i)^6}{(\sqrt{3}-i)^3} &= 2^3\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)\times 2^{-3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\\ &= \cos 2\pi + i\sin 2\pi = 1.\end{aligned}$$

Analysis

If you have problems attempting the questions in this section then you should consider studying M208.

Solution for Exercise 8.1
There is usually more than one way of answering questions about limits; in the following examples we will use L'Hôpital's rule. This method can be applied in certain circumstances, but needs to be used with care.

(a) $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3} = \lim\limits_{x\to 0}\dfrac{\sec^2 x - 1}{3x^2} = \lim\limits_{x\to 0}\dfrac{2\sec^2 x\tan x}{6x}$
$= \lim\limits_{x\to 0}\dfrac{2\sec^4 x + 4\sec^2 x\tan^2 x}{6} = \dfrac{1}{3}$,

(b) $\lim\limits_{x\to\pi/2}\dfrac{\ln(2-\sin x)}{\ln(1+\cos x)} = \lim\limits_{x\to\pi/2}\dfrac{-\cos x/(2-\sin x)}{-\sin x/(1+\cos x)} = 0$,

(c) $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}} = \lim\limits_{x\to\infty}\dfrac{1/x}{x^{-1/2}/2} = \lim\limits_{x\to\infty}\dfrac{2}{\sqrt{x}} = 0$,

(d) $\lim\limits_{x\to 0} x\ln 2x = \lim\limits_{x\to 0}\dfrac{\ln 2x}{1/x} = \lim\limits_{x\to 0}\dfrac{2/(2x)}{-1/x^2} = \lim\limits_{x\to 0}(-x) = 0.$

Solution for Exercise 8.2
From the definition of a limit of a sequence we know that:
for each $\epsilon > 0$, there is an integer N_a such that $|a_n| < \epsilon$, for all $n > N_a$; for each $\epsilon > 0$, there is an integer N_b such that $|b_n| < \epsilon$, for all $n > N_b$.
In particular there is an N'_a such that for $n > N'_a$, $|a_n| < 1$.
Let $N = \max\{N'_a, N_b\}$ then for $n > N, |a_n b_n| = |a_n||b_n| < \epsilon$ and so $\lim\limits_{n\to\infty} a_n b_n = 0$ as required.

Solution for Exercise 8.3
To answer this question we will use the Triangle Inequality in both its usual form and its backwards form. These forms are, respectively:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \text{and} \quad |z_1 - z_2| \geq ||z_1| - |z_2||\,,$$

for any $z_1, z_2 \in \mathbb{C}$.

(a) By the Triangle Inequality,

$$|z^2 - 4z - 3| \leq |z^2| + |-4z| + |-3| = |z^2| + 4|z| + 3,$$

so for $|z| = 2$ we have

$$|z^2 - 4z - 3| \leq 4 + 8 + 3 = 15.$$

(b) By the backwards form of the Triangle Inequality, we have

$$|z^2 - 7| \geq \left| |z|^2 - 7 \right|.$$

So for $|z| = 2$ we have

$$|z^2 - 7| \geq |4 - 7| = 3.$$

(c) Similarly,

$$|z^2 + 2| \geq \left| |z|^2 - 2 \right| = |4 - 2| = 2.$$

Hence we can write

$$\left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| = \frac{|z^2 - 4z - 3|}{|z^2 - 7|\ |z^2 + 2|} \leq \frac{15}{3 \times 2} = \frac{5}{2},$$

and so we can take $M = 5/2$.

Solution for Exercise 8.4

Since

$$-1 \leq \cos(1/x^2) \leq 1, \quad \text{for } x \neq 0,$$

we have

$$-x^2 \leq x^2 \cos(1/x^2) \leq x^2, \quad \text{for } x \neq 0.$$

Hence

$$-x^2 \leq f(x) \leq x^2 \quad \text{for } x \in \mathbb{R}.$$

Thus if we take $g(x) = -x^2$ and $h(x) = x^2$ all three conditions of the Squeeze Rule hold, with $a = 0$ and $I = \mathbb{R}$.

Hence f is continuous at 0, by the Squeeze Rule.